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Mathematics Curriculum

New Brunswick
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Educational Programs & Services Branch

CURRICULUM

Advanced Mathematics with an Introduction to Calculus 120

(Implementation Edition)

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- The teachers and other educators and stakeholders across Atlantic Canada who contributed to the development of the *Advanced Mathematics with an Introduction to Calculus 120* curriculum guide.

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I. Background and Rationale

A. Background

Mathematics curriculum reform in Atlantic Canada is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in an increasingly technological society. Curriculum reform has been motivated by a desire to ensure that students in Atlantic Canada benefit from world-class curriculum and instruction in mathematics as a significant part of their learning experiences.

The *Foundation for the Atlantic Canada Mathematics Curriculum* (1996) firmly establishes the *Curriculum and Evaluation Standards for School Mathematics* (1989) of the National Council of Teachers of Mathematics (NCTM) as a guiding beacon for pursuing this vision, which embraces the principles of students learning to value and become active "doers" of mathematics and advocates a curriculum which focusses on the unifying ideas of mathematical problem solving, communication, reasoning, and connections. These principles and unifying ideas are reaffirmed with the publication of NCTM's *Principles and Standards for School Mathematics* (2000). The *Foundation for the Atlantic Canada Mathematics Curriculum* establishes a framework for the development of detailed grade-level documents describing mathematics curriculum and guiding instruction.

Mathematics curriculum development has taken place under the auspices of the Atlantic Provinces Education Foundation (APEF), an organization sponsored and managed by the governments of the four Atlantic Provinces. APEF has brought together teachers with department of education officials to co-operatively plan and execute the development of curricula in mathematics, science, language arts, and other curricular areas. Each of these curriculum efforts has been aimed at producing a program that would ultimately support the Essential Graduation Learnings (EGLs), also developed regionally. These EGLs and the contribution of the mathematics curriculum to their achievement are presented in the "Outcomes" section of the mathematics foundation document.

B. Rationale

The *Foundation for the Atlantic Canada Mathematics Curriculum* provides an overview of the philosophy and goals of the mathematics curriculum, presenting broad curriculum outcomes and addressing a variety of issues with respect to the learning and teaching of mathematics. This curriculum guide is one of several which provide greater specificity and clarity for the classroom teacher. The *Foundation for the Atlantic Canada Mathematics Curriculum* describes the mathematics curriculum in terms of a series of outcomes—General Curriculum Outcomes (GCOs), which relate to subject strands, and Key-Stage Curriculum Outcomes (KSCOs), which articulate the GCOs further for the end of grades 3, 6, 9, and 12. This guide builds on the structure introduced in the foundation document, by relating Specific Curriculum Outcomes (SCOs) to KSCOs for *Advanced Mathematics with an Introduction to Calculus 120*. Figure 1 further clarifies the outcome structure.

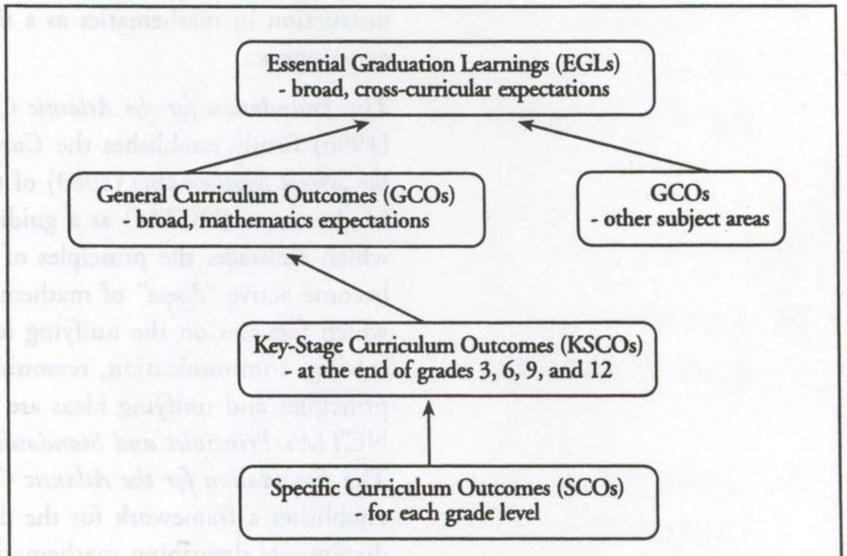


Figure 1: Outcome Framework

This mathematics guide is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice, including the following: (i) mathematics learning is an active and constructive process; (ii) learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates; (iii) learning is most likely when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and nurtures positive attitudes and sustained effort; (iv) learning is most effective when standards of expectation are made clear and assessment and feedback are ongoing; and (v) learners benefit, both socially and intellectually, from a variety of learning experiences, both independent and in collaboration with others.

II. Program Design and Components

A. Program Organization

As indicated previously, the mathematics curriculum is designed to support the Atlantic Canada Essential Graduation Learnings (EGLs). The curriculum is designed to significantly contribute to students meeting each of the six EGLs, with the communication and problem-solving EGLs relating particularly well with the curriculum's unifying ideas. (See the "Outcomes" section of the *Foundation for the Atlantic Canada Mathematics Curriculum*.) The foundation document then goes on to present student outcomes at key stages of the student's school experience.

This curriculum guide presents specific curriculum outcomes for *Advanced Mathematics with an Introduction to Calculus 120*. As illustrated in Figure 2, these outcomes represent the step-by-step means by which students work toward accomplishing the key-stage curriculum outcomes, the general curriculum outcomes, and, ultimately, the essential graduation learnings.

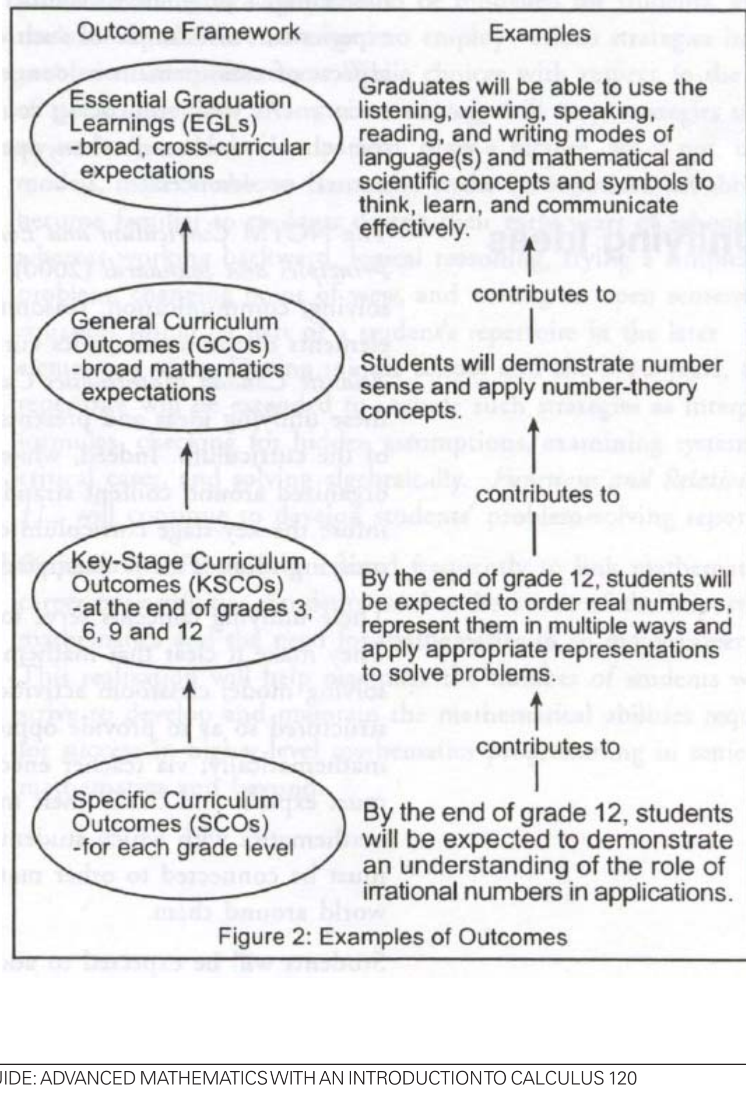


Figure 2: Examples of Outcomes

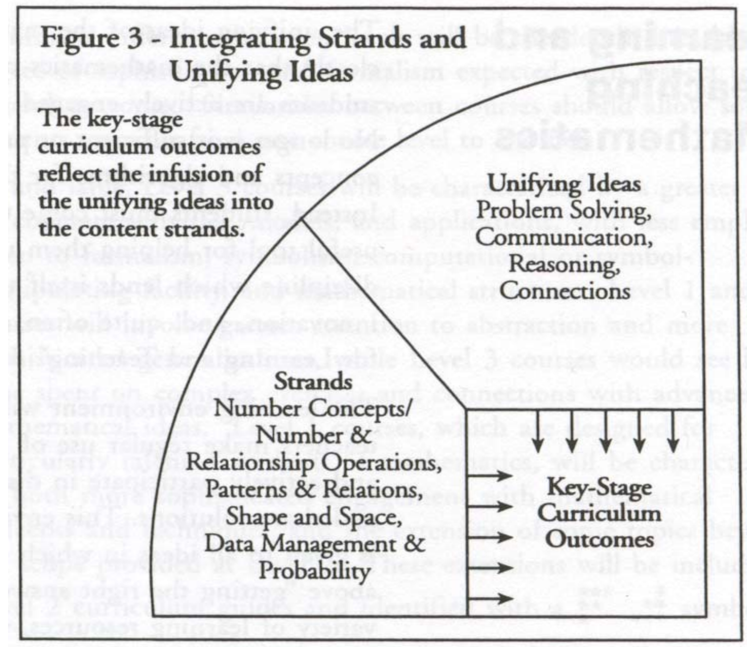
It is important to emphasize that the initial presentation of the specific curriculum outcomes for this course (pp. 15-28) follows the outcome structure established in the *Foundation for the Atlantic Canada Mathematics Curriculum* and does not represent a natural teaching sequence. In *Advanced Mathematics with an Introduction to Calculus 120*, however, a suggested teaching order for specific curriculum outcomes has been given within a sequence of four topics or units (i.e., Sequences and Series; Developing a Function Toolkit Part I; Developing a Function Toolkit Part II; and Complex Numbers). While the units are presented with a specific teaching sequence in mind, some flexibility exists as to the ordering of units within the course. It is expected that teachers will make individual decisions as to what sequence of topics will best suit their classes. In most instances, this will occur in consultation with fellow staff members, department heads, and/or district level personnel.

Decisions on sequencing will depend on a number of factors, including the nature and interests of the students themselves. For instance, what might serve well as a “kickoff” topic for one group of students might be less effective in that role with a second group. Another consideration with respect to sequencing will be co-ordinating the mathematics program with other aspects of the students’ school experience. An example of such co-ordination would be studying aspects of measurement in connection with appropriate topics in science. As well, sequencing could be influenced by events outside of the school, such as elections, special community celebrations, or natural occurrences.

B. Unifying Ideas

The NCTM *Curriculum and Evaluation Standards* (1989) and *Principles and Standards* (2000) establish mathematical problem solving, communication, reasoning, and connections as central elements of the mathematics curriculum. The *Foundation for the Atlantic Canada Mathematics Curriculum* (pp. 7-11) further emphasizes these unifying ideas and presents them as being integral to all aspects of the curriculum. Indeed, while the general curriculum outcomes are organized around content strands, every opportunity has been taken to infuse the key-stage curriculum outcomes with one or more of the unifying ideas. This is illustrated in Figure 3.

These unifying concepts serve to link the content to methodology. They make it clear that mathematics is to be taught in a problem-solving mode; classroom activities and student assignments must be structured so as to provide opportunities for students to communicate mathematically; via teacher encouragement and questioning, students must explain and clarify their mathematical reasoning; and mathematics with which students are involved on a day-to-day basis must be connected to other mathematics, other disciplines, and/or the world around them.



Students will be expected to address routine and/or non-routine mathematical problems on a daily basis. Over time, numerous problem-solving strategies should be modelled for students, and students should be encouraged to employ various strategies in many problem-solving situations. While choices with respect to the timing of the introduction of any given strategy will vary, strategies such as try-and-adjust, look for a pattern, draw a picture, act it out, use models, make a table or chart, and make an organized list should all become familiar to students during their early years of schooling, whereas working backward, logical reasoning, trying a simpler problem, changing point of view, and writing an open sentence or equation would be part of a student's repertoire in the later elementary years. During middle school and the 9/10 years, this repertoire will be extended to include such strategies as interpreting formulas, checking for hidden assumptions, examining systematic or critical cases, and solving algebraically. *Advanced Mathematics with an Introduction to Calculus 120* will continue to develop students' problem-solving reportoires.

Opportunities should be created frequently to link mathematics and career opportunities. Students need to be aware of the importance of mathematics and the need for mathematics in so many career paths. This realization will help maximize the number of students who strive to develop and maintain the mathematical abilities required for success in higher-level mathematics programming in senior high mathematics and beyond.

C. Learning and Teaching Mathematics

The unifying ideas of the mathematics curriculum suggest quite clearly that the mathematics classroom needs to be one in which students are actively engaged each day in the doing of mathematics. No longer is it sufficient or proper to view mathematics as a set of concepts and algorithms for the teacher to transmit to students. Instead, students must come to see mathematics as a vibrant and useful tool for helping them understand their world, and as a discipline which lends itself to multiple strategies, student innovation, and, quite often, multiple solutions. (See the "Contexts for Learning and Teaching" section of the foundation document.)

The learning environment will be one in which students and teachers make regular use of manipulative materials and technology, and actively participate in discourse and conjecture, verify reasoning, and share solutions. This environment will be one in which respect is given to all ideas in which reasoning and sense making are valued above "getting the right answer." Students will have access to a variety of learning resources, will balance the acquisition of procedural skills with attaining conceptual understanding, will estimate routinely to verify the reasonableness of their work, will compute in a variety of ways while continuing to place emphasis on mental computation skills, and will engage in homework as a useful extension of their classroom experiences.

D. Meeting the Needs of All Learners

The *Foundation for the Atlantic Canada Mathematics Curriculum* stresses the need to deal successfully with a wide variety of equity and diversity issues. Not only must teachers be aware of, and adapt instruction to account for, differences in student readiness, but they must also remain aware of avoiding gender and cultural biases in their teaching. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom.

NCTM's *Principles and Standards* (2000) cites equity as a core element of its vision for mathematics education. "All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study – and support to learn – mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students"(p. 12).

This not-withstanding, *Advanced Mathematics with an Introduction to Calculus 120* is an advanced course designed for students expecting to pursue significant studies in mathematics at the post-secondary level. Teachers should continue, however, to understand, and design instruction to accommodate, differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Further, the practice of designing classroom activities to support a variety of learning styles must be extended to the assessment realm; such an extension implies the use of a wide variety of assessment techniques, including journal writing, portfolios, projects, presentations, and structured interviews.

E. Support Resources

This curriculum guide represents the central resource for the teacher of *Advanced Mathematics with an Introduction to Calculus 120*. Other resources are ancillary to it. This guide should serve as the focal point for all daily, unit, and course-long planning, as well as a reference point to determine the extent to which the instructional outcomes should be met.

Nevertheless, other resources will be significant in the mathematics classroom. Textual and other print resources will be significant to the extent that they support the curriculum goals. Teachers will need professional resources as they seek to broaden their instructional and mathematical skills. Key among these are the NCTM publications, including the *Principles and Standards for School Mathematics, Assessment Standards for School Mathematics, Curriculum and Evaluation Standards for School Mathematics*, the *Addenda Series*, *Professional Standards for Teaching Mathematics,* and the various NCTM journals and yearbooks. As well, manipulative materials and appropriate access to technological resources (e.g., software, videos) should be available. Calculators will be an integral part of many learning activities.

F. Role of Parents

Societal change dictates that students' mathematical needs today are in many ways different than were those of their parents. These differences are manifested not only with respect to mathematical content, but also with respect to instructional approach. As a consequence, it is important that educators take every opportunity to discuss with parents changes in mathematical pedagogy and why these changes are significant. Parents who understand the reasons for changes in instruction and assessment will be better able to support their children in mathematical endeavours by fostering positive attitudes towards mathematics, stressing the importance of mathematics in their children's lives, assisting children with mathematical activities at home, and, ultimately, helping to ensure that their children become confident, independent learners of mathematics.

G. Connections Across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating learning experiences—through teacher-directed activities, group or independent exploration, and other opportune learning situations. However, it should be remembered that certain aspects of mathematics are sequential, and need to be developed in the context of structured learning experiences.

The concepts and skills developed in mathematics are applied in many other disciplines. These include science, social studies, music, technology education, art, physical education, and home economics. Efforts should be made to make connections and use examples which apply across a variety of discipline areas.

In science, for example, the concepts and skills of measurement are applied in the context of scientific investigations. Statistical concepts and skills are applied as students collect, present, and analyse data. Examples and applications of many mathematical relations and functions abound.

In social studies, knowledge of confidence intervals is valuable in intrepreting polling data, and an understanding of exponential growth is necessary to appreciate the significance of government debt and population growth. As well, students read, interpret, and construct tables, charts, and graphs in a variety of contexts such as demography.

Opportunities for mathematical connections are also plentiful in physical educaiton, many technological courses and the fine arts.

III. Assessment and Evaluation

A. Assessing Student Learning

Assessment and evaluation are integral to the process of teaching and learning. Ongoing assessment and evaluation are critical, not only with respect to clarifying student achievement and thereby motivating student performance, but also for providing a basis upon which teachers make meaningful instructional decisions. (See "Assessing and Evaluating Student Learning" in the *Foundation for the Atlantic Canada Mathematics Curriculum*.)

Characteristics of good student assessment should include the following: i) using a wide variety of assessment strategies and tools; ii) aligning assessment strategies and tools with the curriculum and instructional techniques; and iii) ensuring fairness both in application and scoring. The *Principles for Fair Student Assessment Practices for Education in Canada* elaborate good assessment practice and serve as a guide with respect to student assessment for the mathematics foundation document. (See also, Appendix A, "Assessing and Evaluating Student Learning.")

B. Program Assessment

Program assessment will serve to provide information to educators as to the relative success of the mathematics curriculum and its implementation. It will address such questions as the following: Are students meeting the curriculum outcomes? Is the curriculum being equitably applied across the region? Does the curriculum reflect a proper balance between procedural knowledge and conceptual understanding? Is technology fulfilling a proper role?

IV. Designing an Instructional Plan

It is important to develop an instructional plan for the duration of the course. Without such a plan, it is easy to run out of time before all aspects of the curriculum have been addressed. A plan for instruction that is comprehensive enough to cover all outcomes and topics will help to highlight the need for time management.

It is often advisable to use pre-testing to determine what students have retained from previous grades relative to a given topic or set of outcomes. In some cases, pre-testing may also identify students who have already acquired skills relevant to the current course. Pre-testing is often most useful when it occurs one to two weeks prior to the start of a a topic or set of outcomes. When the pre-test is done early enough and exposes deficiencies in prerequisite knowledge/ skills for individual students, sufficient time is available to address these deficiencies prior to the start of the topic/unit. When the whole group is identified as having prerequisite deficiencies, it may point to a lack of adequate development or coverage in previous grades. This may imply that an adjustment is required to the starting point for instruction, as well as a meeting with other grade level teachers to address these concerns as necessary.

Many topics in mathematics are also addressed in other disciplines, even though the nature and focus of the desired outcome is different. Whenever possible, it is valuable to connect the related outcomes of various disciplines. This can result in an overall savings in time for both disciplines. The most obvious of these connections relate to the use of measurement in science and the use of a variety of data displays in social studies.

V. Curriculum Outcomes

The pages that follow provide details regarding both specific curriculum outcomes and the four topics/units that comprise *Advanced Mathamatics with an Introduction to Calculus 120.* The specific curriculum outcomes are presented initially, then the details of the units follow in a series of two-page spreads. (See Figure 4 on next page.)

This guide presents the curriculum for *Advanced Mathamatics with an Introduction to Calculus 120* so that a teacher may readily view the scope of the outcomes which students are expected to meet during the year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings in this course are part of a bigger picture of concept and skill development.

Within each unit, the specific curriculum outcomes are presented on two-page spreads. At the top of each page, the overarching topic is presented, with the appropriate SCO(s) displayed in the left-hand column. The second column of the layout is entitled "Elaboration-Instructional Strategies/Suggestions" and provides a clarification of the specific curriculum outcome(s), as well as some suggestions of possible strategies and/or activities which might be used to achieve the outcome(s). While the strategies and/or suggestions presented are not intended to be rigidly applied, they will help to further clarify the specific curriculum outcome(s) and to illustrate ways to work toward the outcome(s) while maintaining an emphasis on problem solving, communications, reasoning, and connections. To readily distinguish between activities and instructional strategies, activities are introduced in this column of the layout by the symbol ❐.

The third column of the two-page spread, "Worthwhile Tasks for Instruction and/or Assessment," might be used for assessment purposes or serve to further clarify the specific curriculum outcome(s). As well, those tasks regularly incorporate one or more of the four unifying ideas of the curriculum. These sample tasks are intended as examples only, and teachers will want to tailor them to meet the needs and interests of the students in their classrooms. The final column of each display is entitled "Suggested Resources" and will, over time, become a collection of useful references to resources which are particularly valuable with respect to achieving the outcome(s).

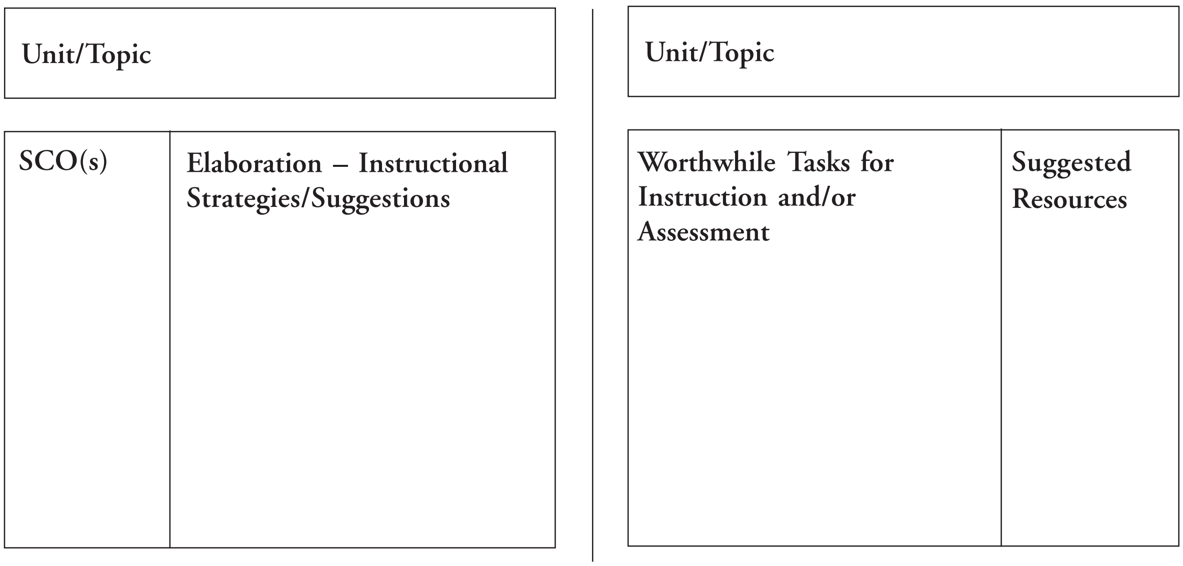


Figure 4: Layout of a 2-Page Spread

SPECIFIC CURRICULUM OUTCOMES (BY GCO)

Specific Curriculum Outcomes
(by GCO)

GCO A: Students will demonstrate number sense and apply number theory concepts.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

ii) *order real numbers, represent them in multiple ways and apply appropriate representations to solve problems*

SCO: By the end of *Advanced Mathematics with an Introduction to Calculus 120,* students will be expected to

Elaboration

A1 **demonstrate an understanding of recursive formulas**

A1 Students will understand that a recursive formula is one that depicts any given term of a sequence in terms of the one (or more) previous term(s) of the sequence. Students will address this outcome in connection with SCOs C1(modeling with and applying recursive formulas), C26 and C25 (infinite sequences and their limits), and A4 (respresentations of series). Unit 1, pp. 30, 32, 44

n, t_n $x \in R$

A2 **determine, describe, and apply the value for 'e'**

A2 By examining rates of change for various exponential functions, students will determine that there is a certain function (i.e., $y = e^x$) such that $f(x) = f'(x)$, for all . Students will recognize *e* as an irrational number with a value slightly greater than 2.7. This outcome will be addressed in conjunction with SCOs B6 (determining derivatives) and C13 (understanding exponential growth and decay). Unit 3, pp. 154, 156

A3 **represent arithmetic and geometric sequences as ordered pairs and discrete graphs**

A3 Students will represent sequences as ordered pairs (of the form ( ), where $n \in N$) and as 2-D graphs depicting these ordered pairs (see SCO C1). They will understand these graphs to be discrete, rather than continuous, given that the first elements of the ordered pairs are restricted to the natural numbers. Unit 1, p. 34

GCO A: Students will demonstrate number sense and apply number theory concepts.

Elaboration

A4 **represent a series in expanded form and using sigma notation**

A4 When the term of a sequence are re-expressed as a sum, the resulting expression (e.g., 5+10+15+20+25+30) is called a series. Students will express series in expanded form (as just shown) and using sigma notation (e.g.,). Unit 1, p. 44

iv) explain and apply relationships among real and complex numbers

A6 **explain the connections between real and complex numbers**

A6 Students will understand that complex numbers are of the form $a+bi$, and that real numbers are the subset of complex numbers that occur when $b=0$. This outcome will be addressed in connection with the representation of complex numbers (SCO C27). Unit 4, p. 160

A7 **translate between polar and rectangular representations**

A7 Since any complex number ,z, is represented graphically by a point on a 2-D surface, it may be described in two ways. First, it may be represented as $z=a+bi$, which provides on ordered pair (a, b) to uniquely describe its position in space. Alternatively, the point in space can be located by specifying distance (r) and direction (θ) to the point, hence the representation $z=r\,(cos\,\theta+i\,sin\quad)$, or $z=r\,cis\quad$. Students will understand and translate between these representations. (See also SCO C27.) Unit 4, p. 170

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

i) *explain how algebraic and arithmetic operations are related, use them in problem-solving situations, and explain and demonstrate the power of mathematical symbolism*

SCO: By the end of *Advanced Mathematics with an Introduction to Calculus 120*, students will be expected to

B1 describe the relationship between arithmetic operations and operations on rational algebraic expressions and equations

Elaboration

B1 Students will add, subtract, multiply and divide rational algebraic expressions, which will require understanding and application of work with numeric fractions (such as finding common denominators for combining terms and removing common factors for division). These techniques will be used in conjunction with solving rational equations (SCO C10) and inequalities (C11), and exploring asymptotic behavior and limits. Unit 3, pp. 118, 124, 126, 128, 130

ii) *derive, analyze and apply computational procedures in situations involving all representations of real numbers*

B2 develop, analyze and apply algorithms to generate terms in a sequence

B2 Students will examine sequences to develop algorithms (i.e., general rules) for describing them. They will then apply these generalizations (e.g., $t_n = t_1 + (n-1)d,\ n \in N$ for arithmetic sequences and $t_n = t_1 r^{n-1},\ n \in N$ for geometric sequences). Algorithms will also include recursive rules (see SCO C18), such as describing the Fibonacci sequence as $t_n = t_{n-2} + t_{n-1},\ n \rangle 2,\ n \in N$. Unit 1, pp. 36, 38

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration

iii) derive, analyze and apply algebraic procedures (including those involving algebraic expressions and matrices) in problem situations

B3 develop, analyze and apply algorithms to determine the sum of a series

B3 Students will develop and apply algorithms for determing the sums of series. In particular, the formulas for the sum of an arithmetic series and $S_n = \frac{t_1\left(r^n - 1\right)}{r-1},\ n \in N$ for the sum of a geometric series will be developed and applied. Unit 1, p. 46

B4 apply convergent and divergent geometric series

B4 In connection with developing an understanding of convergence and divergence (SCO C24) and developing the concept of limit (C25), students will compare convergent and divergent geometric series, distinguish situations when they occur, and develop and apply $S_\infty = \lim_{n\to\infty} S_n = \frac{t_1}{1-r},\ |r|\langle 1$ to determine sums of infinite series. Unit 1, pp. 48, 50

B6 determine and apply the derivative of a function

B6 Students will determine derivatives from first principles (i.e., $f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$ *)* and apply them. This outcome will be addressed in conjunction with several others including understanding the concept of limit (SCO C25), understanding connections among rates of change, slopes of tangent lines and derivatives (C12), analyzing functions for such purposes as determining max/min values and curve sketching (C14), and developing and applying base *e* for exponential functions (A2). Unit 2, pp.76, 96; Unit 3, pp. 154, 156

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

	Elaboration
B7 derive and apply the power rule	**B7** Students will make a generalization of the power rule (i.e., given if $f(x) = x^n$, then $f'(x) = nx^{n-1}$) by examining a number of polynomial functions. As well, the rule will be extended to encompass related situations (e.g., if $g(x) = ax^n + bx^{n-1}$, then $g'(x) = anx^{n-1} + b(n-1)x^{n-2}$). This outcome will be addressed in connection with others such as SCOs C7, C14, C12, and C3. Unit 2, pp. 98, 102, 104
v) apply operations on complex numbers to solve problems	
B9 apply operations on complex numbers both in rectangular and polar form	**B9** Students will add, subtract, multiply, divide, and evaluate powers of complex numbers in rectangular form. As well, they will find products, quotients and powers when complex numbers are expressed in polar form. (See also SCOs C27 and B10.) Unit 4, pp. 162, 164, 172
B10 develop and apply DeMoivre's Theorem for powers	**B10** Students will develop and apply DeMoivre's Theorem to determine powers of complex numbers in polar form. This outcome will be addressed in close association with SCO B9. Unit 4, p. 172

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

i) *model real-world problems using functions, equations, inequalities and discrete structures*

SCO: By the end of *Advanced Mathematics with an Introduction to Calculus 120*, students will be expected to

Elaboration

C1 model problem situations using discrete structures such as sequences and recursive formulas

C1 Students will model situations using sequences (particularly arithmetic and geometric), general rules (such as recursive formulas - see SCO A1), and other discrete structures (e.g., discrete graphs - see A3). While examining sequences students will consider the concepts of infinity (C26) and limit (C25). Unit 1, pp. 30, 32, 34

C2 model problem situations with combinations and compositions of functions

C2 Students will work with situations in which modeling requires combining (e.g., adding or subtracting) or composing functions. Relating combinations and compositions to contexts will help make the patterns/characteristics meaningful. Both irrational and absolute value functions will be considered as compositions of functions. This outcome will be addressed in close conjunction with SCOs C19 and C14. Unit 2, pp. 64, 66, 68; Unit 3, pp. 132, 140, 144, 146

C3 model real-world phenomena using polynomial functions and rational functions

C3 To make determining the equations of polynomial and rational functions (SCO C15), analyzing equations (C10) and functions and their graphs (C14), considering rate of change (C12), and applying the power rule (B7) more meaningful, students will model real-world situations. Unit 2, pp. 84, 94, 102, 104; Unit 3, pp. 108, 114

ii) *represent functions in multiple ways and describe connections among these representations*

C5 use tables and graphs as tools to interpret expressions

C5 Students will regularly use tables and graphs as visual tools to help analyze and make sense of a variety of algebraic relationships and function characteristics. This outcome will be addressed in conjunction with many others including SCOs C7, C14, C10, C15, B6, B7, C19, C11, C8, C16, C23, and C13. Unit 2, pp. 78, 82, 96, 98; Unit 3, pp. 108, 112, 114, 122, 136, 138, 144, 146, 150, 154, 156

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
C6 demonstrate an understanding for asymptotic behavior	**C6** Students will understand asymptotic behaviour in relation to rational functions, examining horizontal, vertical and oblique asymptotes. This outcome will connect with SCOs C14 (analyzing function and their graphs), C16 (parameter changes), C23 (continuity and limits), and C10 (analyzing rational equations). Unit 3, pp. 110, 116, 118, 120, 122
C7 demonstrate an understanding for slope functions and their connection to differentiation	**C7** Students will understand that the slope function, which gives the slope of the tangent to a curve (or the instantaneous rate of change) at any given point on the curve, is derived from the original function, hence the term derivative. This outcome will be addressed in connection with SCOs such as C14, C12, and B7. Unit 2, pp. 78, 80, 98, 100
iii) interpret algebraic equations and inequalities geometrically and geometric relationships algebraically	
C8 explore and describe the connections between quadratic equations and their inverses	**C8** Students will apply previous knowledge regarding functions and their inverses (e.g., reflection across the line $y=x$ and interchanging of x- and y- values) to quadratic functions. They will address the non-functional nature of the inverse. SCOs C14 and C5 will be addressed during this study. Unit 3, p. 136
iv) solve problems involving relationships, using graphing technology as well as paper-and-pencil techniques	
C10 analyze and solve polynomial, rational, irrational, and absolute value equations	**C10** Students will continue to relate solutions to equations (roots) to the x-intercepts (zeros) of the graphs of the corresponding relations. Students will develop a number of algebraic techniques to assist in finding roots, including applying the factor theorem and long division to polynomial equations, eliminating denominators and focusing on numerators when dealing with rational expressions, eliminating square roots to solve irrational equations, and examining cases to deal with absolute value equations. This outcome will be addressed in conjunction with SCOs such as C14, C11, C3 and C2. Unit 2, pp. 82, 84, 86, 88, 92, 94; Unit 3, pp. 112, 120, 122, 124, 128, 130, 132, 134, 138, 140, 142, 144, 148

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
C11 solve polynomial, rational, irrational, and absolute value inequalities	**C11** Students will examine several algebraic and graphical techniques for solving inequalities. These will be developed in close conjunction with SCOs C14(analyzing functions and their graphs) and C10(solving equations). Unit 2, p. 92; Unit 3, pp. 130, 138, 148
v) analyze and explain the behaviors, transformations, and general properties of types of equations and relations	
C12 demonstrate an understanding for the conceptual foundations for limit, the area under a curve, rate of change, slope of a tangent line, and their applications	**C12** Students will understand the concept of limit (see SCO C25) and apply it in situations involving area under a curve and slopes of tangent lines (rate of change). This outcome will be addressed in conjunction with SCOs D2(approximating the area under a curve), B6(determining and applying derivatives), C7(the slope function), and C14(analyzing functions and their graphs), in connection with polynomial (C3) and exponential (C13) applications. Unit 1, pp. 52, 54, 56; Unit 2, pp. 70, 72, 74, 76, 80, 102, 104; Unit 3, pp. 154, 156
C13 extend an understanding for exponential growth and decay through multiple contexts	**C13** Students will renew acquaintance with exponential relations (see also SCOs C14 and C5), consider rate of change (slope) in this context (C12), determine derivatives (B6), and, ultimately, identify and apply the value for *e* (A2). Unit 3, pp. 152, 154, 156
C14 analyze relations, functions and their graphs	**C14** Students will analyze a wide variety of relations and functions using tables and graphs (see SCO C5) as well as algebraic techniques. Analysis will consider parameter changes and transformations (C16), asymptotic behaviour (C6), and continuity and limits (C23). As well, this outcome will be connected to the study of combination and composition of functions (C19), quadratic functions and their inverses (C8), slope functions (C7), derivatives of functions (B6), the power rule (B7), and modeling functions (C2 and C3). Unit 2, pp. 64, 66, 68, 78, 82, 84, 86, 88, 96, 102, 104; Unit 3, pp. 110, 112, 116, 118, 120, 122, 132, 134, 136, 138, 144, 146, 150, 154, 156
C15 determine the equations of polynomial and rational functions	**C15** Students will determine the equations of polynomial functions given the roots or from given data sets using systems of equations. They will also determine equations of rational functions from data using technology. This outcome will be addressed in connection with SCOs C5 and C3. Unit 2, pp. 82, 84, 94; Unit 3, p. 114

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
C16 analyze the effect of parameter changes on the graphs of functions and express the changes using transformations	**C16** Students will connect parameter changes with variations in the graphs of the corresponding functions. In particular, this work will be connected to the analysis (see SCO C14) of rational functions (particularly with respect to the study of asymptotic behaviour (C6)) and quadratic functions and their inverses(C8). Unit 3, pp. 110, 112, 136, 144
C18 demonstrate an understanding for recursive formulas and how recursive formulas relate to a variety of sequences	**C18** Students will not only describe arithmetic and geometric sequences recursively, but also other types of sequences such as the Fibonacci sequence. This outcome will be addressed in connection with SCO B2. Unit 1, p. 38
vi) perform operations on and between functions	
C19 investigate and interpret combinations and compositions of functions	**C19** Students will examine a variety of combinations and compositions of functions, particularly involving linear and quadratic functions. They will describe rational functions as quotients of two functions, and irrational and absolute value functions as compositions. This outcome will be addressed in connection with SCOs C2, C14 and C5. Unit 2, pp. 64, 66, 68; Unit 3, pp. 108, 138, 146
C20 factor polynomial expressions	**C20** Students will add factoring by grouping, sum and difference of cubes, and the use of the factor theorem and long division to their repertoire of techniques for factoring polynomial expressions. Unit 2, pp. 86, 90
vii) describe and explore the concept of continuity of a function	
C23 explore and describe the connections between continuity, limits and functions	**C23** Students will focus on continuity with respect to rational and piece-wise functions. They will associate discontinuities with holes in and/or asymptotes of graphs, and will connect characteristics of the defining algebraic expressions with specific characteristics of the graphs. As well, limits will be used to help identify the nature and location of discontinuities. This outcome will be addressed in connection with CSOs C14, C6 and C25. Unit 3, pp. 116, 118, 150

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

Elaboration

viii) investigate limiting processes by examining infinite sequences and series

C24 demonstrate an understanding of divergence and convergence

C24 In connection with studying sequences and series, students will develop and apply (see SCO B4) an understanding of convergence and divergence. Limits (see C25) will be applied to determine sums of infinite geometric series. Unit 1, pp. 48, 50

C25 demonstrate an intuitive understanding for the concept of limit

C25 Students will develop an understanding of limits in connection with the study of infinite sequences and series (see SCOs C26, C24, and B4), derivatives from first principles (C12 and B6), and continuity and asymptotes (C23 and C6). Unit 1, pp. 32, 40, 42, 48, 50; Unit 2, pp. 72, 74, 76; Unit 3, pp. 116, 118, 150

C26 investigate and apply the concept of infinity by examining sequences and series

C26 Students will understand infinite sequences and series to be those that continue indefinitely. They will determine limits of infinite sequences and use limits to find sums of certain infinite series. This outcome will be addressed in conjunction with SCOs C25, C24 and B4. Unit 1, pp. 32, 40, 42, 48, 50

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ix) make connections among trigonometric functions, polar coordinates, complex numbers and series

C27 represent complex numbers in a variety of ways

C27 Students will represent complex numbers in both rectangular (i.e., $z = a + bi$) and polar (   ) forms, as well as graphically (see SCO C28). This outcome will be addressed in conjunction with such outcomes as A6 (understanding the relationship between real and complex numbers), A7 (translating between rectangular and polar forms), and B9 (operations on complex numbers). Unit 2, p.88; Unit 4, pp. 160, 162, 170

C28 construct and examine graphs in the complex and polar planes

C28 In connection with representing complex numbers (see SCO C27), students will both locate complex numbers on the Argand (complex) plane and construct graphs (with and without the use of technology) of polar equations. Unit 4, pp. 162, 166, 168

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

iv) demonstrate an understanding of the meaning of area under a curve

SCO: By the end of *Advanced Mathematics with an Introduction to Calculus 120*, students will be expected to

D2 demonstrate an understanding of how to approximate the area under a curve using limits

Elaboration

D2 Students will use sequences of rectangles to approximate the area under a curve and, with the use of limits (see SCO C12), determine exact values for areas. Unit 1, pp. 52, 54

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

vi) demonstrate an understanding of the operation of axiomatic systems and the connections among reasoning, justification and proof

SCO: By the end of *Advanced Mathematics with an Introduction to Calculus 120*, students will be expected to

E2 develop and evaluate mathematical arguments and proofs

E3 prove using the principle of mathematical induction

Elaboration

E2 In conjunction with SCO E3, students will prove conjectures using mathematical induction and evaluate such proofs. Unit 1, pp. 58, 60

E3 Students will develop an understanding of the principle of mathematical induction and use this principle to prove conjectures on the natural numbers. Unit 1, pp. 58, 60

Unit 1
Sequences and Series
(15 hours)

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

A1 demonstrate an understanding of recursive formulas

C1 model problem situations using discrete structures such as sequences and recursive formulas

Elaboration—Instructional Strategies/Suggestions

A1/C1 In a previous course, students examined patterns of numbers to determine expressions that could be used to predict more numbers. Students learned that when successive terms in a sequence are subtracted a new sequence of numbers is found. When this new sequence of differences is constant at the first level, the sequence is called arithmetic and the graph is linear. When the sequence of differences (D_1) is not constant then a second sequence of differences (D_2) can be obtained by subtracting successive terms in the first differences sequence. If the terms in the sequence of second differences are constant, then the original pattern is called a quadratic relation, and when graphed looks parabolic. Students also explored constant third-level differences and called these cubic relations. Both the quadratic and cubic relations are examples of power sequences.

Students have also learned that common differences between successive terms may not always occur in a sequence. Sometimes when common differences do not occur, common ratios may result when successive terms are divided. These sequences are called geometric and have graphs and equations that are exponential. When exploring geometric sequences and other sequences, students were introduced to recursive rules or formulas, and ways of representing sequences using these formulas. For example, they may have expressed the Fibonacci Sequence as
$t_1 = 1$, $t_2 = 1$, $t_n = t_{n-1} + t_{n-2}$, $n > 2$, n N.

A1/C1 This course should begin by revisiting recursive formulas and developing a deeper understanding of and facility in their use. For example, students may begin by doing an activity like the following:

- ❑ Ask students to build a sequence of figures using interlocking cubes like those shown below:

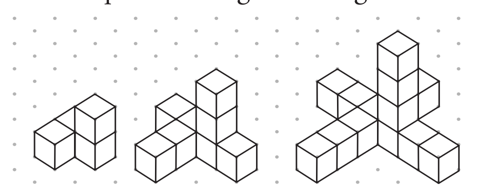


Fig. 1   Fig. 2   Fig. 3

Have students make some observations about the pattern suggested by the figures. Have students describe the pattern in words and numbers, and in recursive language. Students should then continue the pattern.

Students might describe the above pattern as “the number of cubes in Figure 3 is five more than the number of cubes in Figure 2.” They should organize their observations:

Figure 2 = Figure 1 + 5 cubes
Figure 3 = Figure 2 + 5 cubes
Figure 4 = Figure 3 + 5 cubes
assume continuation of the pattern

Figure 10 = Figure 9 + 5 cubes
and then, describe it in general:
Figure n = Figure $(n – 1)$ + 5 cubes

... continued

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper/Technology **(C1/A1)**

1) a) What sequence is created by this recursive routine?
   6 ENTER 1.5 + ANS ENTER ENTER...
   b) Is the sequence arithmetic or geometric? Explain.
   c) Find the 15th term.

2) State the first six terms for the sequence defined by
   $t_1 = 3,\ t_2 = 4\ ,\ t_3 = 2,\ t_n = 2t_{n-1} + t_{n-2} - t_{n-3},\qquad ,\ n \in N$

3) Look for a pattern in this sequence and express it as a recursive relation:
   3, 7, 15, 31, 63, 127…

Performance **(C1/A1)**

4) \$2500 is deposited into a bank paying 8% interest compounded annually.
   a) Complete the table showing how money grows.
   b) How many years will it take for the original deposit to triple?

elapsed time (year)	1	2	3	4...	7	n
balance in \$	2700	2916	t_3	t_4	t_7	t_n

   c) Redo (a) and (b) changing the annual interest rate to 8.5%. Determine, now, how long it will take for the original deposit to triple. Compare your answer to that in (b).
   d) Use the following recursive routine to answer the questions that follow:

$\left(1+\frac{0.084}{12}\right)$ $\left(1+\frac{0.085}{12}\right)$

2500 ENTER ANS ENTER ENTER …

   i) What does represent in this context?
   ii) What is the balance after 1 year? After 4 years? After 7 years?
   iii) How many years will it take to triple the original deposit? Compare this answer to those in (b) and (c). Explain.

5) Between 1960 and 1990, the population of Indian Fields, a rural community somewhere in Atlantic Canada, grew from 391 to 642.
   a) Find the population increase for the 30-year period.
   b) Find the percent increase over the 30-year period.
   c) What do you think the *annual* growth rate was during this period?
   d) Check your answer to 5 (c) by writing and using a calculator recursive routine. Explain why that answer does or does not work out to a population of 642 people over the period.
   e) Using guess-and-check, find a growth rate, to the nearest 0.1%, that comes closest to producing the growth experienced.
   f) Use the answer to 5 (e) to estimate the population in 1980. How does this compare with the average of the population of 1960 to 1990? Why is that?

6) Imagine you have won the Clearance House Sweepstakes. You can choose between a lump sum of \$2 000 000, or \$10 000/mo for the rest of your life. You decide that in either case, you will deposit half the money in a savings account that compounds interest monthly at 6%/a. Which is the better deal? How long would it be before the amount you have from the \$10 000/mo plan exceeds the \$2 000 000 lump sum?

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

A1 demonstrate an understanding of recursive formulas

C1 model problem situations using discrete structures such as sequences and recursive formulas

C25 demonstrate an intuitive understanding of the concept of limit

C26 investigate and apply the concept of infinity by examining sequences and series

Elaboration—Instructional Strategies/Suggestions

... continued

A1 Lists of numbers that occur according to some pattern are number sequences, and the numbers in a sequence are its terms. For example, the list 1, 3, 5, 7, 9, 11, ... is the sequence of odd numbers and can be described recursively as $t_1 = 1$, $t_n = t_{n-1} + 2$, and $n > 1$, . The 10th term of the above sequence, t_{10}, can be compared to the preceding term, , by writing the recursive formula $t_{10} = t_9 + 2$.

Ask students to form another sequence by looking at the surface area of each of the figures formed with interlocking cubes, and describe the pattern recursively.

A **recursive formula** for a sequence is a set of statements that specifies one or more initial terms, and defines the *n*th term, t_n, in terms of one or more of the preceding terms. For example, the Fibonacci sequence is defined as $t_1=1$, $t_2=1$, $t_n=t_{n-1} + t_{n-2}$, $n > 2$, $n \in N$. An **arithmetic sequence** is a sequence where each term is equal to the preceding term plus a constant. This constant is called the **common difference**.

A **geometric sequence** is a sequence where each term is equal to the preceding term multiplied by a constant, or $t_n = r \cdot t_{(n-1)}$. The constant, r, is called the **common ratio.**

Students could become familiar with the use of calculators to define (and graph) sequences given recursively or otherwise (see the extension problem 5 on the next page).

C1/C25/C26 Sequences are said to have limits when the amount of change between terms slowly decreases as the sequence approaches a certain value (i.e., the amount of change becomes less as the terms get closer and closer to a specific value). This specific value is called a limit.

If a cup of hot coffee was left to cool and its temperature was recorded at one-minute intervals, this sequence of temperatures would approach room temperature. Sequences whose terms continue to grow (or decay) indefinitely approach infinity. A bank balance which continues to increase due to monthly compounding is an example of a sequence whose terms get larger and larger. Is there a limit to how tall a tree can grow? Can people continue to build taller buildings, run faster, jump higher?

Medication and its elimination in the human body, a water supply and pollution control system, a contaminated lake and cleanup processes, are real-world examples of changing systems. The idea of determining the limiting value in order to determine the growth rate at an instant is a very important idea. This idea which students began to examine in a previous course will be examined further in this course.

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**A1/C1/C25/C26**)

1) a) Find t_5 and t_{10} of the sequence defined by the notation below:
   $t_1 = 450$, $t_n = 0.75t_{n-1} + 210$, when $n > 1$;
   b) Describe what happens to this sequence as it continues to grow.
   c) Invent a situation that could be modeled with this recursive routine.

2) Harry's medication is one tablet per day. Each tablet contains 40 milligrams (mg) of a prescribed drug. Harry takes the tablet with his dinner. By dinner one day later, his body eliminates 20% of the drug. Harry forgets to take any more medication, but his body continues to eliminate 20% of the remaining drug each day. Write a recursive routine that shows the daily amount of this drug in Harry's body. How long will it take before there is less than 0.5 mg of the drug present?

3) Sarah has found the car of her dreams but must arrange financing for $11 500. The annual interest rate will be 13.8% of the unpaid balance, compounded monthly.
   a) Write a recursive formula that shows the amount still owing for a loan that has a monthly payment of $313.10.
   b) List the first seven terms of this sequence.
   c) When will the loan be paid off?
   d) What is the total cost of the new car paid over this time period?

Journal (**C1/C25/C26**)

$n \in N$

4) Describe how the sequences of numbers involved in the following situation are different.
   a) The temperatures of a container of hot water as it cools.
   b) The number of lily pads in a pond doubles each day.

Extension (**C1/C25**)

5) Graphing technology can be used so that students can visualize the limiting process. For example:
   A small forest of 4000 trees is under a new forestry plan. Each year 20% of the trees will be harvested and 1000 new trees will be planted. Will the forest eventually disappear? Have students model this problem using graphing technology and trace along the graph to observe the behaviour of the forest. Have them explain the behaviour. (Students could use their TI-83 in Sequence MODE, using the TIME axes FORMAT (the variable "*t*" for term is now "*u*")).

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

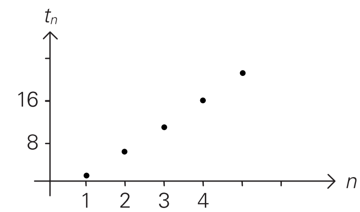
A3 represent arithmetic and geometric sequences as ordered pairs and discrete graphs

C1 model problem situations using discrete structures such as sequences and recursive formulas

Elaboration—Instructional Strategies/Suggestions

A3/C1 By using a recursive routine, students can display a sequence of numbers quickly and efficiently. They can also draw graphs to help visualize sequences.

Written recursively, a sequence is defined by the notation below:



Each point in the graph is a geometric representation of (n, t_n) for some choice of n. For example, $t_4 = 16$ is pictured as the point (4, 16). A table of the early points of this sequence looks like this.

n	1	2	3	4	5
t_n	1	6	11	16	21

The discrete points for this sequence should appear to be **linear** (on the path of a straight line) since the term values increase at a constant rate. Ask students to find the slope of the line suggested by this sequence and interpret it.

Students will find graphs are very useful tools in helping them to understand and explain situations, and to visualize the mathematics. Students should look for connections between the graph and the mathematics used to create the graph. For example, by examining the shape of the relationship on the graph students might be able to discuss the nature of the growth, whether it's increasing or not, and whether the sequence of terms is heading to a limiting value or not.

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Performance **(A3/C1)**

1) The sequence below models the population growth of a foreign country each decade (n represents decades) since 1979.

$$t_1 = 1\ 360\ 500,\ t_n = (1 + 0.12)t_{n-1}\ ,\ \text{for } n > 1,\ n \in N$$

Copy and complete a table of values for the first six terms of the sequence. What is the meaning of 1 360 500?

n (decades)	1	2	3	4	5	6
t_n						

a) What is the growth rate over each decade?
b) Is this an arithmetic or geometric sequence, or neither?
c) Sketch a graph using an appropriate domain and range.
d) Create the recursive formula that models the population growth of Canada each decade since 1867 (try www.statcan.ca). How does the graph compare to the graph for the foreign country referred to earlier?

2) My uncle's farm has approximately 7200 white pine trees. Each year, the nursery plans to sell 12% of its trees and plant 500 new ones. Determine the number of pine trees on the farm after ten years.
a) Draw a graph that represents the number of trees each year during the first ten years and indicate the graphing window used. Sketch this graph on paper.
b) Determine the number of pine trees after many years (in the long run), and explain what is occurring.
c) Try different starting totals in place of the 7200 trees. Describe any changes to the long-run totals.
d) Draw graphs for 20-year and 30-year periods.
e) Describe what happens in the long run.
f) Rework the problem again, but this time build in a catastrophe of some sort in the fifth year. Describe the catastrophe.
g) How does the solution change?

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

B2 develop, analyse, and apply algorithms to generate terms in a sequence

Elaboration—Instructional Strategies/Suggestions

B2 Students should be able to generate terms for an arithmetic sequence. For example: For a summer job, Matthew is going to sell ice cream. Following local municipality bylaws, Matthew must build and equip a kiosk. He wants to determine how many ice creams he must sell to begin to make a profit. His initial expenses, before any sales, amount to \$1275. He expects to make \$0.75 profit per ice cream sale.

This relationship produces the following sequence of terms:

$$-1275,\ -1274.25,\ -1273.5,\ -1272.75,\ \ldots$$

A sequence where the difference between successive terms is a constant (called "d"), is an arithmetic sequence. Every arithmetic sequence defines a linear function with domain N.

If the first term is t_1, then the second must be $t_2 = t_1 + d$, the third $t_3 = t_1 + 2d$, the fourth $t_4 = t_1 + 3d$. Students should see that the nth term would be $t_n = t_1 + (n - 1)d$.

Matthew's profit sequence can be expressed this way:

$$-1275 + 0(0.75),\ -1275 + (1)(0.75),\ -1275 + (2)(0.75),\ \ldots$$

And in general this can be modeled with an equation: profit = $-1275 + 0.75(n - 1)$

where .

Some sequences, instead of having a constant difference between successive terms, have a constant ratio between successive terms.

Once Matthew begins to make money, he deposits it into a savings plan. Initially, he invests his first \$100 at 8% compounded annually. A sequence can be formed:

$$100,\ 100(1.08),\ 100(1.08)^2,\ 100(1.08)^3,\ \ldots$$

giving the values 100, 108, 116.64, 125.97, ... representing the growth of the initial \$100 at the end of each successive year.

If the first term is t_1, then the second term must be $t_2 = t_1 r$, and term 3 is

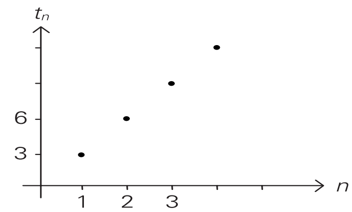
$t_3 = t_1 r(r)$, term 4 is $t_4 = t_1 r(r)(r)$. This leads to the fifth term being $t_5 = t_1 r^4$ and the nth term $t_n = t_1 r^{n-1}$. Such a sequence, where the ratio in successive terms is a constant ratio, r, is called geometric, and can be represented with an exponential function. Terms in arithmetic sequences can be generated with $t_n = t_1 + (n - 1)d$ where n represents the term number. Terms in geometric sequences can be generated with $t_n = t_1 r^{n-1}$ where n represents the term number. The n^{th} term t_n is called the general term.

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**B2**)

1) a) State the terms in the sequence pictured.
   i) What is the value for t_4?
   ii) Is there a common difference? Explain.
   iii) Continue the pattern to bdetermine t_{10}.
   iv) Determine the slope of the suggested line that passes through the points?
   v) What is the value of the vertical axis intercept?
   vi) Write the formula for the sequence.
   vii) Write the equation of the line that contains these points.



   b) Show how to find the 15th term in this sequence.

2) Dorothy was asked to find the three arithmetic means between 3 and 6. This suggested to her that the sequence would be arithmetic and look like
   3, __, ___, ___, 6. To find the three means
   a) Name two points on the graph of this sequence.
   b) Find the slope of the line connecting the points.
   c) Use the slope to find the missing terms, and also the terms just prior to the 3.
   d) Plot all the points, and write the equation of the line that contains them.
   e) Write the formula for the sequence t_n.
   f) Dorothy wants to know what the connection is between the slope and the common difference in this sequence. What would you tell her?
   g) Describe how you could find the three arithmetic means in a different way.

Performance (**B2**)

3) A store owner received hundreds of vacuum cleaners that he wanted to sell for $650.00. The next month, he still had many of vacuum cleaners in his store, so he told his manager to discount the price 25% each week until they were all sold.
   a) Write a formula to find the price of the vacuum cleaners in successive weeks.
   b) What was the price in the second week?
   c) What was the price for the vacuums in the fourth week?
   d) When will the vacuum sell for less than $200.00?

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

C18 demonstrate an understanding for recursive formulas and how recursive formulas relate to a variety of sequences

B2 develop, analyse, and apply algorithms to generate terms in a sequence

Elaboration—Instructional Strategies/Suggestions

C18/B2 Besides arithmetic sequences and geometric sequences, students should have experiences discovering and applying other sequences such as Fibonacci and harmonic.

The Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, ...) could be generated recursively (if $t_1 = 1$, and $t_2 = 1$, then $t_n = t_{n-2} + t_{n-1}$; $n > 2$,). There are many examples of Fibonacci numbers in nature, such as the petal count on flowers (many flowers have petal counts of 5, 8 or 13 petals), the spirals on pineapples (5 spirals going one way, and 8 the other), and turtle shells. Students could connect the Fibonacci sequence and the golden ratio. For example, if each term of the Fibonacci sequence is divided by its preceding term, the resulting sequence of numbers get closer and closer together, appearing to approach the approximate value for the Golden Ratio (or approximately 1.618...). The Golden Ratio and the Fibonacci sequence are related in several other ways. For example, a special geometric sequence is formed by the powers of the Golden Ratio, whch has decimal approximations equivalent to: This sequence can be used to develop the logarithmic spiral (see the activities on the next page).

Music is based on the harmonic sequence (a sequence with reciprocals forming an arithmetic sequence). In music, for example, strings of the same material, same diameter, and same torsion, with lengths proportional to terms in a harmonic sequence, produce harmonic tones. The terms in the sequence form a harmonic sequence. Students could explore the application of the harmonic sequence on instrument production.

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Activity **(C18/B2)**

1) Did you know that the powers of the $\sqrt[12]{2}$ (about 1.0595) are related to the notes of the musical scale? Create a table of values for the polar coordinates

where n = 1 corresponds to the first ordered pair and n = 2 corresponds to the second . Plot the points in the table and connect them. To plot the points begin on the positive horizontal axis, rotate counter-clockwise 15º and mark the point units from the origin. Return to the horizontal axis, rotate 30º and mark the point from the origin, and so on. The result should be a musical spiral, sometimes called the logarithmic spiral. Logarithmic spirals are not only musical, they're golden (Golden ratio). Follow these steps to generate an approximate logarithmic spiral:

a) Construct a Golden Rectangle *PQRS* (*PQ* is to *QR* as is to 2)
b) Construct diagonal *SQ*.
c) *M* is on *RS*, and *N* on *PQ* so that figure *PNMS* is a square, creating another golden rectangle *NQRM*.
d) Construct diagonal *RN*.
e) *G* is on *RQ*, and *H* on *MN* so that figure *MRGH* is a square, and *HGQN* another golden rectangle.

$$\left(\left(\sqrt[12]{2}\right)^n, n(15^\circ)\right)$$

f) Continue creating squares and golden rectangles as long as there is space.
g) Draw the spiral by joining *P* to *M* with the arc of a circle with radius *PN*. Continue with an arc, radius *MH* joining *M* to *G* ... .
h) Create a recursive sequence that represents the spiral drawn above.
i) Create a series that would represent the length of the spiral if the number of individual arcs was known. If this is not possible, explain why not.

Performance **(C18/B2)**

2) Many centuries ago an interesting myth arose that the Greeks considered the golden ratio essential to beauty and symmetry. Many famous people's names (such as Euclid, Leonardo Pisano Bigollo (Fibonacci), Leonardo da Vinci, Gustave Caillebotte, and George Seurat) are connected to interest in, and in use of, the golden ratio. Some of this interest and use of the golden ratio involved the design of buildings and paintings, spirals, and shapes in nature, and the observation that the human body exhibits ratios close to the golden ratio.

   Ask students to research the golden ratio, its connection to spirals and to nature. Ask them to also research the people connected to the golden ratio and spirals to find how they used the golden ratio or how the golden ratio affected their lives, and the lives of us all.

3) a) Determine a value for the square of the golden ratio. Determine a value for the reciprocal of the golden ratio. Describe your observations in words, then using symbols.
   b) Solve the equation $x^2 = x + 1$.

Suggested Resources

Garfunkel, Solomon, *For All Practical Purposes*. New York: W.H. Freeman, 4th ed., 1997.

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

C25 demonstrate an intuitive understanding of the concept of limit

C26 investigate and apply the concept of infinity by examining sequences and series

Elaboration—Instructional Strategies/Suggestions

C25/C26 Students have been examining sequences with a specified number of terms. Now, they will examine sequences that go to infinity. For example, in a paper folding activity, each time the students fold a piece of paper in half, the area that they are looking at is halved. If it were possible to continue the process indefinitely, each term in the sequence $a, \frac{a}{2}, \frac{a}{4}, \frac{a}{8}, \frac{a}{16}, \ldots$ would respectively describe the area of the paper.

An infinite sequence is a function defined on the natural numbers, and has an unlimited number of terms. Consider the harmonic sequence (a sequence whose reciprocals make up an arithmetic sequence) . Notice that as n increases, the value of each fraction gets smaller and smaller, and the terms of the sequence approach zero. Zero is considered to be the limit of the sequence. This is denoted by and is read "the limit of one-over-n, as n approaches infinity, equals zero". Students should graph $y = \frac{1}{n}$ and examine the curve as the values in the domain get larger and larger.

Not all sequences have a limit of zero. For example, students may be asked to find the limit of an infinite sequence as generated by . This generates the sequence

. Students may try some larger values of n in the sequence to obtain

$$t_{50} = \frac{1-150}{200} = \frac{-149}{200} = -0.745$$
$$t_{100} = \frac{1-300}{400} = \frac{299}{400} = -0.7475$$
$$t_{200} = \frac{1-600}{800} = \frac{599}{800} = -0.74875$$

Students should observe that the sequence seems to approach the value –0.75 or $-\frac{3}{4}$, and write .

Students should also be able to determine the limit of a sequence by using an algebraic approach(see below); but this will be given more stress in Unit 3 (see p. 118).

$$\lim_{n\to\infty}(t_n) = \lim_{n\to\infty}\left(\frac{1-3n}{4n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n}-\frac{3n}{n}}{\frac{4n}{n}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{\frac{1}{n}-3}{4}\right), \text{ and } \lim_{n\to\infty}\left(\frac{1}{n}=0\right), \text{ so } \lim_{n\to\infty}\left(\frac{\frac{1}{n}-3}{4}\right) = -\frac{3}{4}, \text{ or } -0.75$$

... continued

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Paper and Pencil (C25/C26)

1) Determine if the following sequences have a limit. If the sequence does have a limit, state it, if not, show that no limit exists.

   a) $4, 6\frac{1}{2}, 9\frac{1}{3}, 12\frac{1}{4}, ..., \frac{3n^2+1}{n}$

   b)

   c)

   d)

2) Given the sequence {12, 19, 26, 33, ... }. Change the given sequence into an harmonic series and approximate its limit, if possible.

3) The harmonic series keeps getting larger and larger as the number of terms, n, increases. By associating terms, show that

[illegible] 4) Given the alternating harmonic series,

   a) By associating the first and second terms, the third and fourth terms, and so on, show that $S > 0$.
   b) By associating the second and third terms, the fourth and fifth terms, and so on, show that $S < 1$.
   c) Find the sum of the first 100 terms of S.

... continued

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

C25 demonstrate an intuitive understanding for the concept of limit

C26 investigate and apply the concept of infinity by examining sequences and series

Elaboration—Instructional Strategies/Suggestions

.. continued

C25/C26 Using technology, students might produce the graph of , or use the table feature to observe the value of y as $x \to \infty$. At this time students will only study limits as . Later in this course, students will study rational functions and explore the idea that has a horizontal asymptote of $y = -\frac{3}{4}$, and that the sequence generated by the general term has a limit of , as .

When the terms of a sequence become increasingly very large, the sequence is said to go to a limit of infinity. For example, the sequence defined by the expression has a limit at infinity since , and this can be written $\lim_{n\to\infty}(4n) + \lim_{n\to\infty}\left(\frac{1}{n}\right)$. As approaches zero and n approaches , $4n$ also approaches , and the limit is infinite.

When the terms of a sequence approach two different values, then the sequence is said to have no limit. For example, the expression generates the sequence {5, 8.5, 12.3, ...} when $n > 0$, but {–5, –8.5, –12.3, ...} when $n < 0$. Thus, the terms of the sequence generated by alternate between approaching positive infinity and negative infinity depending on whether n is even or odd. This results in their being no limit.

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

... continued

Pencil and Paper **(C25/C26)**

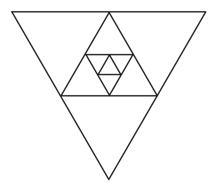
5) A sequence of terms can be generated by the expression $\frac{1-3n}{2n}$.
   a) Find the limit, if it exists.
   b) Plot the points in the sequence and explain how the limit can be seen on the graph.

6) Write an expression for the n^{th} term and determine the limit of the sequence if it exists.
   a) 1, 5, 25, 125, 625, ...   b) 5, 11, 17, 23, 29, 35, ...
   c)   d) –8, 6.5, –6, 5.75, –5.6, ...

Performance **(C25/C26)**

7) Express each term of the sequence ... as a power of 2.
   Determine the limit of this sequence.

8) A sequence of equilateral triangles is constructed as shown. Each successive triangle is formed by joining the midpoints of the sides of previous triangle. The original triangle has a side length *x*.



$3, \frac{7}{2}, \frac{11}{3}, \frac{15}{4}, \ldots$

$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}$

   a) Determine the perimeter of the fourth triangle.
   b) Determine the sum of the perimeters of the first four triangles.
   c) Does the sum of the perimeters approach a specific number? Explain.

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

A4 represent a series in expanded form and using sigma notation

A1 demonstrate an understanding of recursive formulas

Elaboration—Instructional Strategies/Suggestions

A4 A series is formed when the terms of a related sequence are expressed as a sum. In general, the sum of n terms in a series is written $t_1 + t_2 + t_3 + \ldots + t_n = S_n$. The expression $t_1 + t_2 + t_3 + \ldots + t_6 = S_6$ indicates the sum of the first six terms. The series $1 + 3 + 5 + 7 + 9 = S_5$ is a simple example where S_5 represents the sum of the first five terms of the sequence of odd integers. Students should discover that $S_n = 1 + 3 + 5 + \ldots + t_n = n^2$. A decimal like 0.44444 shows how the geometric series $S_5 = 0.4 + 0.04 + 0.004 + 0.0004 + 0.00004$ is formed by combining five consecutive terms of a geometric sequence with a common ratio

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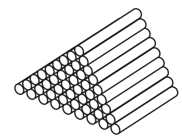


Finding the sum of a series has been a problem that has intrigued mathematicians throughout history. In early times, some would model the sum $1 + 2 + 3 + \ldots + n$ with a "pile of reeds".

Another notation that is often used to indicate summation is the Greek letter (sigma, corresponding to the Roman letter *S*). **Sigma notation** involves the use of an index of summation, much like the counter in the loop of a computer program. For example, is read as "the sum of $3i^2$ for i=1 to i=4, $i \in N$."
In this example, i is called the index of summation. The index can be represented by any letter. The number 1 is the lower limit or the initial value, and the number 4 is the upper limit or final value. For example, when students are asked to write in expanded form, they would substitute i=1, 2, 3, and 4 respectively, into $3i^2$, then add $(3(1)^2 + 3(2)^2 + \ldots + 3(4)^2)$.

As a second example, using k for the index:

$$\begin{aligned}&\sum_{k=3}^{8}(2k+4)\\&=(6+4)+(8+4)+(10+4)+(12+4)+(14+4)+(16+4)\\&=10+12+14+16+18+20\\&=90\end{aligned}$$

In general, the expression $\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$, where n is a natural number.

A1 Using recursive notation, students could write this sum as $S_n = t_n + S_{n-1}$, where t_n is the n^{th} term. The sum $t_n + S_{n-1}$, for $n = 100$ means "the 100^{th} term plus the sum of the first 99 terms."

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**A4**)

1) Write the series, and find the sum given by

a) $\sum_{k=1}^{5}\left(k^2+1\right)$

b) $\sum_{k=2}^{7}\frac{k}{k+1}$

c) $\sum_{k=1}^{4}(k+1)^2$

2) Write each of the following series in sigma notation
   a) 1 – 1 + 1 – 1 + 1
   b) 0 – 2 + 4 – 6 + 8
   c) 1 + 3 + 5 + 7 + 9 + 11

Performance (**A4/A1**)

3) When the famous mathematician Karl Friedrick Gauss (1777–1855) was nine years old, his teacher asked the class to add all the whole numbers between and including 1 and 100. Karl didn't have a calculator but was able to obtain the answer very quickly without adding the numbers one by one.
   a) Ask students to represent the problem using a recursive formula.
   b) Ask students to represent the problem using sigma notation. Use this to solve the problem.
   c) Using a calculator, ask students to find some partial sums (e.g., the sum of the first 5 terms, first 7 terms, first 11 terms ...), graph them as discrete points and find the equation for the curve of best fit. Compare this formula to the one in (b).

$\sum_{i=1}^{n} p = np$

4) a) Write each series in expanded form. Do the series have the same number of terms? Do they have equal sums? What might be concluded?

   i) $\sum_{k=1}^{5}(2k)$ ii) $\sum_{k=2}^{6}\left(2(k-1)\right)$ iii) $\sum_{k=5}^{9}\left(2(k-4)\right)$

   b) Use your answer from (a) to compose an expression that is equivalent to $\sum_{p=3}^{8}\left(p^2-4p+2\right)$ and has an initial value of 1. Then, test as to whether the two expression are equivalent. Does your conclusion in (a) appear valid?

5) Show that

a) $\sum_{p=2}^{6}(ap)=a\sum_{p=2}^{6}p; a\in R$ b) $\sum_{i=1}^{5}p=np$

c) $\sum_{i=1}^{6}(x_i+y_i)=\sum_{i=1}^{6}x_i+\sum_{i=1}^{6}y_i$ d)

Suggested Resources

Sequences and Series

Outcomes

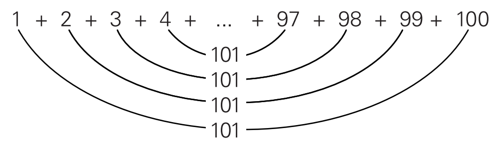
SCO: In this course, students will be expected to

B3 develop, analyse, and apply algorithms to determine the sum of a series

Elaboration—Instructional Strategies/Suggestions

B3 An **arithmetic series** results from expressing the terms of a related arithmetic sequence as a sum.

"In 1786, when mathematician Karl F. Gauss was 9, he was asked to find the sum of the first 100 natural numbers." While his classmates calculated for hours, young Karl Friedrich Gauss quickly wrote the answer on his slate. How did he do it? Gauss said that he paired the numbers.



The sum of each pair was 101. Because there were 100 numbers in all there must be 50 pairs. Therefore, the total sum was $50 \times 101 = 5050$. Notice that each pair of numbers adds up to the same sum as that of the first and last terms. Students can use Gauss's strategy to determine a formula for S_n for any arithmetic series. When n is the number of terms in the arithmetic series, then is the number of pairs.

Therefore, where t_1 is the first term and t_n, the n^{th} term, represents the sum of an **arithmetic series**. Students could write this using sigma notation as $\sum_{i=1}^{n} i$.

Students will also want to find an expression for the sum of the first n terms of a **geometric series**. Beginning with the series where or 1:

$S_n = t_1 + t_1 r + t_1 r^2 + t_1 r^3 + ... + t_1 r^{n-1}$, then multiply both sides by r ...

$rS_n = t_1 r + t_1 r^2 + t_1 r^3 + ... + t_1 r^{n-1} + t_1 r^n$, then subtracting from line one ...

$S_n - rS_n = t_1 - t_1 r^n$

$S_n (1 - r) = t_1 (1 - r^n)$

, when r is the common ratio, $r \neq 1$ and t_1 is the first term, and n is the number of terms. Equivalently, this can be written . This might also be written in sigma notation: $\sum_{i=1}^{n} t_1 r^{i-1}$

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Paper and Pencil (B3)

1) Find the sum
   a) 1 + 4 + 7 + ... + 106
   b) 3 + 7 + 11 + ... + t_{20}
   c) $\sum_{p=1}^{6}(3-2p)$

2) Determine the sums.
   a) 4 + 8 + 16 + ÿ + 1024
   b) $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots-\frac{1}{256}$
   c) $p+p^2+p^3+\ldots$ (n terms)
   d) $1+x^2+x^4+x^6\ldots$ (n terms)

3) Fred is a parachute jumper. He falls 4.9 m in the first second, 14.7 m in the next, 24.5 m in the third second, and so on.
   a) How far does Fred fall in his first 10 seconds of free fall?
   b) If Fred jumps at 3000 m and opens his chute at 1000 m, how many seconds should he count before he opens his chute?

4) Suppose you drop a rubber ball from a window 15 m above the ground. The ball bounces back up to 80% of its previous height on each bounce. How far has the ball travelled by the time it touches the ground the fifth time?

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

C26 investigate and apply the concept of infinity by examining sequences and series

C25 demonstrate an intuitive understanding for the concept of limit

C24 demonstrate an understanding of divergence and convergence

B4 apply convergent and divergent geometric series

Elaboration—Instructional Strategies/Suggestions

C26/C25/C24/B4 The expression , where n is a positive integer, is a finite series, since a_n represents some specific ending term. However, the expression is an infinite series (the ellipsis symbol (...) suggests it continues indefinitely).

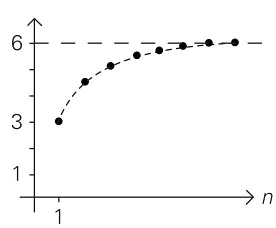
The series 1+3+5+9+... has partial sums that keep getting larger and larger (S_2 = 1+3 = 4, S_3 = 1+3+5 = 9, ...) as the number of terms increase. However, the series $3+\frac{3}{2}+\frac{3}{4}+\frac{3}{8}+K$ behaves differently.

The first few partial sums are:

As n gets larger, the partial sums (S_n) are getting larger but seem to be approaching a value of six. Try a very large value for n, and use the sum of geometric series formula $S_n=\frac{t_1\left(1-r^n\right)}{1-r}$.

$$S_{20}=\frac{3\left(1-\left(\frac{1}{2}\right)^{20}\right)}{1-\frac{1}{2}}=\frac{3(0.9999990463)}{0.5}\ \text{B}\,6(.9999990463)\ \text{B}\,5.999994278$$

The above calculations clearly suggest that S_n will never be 6, since it will always be 6 multiplied by some number slightly less than 1. Since the values get closer and closer to six, the series is said to "converge to 6". Visually displayed is the graph of the partial sums.



A series converges to a number S, if the partial sum, S_n, is close to S, and continues to get closer to S as n gets very, very large.

Students should explore and discover that a geometric series converges when the common ratio (r) is between –1 and 1, i.e., $|r|<1$.

... continued

Sequences and Series

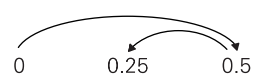
Worthwhile Tasks for Instruction and/or Assessment

Activity (C26/C25/B4/C24)

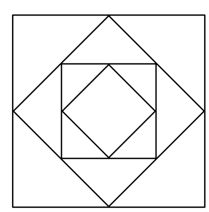
1) Suppose $g(x)=\frac{x+3}{2}$. Evaluate
   a) $g(g(1))$
   b) $g(g(g(1)))$
2) Investigate what happens if this process is continued, for example, $g(g(g(g(1))))$, and so on.
3) Now, investigate the same phenomenon using values other than 1. For example, $g(g(g(10)))$ use "10" instead of 1. Try values that are both large and small as well as negative and positive.
4) Why does the sequence always seem to approach 3? (Hint: investigate $g(g(x))$, $g(g(g(x)))$, ... and look for patterns that can be generalized.)
5) Graph $y = g(x)$ and $y = x$. Use the $y = g(x)$ to evaluate $g(8)$. Use the $y = g(x)$ graph to evaluate $g(8)$. Describe your process.
6) Use both $y = g(x)$ and $y = x$ to evaluate $g(g(8))$. Describe your process. Continue this process to investigate as you did in 2) and 3) above. Describe and explain your findings.

Performance (C26/C25/B4/C24)

7) A flea jumps 0.5 m, then 0.25 m, then 0.125 m and so on. Its first jump was to the right, its second jump from that point to the left, then right, and so on.
   a) Which way and how far is the 7th jump?
   b) Which way and how far is the 16th jump?
   c) What point is the flea zooming in on?



8) A set of squares is drawn as shown. The side length of the outside square is 1 unit and the corners of each successive square are the midpoints of the preceding square.
   a) Show that the perimeter of the sides of the squares form a geometric sequence
   b) Show that the areas of the squares form a geometric sequence.
   c) Show that the sum of the areas of the squares approaches a finite number. Determine the number.
   d) Does the sum of the perimeters approach a finite number? Explain.



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Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

C26 investigate and apply the concept of infinity by examining sequences and series

C25 demonstrate an intuitive understanding for the concept of limit

C24 demonstrate an understanding of divergence and convergence

B4 apply convergent and divergent geometric series

Elaboration—Instructional Strategies/Suggestions

... continued

C26/C25/C24/B4 Students should examine the following situation:

$\left(\frac{1}{2}\right)^{5} = 0.03125$, $\left(\frac{1}{2}\right)^{50}$ B 8.88×10^{-16}, $\left(\frac{1}{2}\right)^{100}$ B 7.89×10^{-31}

They should conjecture that a proper fraction raised to a large number approaches the value zero (converges to the value zero). To find the number to which a geometric series with $|r|<1$ converges, examine the formula:

as $n \to \infty$. If , then r^n approaches 0 as , so

. This number is called the limit of the series as n approaches infinity.

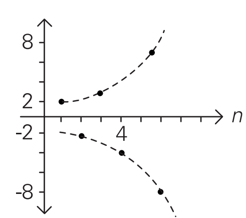
Students should understand from the example on the previous two-page spread that if they add a finite number of terms their answer should be less than 6, but if they could add an infinite number of terms, the answer would be written as 6.

B4/C25 The definition allows students to find exact values for repeating decimals such as . For example, express the repeating decimal as 0.1358 + 0.00001358 + 0.000000001358 + ... This is a geometric series with t_1 = 0.1358 and a common ratio of r = 0.0001.

So, the sum of an infinite number of terms .

C24 As students explore convergence of geometric series they may ask themselves what happens when $|r|>1$. For example, does the geometric series 2 – 3 + 4.5 – ... converge? In their solution: or –1.5. Students will find the series does not converge. That is, successive partial sums do not converge to a certain value, but actually alternate positive and negative, and grow in magnitude.

$S_2 = 2-3=-1$, $S_3 = 2-3+4.5=3.5$, $S_4 = 2-3+4.5-6.75=-3.25$



Looking at the graph of the partial sums, the successive points are alternating back and forth across the *x*-axis, with the distance from the x-axis getting larger and larger. Students should conclude that if a series does not converge, it is said to diverge. When the common ratio , the series will not converge.

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

... continued

Pencil and Paper **(C26/C25/B4/C24)**

9) Cory found the limit of the series 1 – 1 + 1 – 1 + … by
. Is Cory correct? Explain.

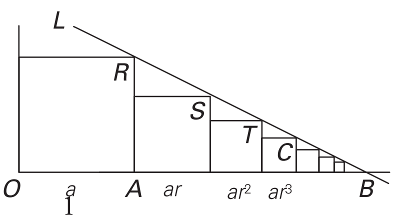
10) Use what you know about convergent geometric series to determine the number to which the series $1+2\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+4\left(\frac{1}{3}\right)^3+5\left(\frac{1}{3}\right)^4+\ldots$ converges.

11) Express $0.\overline{456}$ as a quotient of two integers (e.g., exact value).

12) Express as a quotient of two integers (e.g., exact value).

Activity **(C25/B4/C24)**

13) In the Cartesian rectangular coordinate system, consider the squares with sides from the geometric series.



a) Express the points *R*, *S*, and *T* in terms of *a* and *ar*, etc ...
b) Determine the slope of the line *LB* in terms of *r* and state the equation for *LB* in terms of *r*.
c) Express *OB* as a series.
d) Find *x* in terms of *a* and *r*.

$0.2\overline{150}$ $S_\infty = \frac{a}{1-r} = \frac{1}{1-(-1)} = \frac{1}{2}$ $\frac{2}{27}$

Performance **(C26/C25/B4/C24)**

14) Investigate The Koch Snowflake:
a) An equilateral triangle of side *a* is cut out of paper (Fig 1).
b) Next, 3 equilateral triangles, each of side , are cut out and placed in the middle of each side of the first triangle (Fig. 2).
c) 12 equilateral triangles, each of side , are placed in the middle of the sides of this figure (Fig 3).
d) Figure 4 shows the result of adding 48 equilateral triangles, each of side , to the previous figure.
e) Assume that this procedure can be repeated indefinitely.
f) Find the perimeter of the figure we obtain and the area of the same figure.

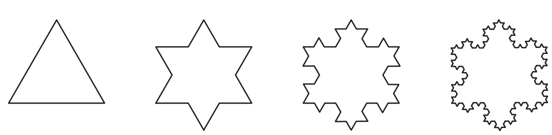


Figure 1 Figure 2 Figure 3 Figure 4

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

C12 demonstrate an understanding for the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of the tangent line and their applications

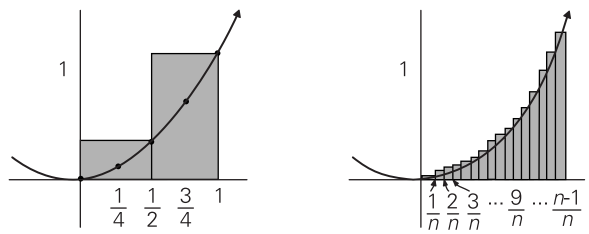
D2 demonstrate an understanding of how to approximate the area under a curve using limits

Elaboration—Instructional Strategies/Suggestions

C12/D2 Students can use sequences, series and limits to calculate the areas of certain regions, in particular, the regions formed between curves and the horizontal axis on a graph.

To begin to investigate the area-under-a-curve concept, have them find the area between $y = 3x$ and the x-axis (area under the curve $y = 3x$) using several rectangles, for the interval between zero and one on the horizontal axis. From previous study they know that the area is one-half square unit. Using an ever increasing number of rectangles (as in Activity 1 on the next page), students should be able to see the area approaching one-half square unit.

Students should then perform the same task using a function such as $y = x^2$.



Students should do this very concretely to start with, then gradually proceed toward the infinite case. For example, they might be asked to complete a table like the one in activity 1)e) on the next page.

Students should observe that as the number of rectangles increases, the amount of area above the curve $y = x^2$, but within the rectangles, decreases. In fact, as the number of rectangles approaches infinity, the amount of area inside the rectangles and above $y = x^2$ nears zero.

Students will use series and limits to determine the area. Students might find the area for n rectangles as . Since there are n rectangles, then each width is $1/n$ of the first unit along the horizontal axis. The heights of the rectangles would be found by evaluating $y = x^2$ for each rectangle along the way (the first rectangle would have height , the second , and so on ...). The heights of the rectangles would form a sequence: . The area would then be found by summing all the areas of the rectangles:

$$A = \frac{1}{n}\left(\frac{1}{n^2}\right)+\frac{1}{n}\left(\frac{4}{n^2}\right)+\frac{1}{n}\left(\frac{9}{n^2}\right)+\ldots+\frac{1}{n}\left(\frac{(n)^2}{n^2}\right)$$

$$= \frac{1}{n^3}+\frac{4}{n^3}+\frac{9}{n^3}+\ldots+\frac{(n-1)^2}{n^3}+\frac{n^2}{n^3}$$

$$= \frac{1}{n^3}\left(1+4+9+\ldots+(n-1)^2+n^2\right)$$

continued ...

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

Activity **(C12/D2)**

1) Find the area under $y = 3x$ for $0 \le x \le 1$, and above the x-axis.
   a) Begin by dividing the x interval into 2 equal parts. Form two rectangles whose heights are drawn by joining the point $x = 1$ vertically to $y = 3x$, and

      vertically to $y = 3x$.
   b) How wide is each rectangle? Calculate the areas of the two rectangles.
   c) Why is this not a good approximation of the area asked for?
   d) Divide the x-interval into four equal parts, make four rectangles and calculate the area inside the four rectangles.
   e) Continue this process and record findings in a table whose headings would be: *number of rectangles, width of each rectangle, area (units^2).*
   f) Describe what is happening to the area inside the rectangles above the line $y = 3x$ as the number of rectangles increases.
   g) What would be the approximate area "under the curve" if the x-interval is divided into 10 equal parts? 20 equal parts?
   h) If the x-interval is divided into n equal parts, what would be the width of each rectangle? Describe how you would find the height of each rectangle.
   i) Create a sequence of the heights with the first height being , and the last

      being .

[illegible]

   j) Complete this series to represent the sum of the areas:
   k) Factor out the common $\frac{3}{n^2}$, write the resulting arithmetic series, then find its sum, and simplify to find the area of the n rectangles.
   l) Finally, use (area of n rectangles) = (area of region) to find the requested area, and explain why this limit is needed.

Journal **(C12/D2)**

2) How would the above activity be different if the area to be determined was the area below the curve in the interval ?

Performance **(C12/D2)**

3) Determine the area between the given curves and the x-axis, and between $x = 0$ and $x = 3$. Do this using 5 rectangles, and again using ten rectangles. Explain how you might now be able to predict the actual area indicated.

   a)

   b)

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

C12 demonstrate an understanding for the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of the tangent line and their applications

D2 demonstrate an understanding of how to approximate the area under a curve using limits

Elaboration—Instructional Strategies/Suggestions

... continued

C12/D2 Realizing the series $(1 + 4 + 9 + 16 + \ldots + (n-1)^2 + n^2)$ is a power series, students could use difference tables to find an expression that would represent the sum. The sum of the first term is 1, sum of the first 2 terms is 5, sum of the first 3 terms is 14, etc.

x	y			
1	1			
		> 4		
2	5		> 5	
		> 9		> 2
3	14		> 7	
		> 16		> 2
4	30		> 9	
		> 25		
5				

From the table students would determine that this is a degree—three relationship, and so use the model $ax^3+bx^2+cx+d = y$. They would know from a previous

course that the leading coefficient a will be of 2, or

. Substituting into $y = ax^3 + bx^2 + cx + d$, and using the coordinates (1, 1), (2, 5), and (3, 14) students would form this system:

and solve for b, c, and d.

Alternatively, students could use the finite difference method, learned in a previous course, by creating a corresponding table for a cubic:

x	y	D_1	D_2	D_3
1	$a + b + c + d$			
		> $7a + 3b + c$		
2	$8a + 4b + 2c + d$		> $12a + 2b$	
		> $19a + 5b + c$		> $6a$
3	$27a + 9b + 3c + d$		> $18a + 2b$	
		> $37a + 7b + c$		
4	$64a + 16b + 4c + d$			
5	$125a + 25b + 5c + d$			

Students would get , and $d = 0$. Students would conclude that the

series $(1 + 4 + 9 + \ldots + n^2) =$.

Substituting this series back into the area calculation: A B $\frac{1}{n^3}\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\right)$,

and, simplifying, A B $\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$. This represents the approximate area.

To obtain the actual area, the number of rectangles must approach infinity, i.e., $n \to \infty$,thus students will take the limit:

Therefore, the area of region between $y = x^2$ and the x-axis for the

interval is . In general then, (area of n rectangles) = (area under the curve), i.e., the area of the region between the curve and the x-axis.

continued ...

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Paper and Pencil **(C12)**

1) Using the following headings: number of rectangles, width of rectangles, area (units^2); construct and complete the table for up to 10 rectangles for the following areas:
   a) area under the curve $y = 2x + 1$, for $x = -2$ to $x = 0$.
   b) area under the curve $y = x^2 - 2x$, for $x = 2$ to $x = 3$.

Paper and Pencil **(C12/D2)**

2) Determine the area under the curve.
   a) $y = x^2 + x + 1$, for $0 \le x \le 1$
   b) $y = 3x^2$, for
   c) $y = x^3$, for
   d) $y = 2x^2 + 5$, for

Journal **(C1/D2)**

3) When calculating the area bounded by the curve and the x-axis, explain the difference in area when rectangles with heights over the curve are used, rather than rectangles with heights under the curve.

$0 \le x \le 2$

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

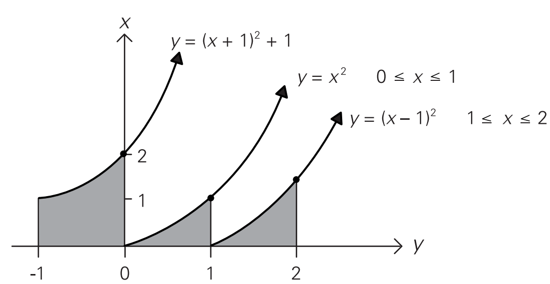
C12 demonstrate an understanding for the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of the tangent line and their applications

Elaboration—Instructional Strategies/Suggestions

C12 Students can use their knowledge of transformations to simplify these area calculations. For example: if students were asked to find the area under the curve $y = (x - 1)^2$ for they could say $y = (x - 1)^2$ is $y = x^2$ translated one unit right, so the area between $y = x^2$ and the x-axis for is the same, thus square units.

Another example: Find the area under $y = x^2 + 2x + 2$ for . Transforming $y = x^2 + 2x + 2$ to $y - 1 = (x + 1)^2$ and drawing the graph helps students realize that this is the same area as that under the curve of $y = x^2$ for , except that there would be an extra square unit of area because of the vertical translation.

Thus, there would be square units of area.



Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

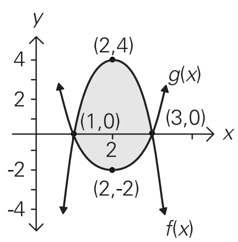
Pencil and Paper **(C12/D2)**

1) a) Ask students to find the area under the curve $y = x^2 + 4x$ from $x = 0$ to $x = 1$.
   b) Ask students how they could apply transformations to find the area under the curve $y = x^2 - 4x$ from $x = 4$ to $x = 5$
   c) Ask students to find the area asked for in (b).

2) Describe what transformations would make it easier to calculate the area under the curve . Explain with the aid of a diagram.

3) Find the area under the curve
   a) $y = (x + 1)^3$ for
   b) $y = 2x^2 - 12x + 18$ for

Extension **(C12/D2)**



$3 < 1 \le x \le 4$ 0^3 $f(t) = 2\sin\left(\frac{-2p^2 t}{5}\right)$ 4 p $+ 16; 1 + 2 \le p \le 1$

4) Calculate the area of the shaded region bounded by the two parabolas.

Performance **(C12/D2)**

5) A special set of fish tanks has water flowing in and out at regular intervals in order to properly monitor oxygen levels. The rate of flow of the water in and out is sinusoidal, and can be modeled by the function .

   Starting with an empty tank, what volume of water would be in the tank after approximately 2.5 seconds?

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

E3 prove using the principle of mathematical induction

E2 develop and evaluate mathematical arguments and proofs

Elaboration—Instructional Strategies/Suggestions

E3/E2 The Principle of Mathematical Induction

If students were asked to prove a statement like

, they should notice that the series is neither arithmetic nor geometric. They cannot therefore, use the sum formulas developed earlier. It can, however, be demonstrated to be true for particular values. If $n = 1$, then $\frac{1}{1(2)} = \frac{1}{1+1}$. If $n = 2$, . Using this strategy, however, it is impossible to show this for all values of . Instead, students will use the Principle of Mathematical Induction.

A proposition is true for every natural number n if

- it is true for $n = 1$, and
- it follows from the truth of the proposition for $n = k$ that the proposition is true for $n = k + 1$

In practice, students should use *four steps in a proof by mathematical induction*.

Step 1 : Show the statement is true for $n = 1$
Step 2: Assume that the statement is true for $n = k$
Step 3: Prove the statement is true for $n = k + 1$, using the result of step 2
Step 4: Write a conclusion

Have students use mathematical induction to prove formulas like the following

for $n \in N : 1+3+5+K\,(2n-1)=n^2$

continued ...

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Paper and Pencil (E3/E2)

1) Explain the process of mathematical induction to a student who has missed class.
2) Conjecture and prove, by mathematical induction, the formula for the sum of S_n of the following series:

   a) $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\frac{1}{7\times9}+\dots+\frac{1}{(2n-1)(2n+1)}$

   b)

   c)

3) The principle of mathematical induction can be used to prove the truth of certain inequalities. Examine the following proof that $3^n < n!$ for $n \in N, n \geq 7$.

   Step 1 Prove the statement is true for $n = 7$ ... $3^7 = 2187 < 5040 = 7!$

   Step 2 Assume that $3^k < k!$

   Step 3 Prove $3^{k+1} < (k+1)!$

   Step 4 Write your conclusion

   Proof:

$2+7+12+\dots+(5n-3)$

$\sum_{s=1}^{k}\frac{1}{(3s-2)(3s+1)}$

$3^k < k!$ (assumed to be true)

$3\times3^k < 3\times k!$ (multiply by 3)

$\therefore \quad 3^{k+1} < 3k!$ (simplify) #1

But $3 < k+1$ $(k \geq 7)$

$3k! < (k+1)k!$ (multiply by k!)

$\therefore \quad 3k! < (k+1)!$ #2

$\therefore \quad 3^{k+1} < 3k! < (k+1)!$ (combine #1 and #2)

$\therefore \quad 3^{k+1} < (k+1)!$

4) Write proofs by induction to verify the formulas for sums of
   a) consecutive odd numbers
   b) consecutive even numbers
   c) consecutive natural numbers
   d) consecutive square numbers

Suggested Resources

Sequences and Series

Outcomes

SCO: In this course, students will be expected to

E2 develop and evaluate mathematical arguments and proofs

E3 prove using the principle of mathematical induction

Elaboration—Instructional Strategies/Suggestions

... continued

E2/E3 Prove: $1 + 3 + 5 + ... + (2n - 1) = n^2$ for $n \in N$. Students should record the following:

Step 1: For $n = 1$, *L.S.* = 1, *R.S.* = $1^2 = 1$. Since *L.S.* = *R.S.*, the statement is true for $n = 1$.

Step 2: Assume it is true for any number k

$$1 + 3 + 5 + \ldots + (2k - 1) = k^2$$

Step 3: Prove that it is true for (k + 1), that is, prove

$$1 + 3 + 5 + \ldots(2k - 1) + (2(k + 1) - 1) = (k + 1)^2$$

Proof:

(substituting step 2)

Step 4: Thus, by the principle of mathematical induction, $1+3+5+...+(2n-1)=n^2$, for all $n \in N$.

Sometimes, instead of a formula being given, a series is given, such as one similar to that on the previous page:

and students will be expected to determine the formula that represents the series. Students might examine this series and note that the sum of the first two terms is

, the sum of the first three terms is , and so on. This might lead them to

use inductive reasoning to predict the sum of the first n consecutive terms. They could then prove this prediction using the principle of mathematical induction.

Sequences and Series

Worthwhile Tasks for Instruction and/or Assessment

Performance (E2/E3)

1) Prove the following:
   a) The nth odd number is $2n - 1$.
   b) The sum of the first n odd natural numbers is n^2.
   c) $n^2 + n$ is divisible by 2.
   d) $n^3 - n$ is divisible by 6.
   e) The sum of the cubes of three consecutive natural numbers is divisible by 9.

2) Let

$$s_n = \sum_{i=1}^{n} \frac{i}{(i+1)!}$$

   a) Calculate S_1, S_2, S_3, and S_4.
   b) Write a formula from a. for S_n.
   c) Prove that your formula is correct.

3) An ancient legend has it that the inventor of the game of chess was offered a reward of his own choosing for the delight the game gave the king. The inventor asked for enough grains of wheat to be able to place one grain on the first square of the chessboard, two on the second, four on the third, and so forth, doubling the number of grains each time. Find a formula for the total number of grains on a chessboard after the k^{th} square has been filled.

Suggested Resources

Unit 2
Developing a Function Toolkit
Part I
(25-30 hours)

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C19 investigate and interpret combinations and compositions of functions

C2 model problem situations with combinations and compositions of functions

C14 analyse relations, functions, and their graphs

Elaboration—Instructional Strategies/Suggestions

C19/C14 Students' facility with functions becomes increasingly more sophisticated as they learn to reason about new functions derived from familiar ones via composition, inversion, transformation, and combination. For example, a profit relationship can be determined by combination, subtracting the functions that represent revenue and expenses. Students should explore visually the effects upon the domains and ranges of combination situations like the following:

linear ± linear, linear ± quadratic,quadratic ± quadratic, linear · linear

Students should also explore composition of functions in situations like the following:

1) $f(x) = g(h(x))$
2) $f(x) = h(g(x))$

if a) g and h are linear; b) one of g or h is linear and the other quadratic; c) both are quadratic.

Students should make predictions about what the resulting combinations and compositions would look like, study the tables, examine the graphs, note any changes in the domain and range, and interpret the results.

❒ Ask students what they would expect under the following operations:
given $g(x) = 3x - 4$ and $h(x) = 5 - 5x$

i) $f(x) = g(x) + h(x)$ ii) $f(x) = g(x) \cdot h(x)$ iii) $f(x) = g(h(x))$

From i) above they should be able to visualize the graph for f appearing between the graphs for g and h. This might be a good opportunity for mental math with students. In (ii), students should visualize a parabolic shape (product of two linear binomials). Its position is harder to determine, but it does concave down, and has a large vertical stretch factor, making the graph quite narrow. Students should know and be able to explain why the graph of the function passes through the zero of each of the given linear functions (see p. 68). In (iii), students, again using mental math, should be able to determine that this composition is linear, why it is linear, and approximate position of the graph.

C2 Teachers should work with combinations and compositions of functions in context to help students make meaning for the patterns they conjecture. For example, from the activity in the next column, students, working in the context of a business application, might conjecture that the profit made in running a business is equal to the revenue minus the cost [$P(x) = R(x) - C(x)$]. If $R(x)$ and $C(x)$ are linear, so is $P(x)$. In the business context domain and range, and independent and dependent variables, are reasonable quantities to discuss (e.g., $x \in W$). The break even point is discussed as the point of intersection of $R(x)$ and $C(x)$. This value is significant as the zero of the profit function. Trying to find when there is a profit, $P(x) > 0$, is a practical application of analysis of a linear function.

The next two left-hand pages (pp. 66 and 68) present a more detailed examination of combinations and compositions.

continued ...

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

Activity (C2/C19/C14)

1) a) During the summer holidays, you consider starting a lawn-mowing business. You will need to buy a power mower for $634. Every time you mow a lawn, you will spend $0.40 on gasoline. You charge a flat rate of $11 for mowing a lawn.

i) Set up a table in which one column corresponds to the number of lawns mowed—start with 0 and choose large as well as small numbers—and a second column corresponds to the total cost of the business when that number of lawns has been mowed. Be sure to include cost of mower.

Number of lawns	Cost
0	
1	
2	
10	
100	
200	

ii) When *x*, the number of lawns mowed, increases by 1, how does *y*, the total cost of the business, increase?

iii) Graph this set of points.

iv) Find the linear cost equation. To what business costs do the slope and the *y*-intercept correspond?

b) Make a similar table that gives the total revenue accumulated in relation to the number of lawns mowed. Use this table to find your revenue equation, and create a graph.

Number of lawns	Revenue
0	
1	
2	
10	
100	
200	

c) Profit is found by subtracting cost from revenue. Find the appropriate difference and create a table that shows profit as a function of the number of lawns mowed. Use the table of the equations found for revenue and cost to derive the profit equation.

Number of lawns	Profit
0	
1	
2	
10	
100	
200	

d) Graph the three equations on one set of axes, choosing appropriate scales.

i) What does *x* mean in each case?

ii) What does *y* represent in each case?

iii) What do the individual slopes represent?

iv) Which lines appear parallel? Are they parallel? Explain.

v) How many lawns would you need to mow to break even?

vi) How many lawns do you think you can actually mow in one week? What would be your weekly revenue?

vii) If you needed to make $1000 profit to go on a trip, how many lawns would you have to mow?

viii) Do you think it is a good idea to go into this business? Explain your answer.

... continued

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C19 investigate and interpret combinations and compositions of functions

C2 model problem situations with combinations and compositions of functions

C14 analyse relations, functions, and their graphs

Elaboration—Instructional Strategies/Suggestions

... continued

C19/C2/C14 Sometimes students will encounter two functions that are related where both functions are needed to answer a question or analyse a problem. Below, Figure A shows the radius r of a spreading ripple in the water after the pebble hits the water, as a function of time t, $r = f(t)$. Figure B shows the area A of the circular region as a function of its radius r, $A = g(r)$.

The graphs can be used to find the area of the circular region after 4 seconds. From Figure A, the radius is 2 cm when $t = 4$ s. From Figure B the radius of 2 cm indicates an area of approximately 12.56 cm^2.

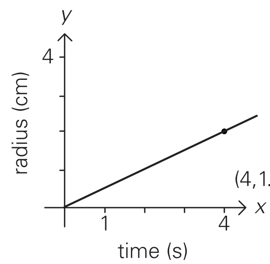


Figure A

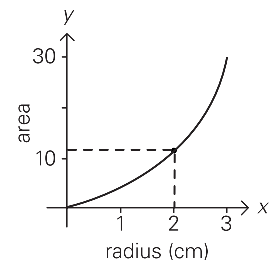


Figure B

In the above example students have used the output from the first function as the input for the second function. The domain of the second function has to connect to the first function. This is an example of the composition of two functions. The new functional relation between area and time is area = $g(f(t))$. Students should notice that $f(t)$ is the output from Figure A and the input in Figure B.

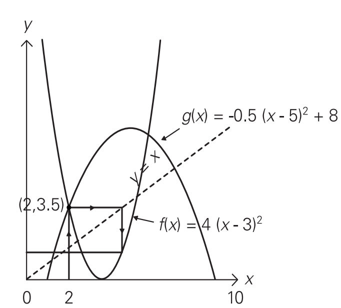
The symbol $g(f(t))$ is a composition of the two functions f and g. Students should understand that the composition $g(f(t))$ gives the final outcome when the independent value is substituted into the inner function f, and its value $f(t)$ is then substituted into the outer function g.

Students have actually been composing functions when they transformed graphs using two or more steps. The function $3f(x) - 1$ is obtained by first stretching $f(x)$ by a factor of 3 to get a new image, and then subtracting 1 from these new dependent values to slide the graph down one unit.

Students can visualize the composition of functions using graphs:

Given $f(x) = 4(x - 3)^2$ and $g(x) = -0.5(x - 5)^2 + 8$, evaluate $f(g(2))$:

- evaluate $g(2)$: trace a vertical line from $x = 2$ to g
- trace a horizontal line to y = x changing the y-value 3.5 to an x-value 3.5 ($y = x$)
- trace a vertical line from this point to intersect the graph f (other parabola)
- find y-value of this point by tracing a horizontal line to y-axis



Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

... continued

Activity **(C19/C14/C2)**

2) Students will need a worksheet with graphs of $f(x) = a(x - x^2)$ for various values of a. Begin each graph with an x-value of 0.3. Then use a graphical method similar to the one described in the previous column to find $f(f(f(f(\ldots f(x) \ldots))))$. Carefully draw each graph. Repeat the graphical steps enough times to be able to predict what is going to happen. Compare graphs with others to see the affect of different values of a. Ask students to describe what happens in each case.

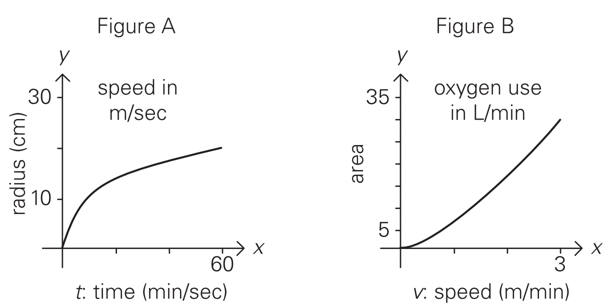
Performance **(C19/C14/C2)**

3) Consider a line as an inner function, given by . Suppose g is the absolute value function. Then $g(f(x))$ will be absolute value of the inner linear function $f(x)$. What will $g(f(x))$ look like?

4) Suppose $g(x)=2x - 7$ and . Determine the expression for each function composition.
   a) $f(g(x))$ b) $g(f(x))$

$f(x)=\frac{3x}{4}-3$

5) Figure A shows a horse's speed as a function of time as it swims in a large training pool. Figure B show the horse's oxygen consumption as a function of



its speed. Time is measured in seconds, speed in metres per second, and oxygen consumption in litres per minute.

Have students:

a) Use the graphs to approximate the horse's oxygen consumption after 20 seconds of swimming.
b) Draw segments on both graphs that verify your thinking.
c) How many seconds have elapsed if the horse's oxygen consumption is 15 L/min?

Suggested Resources

Murdock, Jerald et al. *Advanced Algebra Through Data Exploration*. Berkley, CA: Key Curriculum Press, 1998.

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C19 investigate and interpret combinations and compositions of functions

C2 model problem situations with combinations and compositions of functions

C14 analyse relations, functions, and their graphs

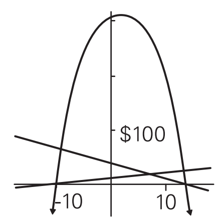
Elaboration—Instructional Strategies/Suggestions

C19/C2/C14 One way to generate a quadratic function is by finding the product of two linear functions. For example, given $g(x) = x + 2$ and $h(x) = x - 3$, then $f(x) = (x + 2)(x - 3) = x^2 - x - 6$.

Students will associate the roots of a quadratic function with its linear factors. The parabola of the function passes through the zeros of its linear factors. For example:

A clothing store sells ten dress shirts per week at a price of $35 each. Sales predictions suggest that each $2.50 decrease in price will increase sales by one shirt per week. What price will maximize revenue?

By letting x represent the number of times the price is decreased by $2.50, students would represent this problem with the function $R(x) = (35 - 2.5x)(10 + x)$, (a combination of two linear functions).



Drawing the graph of each component, $y_1 = 35 - 2.5x$, $y_2 = 10 + x$, and the graph of the combination $y_3 = (35 - 2.5x)(10 + x)$ students can observe the connection between the three graphs. (The roots (zeros) for each linear component are the roots (zeros) of the quadratic). Because of the symmetry property of parabolic curves, students could use the roots (x-intercepts) to determine the x-value of the maximum point. Students should be able to explain this. Optionally, students could TRACE the quadratic or use the "4:maximum" feature in the "CALC" menu to determine the maximum revenue. Students should determine and interpret the intercepts, and explore graphically and algebraically how many times the price should be decreased if the revenue is to be $300, $355, $360, or $370.

Students should discuss the domain and range. They should analyse the appropriate restrictions on each, and whether the situation should be represented as a continuous or discontinuous graph. Students may choose to solve quadratic equations by graphing, by factoring, or by using the quadratic formulas (all addressed in previous courses). Students should note the inadmissible root at –10. Inadmissible roots are those that must be rejected due to the conditions stated in the problem.

As students continue to examine the situation described above, they may want to investigate and describe the rate at which the revenue changes with respect to each decrease in price. They could do this by sketching tangent lines at various points along the parabola (see p. 70).

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance (C19/C2/C14)

1) The local corner store currently sells 240 bags weekly of Snax-Treat at a price of $3.29 each. Sales predictions indicate that each 25 cent decrease in price will increase sales by 60 bags weekly. If the store pays $2.00 for each bag, what prices will maximize the revenue? Show how a graph of a composition of functions could be used to find the x-intercepts, and how they can be used to find the price that will maximize the revenue.
2) Determine an equation for the entire surface area of a cylinder whose radius is r, and height, h. Express this formula as the sum of a linear and quadratic function.
   a) Draw a graph for the linear and quadratic function on the same axes if the height of the can is 10.0 cm. Interpret the meaning of each function.
   b) Predict the shape and location of the function that represents the entire surface area. Explain the common intersection point.
   c) How much paint will I need to spray the entire can if its radius is 7.0 cm?
   d) How much paper is needed to create the label for the same can, assuming the label covers the entire body of the can except 0.75 cm at the top and 0.90 cm at the bottom, and there is no overlap?
3) Building zones and safety regulations mandate that the new "Mumbley's Foodmarket" must provide a no-parking strip of uniform width surrounding its new store. The total area of the strip must be twice that of the building which measures 96 m by 84 m. Determine the width of the no-parking strip.
4) The property that Tom owns has an irregular shape. However, Tom knows that the area can be calculated by adding the length of one particular side to three times the square of that same side. The problem is that that side cannot be measured. Tom does know that it is roughly 25 m shorter than the width of his property.
   a) Find a function that relates the area A, of his property to its width, w.
   b) Show how to write this function in (a) in function notation using composition.
   c) Find the width if the area is 1200 m^2.

Suggested Resources

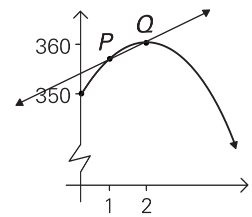
Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C12 demonstrate an understanding for the conceptual foundations for limit, the area under a curve, rate of change, slope of a tangent line, and their applications

Elaboration—Instructional Strategies/Suggestions



C12 As students are investigating the graph of $R(x) = (35 - 2.5x)(10 + x)$ to find how many times the price should be lowered in order to maximize revenue, they should consider when the rate of change is zero because that is when the revenue maximizes. Using their knowledge of secant lines and tangent lines from a previous course, students should select two points, *P* and *Q*, say at $x = 1$ and $x = 2$ respectively on the curve. To find the slope of the secant line (the average rate of change):

Thus, 2.5 is the slope of the secant *PQ*.

As the point *P* approaches the point *Q* the slope of the secant will get less steep and less steep and in this example, will approach a horizontal line since *Q* is very close to, or is the point that represents the maximum value. Eventually when *P* approaches *Q* the slope of the secant approaches zero (since $R(2) - R(2) = 0$). The secant now touches the curve at only one point ($x = 2$), thus, is tangent to the curve $R(x)$ at 2. This approximates the instantaneous rate of change of the revenue with respect to the number of price changes when the number of price changes is 2.

Since the slope of the tangent line is seen to be zero at the maximum point, then the slope of the tangent line is 0 when $x = 2$.

So, the maximum revenue occurs when there have been two decreases of \$2.50 in price, giving a maximum revenue:

$$R(2) = 350 + 10(2) - 2.5(2)^2 = 360, \text{ or } \$360.$$

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance **(C12)**

1) While working 100 metres above the street in the city's downtown, a construction worker loses her hammer. The distance an object in freefall travels downward is $d(t) = -4.9t^2$. The negative sign indicates the downward direction.
   a) Find an expression $v(t)$ for the velocity of the hammer after it has been falling for 2 seconds.
   b) What is the velocity of the hammer as it hits the ground?

Activity **(C12/25)**

2) Consider the problem of finding the tangent to the parabola $y = x^2$ at the point (1,1).
   a) Draw the graph on graph paper. Use a ruler to draw an approximate tangent to the curve at (1,1).
   b) Find the slope m_{tan} and y-intercept, b. Write an equation for the tangent in slope-intercept form. How close was your estimate to the equation you found?
   c) Draw the secant from (1,1) through (–1,1). Use these two points to find the equation of the secant in slope-intercept form. Label this line #1.
   d) Draw, label, and find equations for the secants from (1,1) through points (–0.5, 0.25), (0,0), (0.5, 0.25), and (0.7, 0.49). Label these lines #1, #2, ... #5.
   e) Copy and complete this table in your notebook.

Secant #	Slope	Intercept
1	2	
2		
3		
4		
5		

   f) Now look for a trend in these columns. As the x-coordinate comes closer and closer to 1, what happens to the slope and intercept of the tangent?
   g) Repeat this investigation, taking a sequence of secants from (1,1) through #1, (4,16), #2 (3,9), #3 (2,4), #4 (1.5, 2.25), and #5 (1.2, 1.44). Make a table and look for trends. What do you conclude?

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C25 demonstrate an intuitive understanding of the concept of limit

C12 demonstrate an understanding for the conceptual foundations for limit, the area under a curve, rate of change, slope of a tangent line, and their applications

Elaboration—Instructional Strategies/Suggestions

C25/C12 In a previous course, students have demonstrated that the slope of a curve at a given point can be approximated by finding the slope of the straight line joining two points on the curve *P* and *Q*. As *P* approaches *Q* and eventually becomes *Q*, the slope of this tangent line to the curve is interpreted as the approximation of the instantaneous rate of change at that point. Students have obtained an expression for the slope in a previous course by using:

$$m = \frac{f(x+h) - f(x)}{h} \text{ for } h = \{0.01,\ 0.001,\ 0.0001,\ \dots\}$$

On the preceding two-page-spread, when given $R(x) = 350 + 10x - 2.5x^2$, students found the slope of the secant line *PQ* (the average rate of change of the function between the points *P* and *Q*). As the point *P* approaches the point *Q*, the secant line gets closer and closer to being the tangent line to the curve at *Q* and becomes the tangent line at *Q* when *P* lies on *Q*. The instantaneous rate of change at a point on the curve is the slope of the tangent to the curve at this point. The expression for this rate of change, or this slope of the tangent line can be found by selecting a point *P (x, R (x))*, and a point *Q (a, R(a))*, and finding the expression for the slope of *PQ* to the function *R*:

$$mPQ = \frac{R(x) - R(a)}{x - a}$$
$$= \frac{\left(350 - 10x - 2.5x^2\right) - \left(350 + 10a - 2.5a^2\right)}{x - a}$$
$$= 10 - 2.5(x + a)$$
$$= 10 - 2.5x - 2.5a$$

As $P \to Q$, the value for , and so:

the slope of the tangent at $Q = 10 - 2.5x - 2.5x$

$= 10 - 5x$

In general, to predict the instantaneous rate of change of some function $y = f(x)$ at some specific value of *x*, say $x = a$, students should calculate the slope of the secants over ever-decreasing intervals between *x* and *a* that contain the specific value $x = a$ as an endpoint. The specific number that the slopes approach as the difference between *x* and *a* approaches zero is called the limit of the slopes.

... continued

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance (C25/C12)

1) In a baseball game a ball is popped into the air so that the vertical height of the ball in metres at time t seconds is $f(t) = -4.9t^2 + 24.5t + 1$.
   a) Predict the vertical speed of the baseball in m/s at the following instants: $t = 1, 2, 2.5, 3$ and 4 seconds.
   b) Why should the vertical speed at $t = 1$ be greater than at $t = 2$ seconds?
   c) What is the significance of the answer at $t = 2.5$?
   d) Why should the vertical speed downward be greater at t = 4 than at t = 3?

2) Throw a pebble into a still pond and watch the circular ripples that the pebble causes.
   a) What is the rate of change of the area enclosed by a circular ripple with respect to the radius?
   b) What is the rate of change when the radius of the circle is 5 m?

3) The area of a square increases as its sides increase. Find the rate of change of the area with respect to the length of the sides when the sides are 3 units long.

4) By selecting a point h units to the right or left of a given point on a graph
   a) Show how the expression for the average rate of change can be obtained.
   b Describe how this is used to determine the instantaneous rate of change at any given point on the curve.
   c) Determine the instantaneous rate of change at the given point of the three given functions that follow.
   d) Finally, find the equation for the tangent line at the given point. Check using graphing technology.

   i) $y = 2x^2$; (1,2) ii) $y = x^2 + 1$; (2,5) iii) $y = -3x^2 + 2x - 5$: at $x = 2$

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

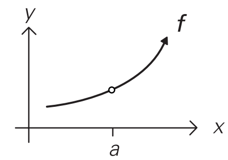
C25 demonstrate an intuitive understanding of the concept of limit

C12 demonstrate an understanding for the conceptual foundations for limit, the area under a curve, rate of change, slope of a tangent line, and their applications

Elaboration—Instructional Strategies/Suggestions

... continued

C25/C12 Students should explore another way to examine the concept of limit: When students try to determine the limit of a function at a particular value "*a*", they should examine values for *x* that approach *a* from the left (smaller values than *a*), and from the right (larger values of *a*).



When $f(x)$ approaches as *x* approaches *a* "from the left", students should write $\lim_{x\to a^-} f(x) = L$

and say: "*L* is the left-hand limit of $f(x)$ as *x* approaches *a*".

When $f(x)$ approaches as *x* approaches *a* "from the right", we write

$\lim_{x\to a^+} f(x) = L$ and say: "*L* is the right-hand limit of $f(x)$ as *x* approaches *a*".

When the left-hand limit, and the right-hand limit are the same, students can write: .

❐ Have students consider the behaviour of $f(x) = 2x^2 + 1$ as *x* approaches 3.
   a) Have them try several values approaching 3 from the left (2.9, 2.99, 2.999, ...), then from the right (3.1, 3.01, 3.001, ...).
   b) Have students use the format given above to record their findings, and make their conclusion.

The process described above for finding the limit appears to be different than that learned previously. At that time students were finding limits as *x* approached infinity, and essentially were calculating the left-hand limit (trying values that approached infinity from the left). Students will quickly see that the procedure to determine the limit of a polynomial function for a specific value *x* is essentially an evaluation of the function for that value. They should understand that this is so because this function has a parabolic shape with an unrestricted domain and the function is continuous.

The limit concept is the central idea of calculus and must be grasped, at least intuitively, before going on to develop differential calculus. The limit concept is revisited again on p. 96 for cubic functions.

Have students consider this Definition of Limit:

If and $f(x)$ is a function, and, as *x* approaches *a* from the right and from the left, $f(x)$ approaches *L*, then we define .

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

... continued

Pencil and Paper (C25/C12)

5) For each question find the left-hand limit, the right-hand limit, and make a concluding statement about whether the limit exists.

   a)

   b) $\lim_{x\to 1}\left(2x^2+5x-1\right); x = 0.09,\ 0.99,\ 0.999,\ 1.1,\ 1.01,\ 1.001$

   c) $\lim_{x\to 3}\left(-\frac{1}{3}x^2-5x-3\right); x=-3.1,\ -3.01,\ -3.001,\ -2.9,\ -2.99,\ -2.999$

6) For the top graph, use limit notation to describe the behaviour of *f(x)*
   a) as $x \to 2^-$
   b) as $x \to 2^+$
   c) as $x \to 2$

7) For the lower graph, which statement is correct?

   a)

   b)

   c) [illegible] 1.99, 1.999, 2 [illegible] 2.01, 2.0001 does not exist

   d) $\lim_{x\to\pi} f(x)$ does not exist

Performance (C25/C12)

8) a) Graph $f(x) = 2x^2 + 1$, and $g(x) = 3x - 2$
   b) Graph $f(x) + g(x) = h(x)$
   c) Find the limit as $x \to 0$ for each of *f, g,* and *h*. Explain how this could have been predicted without finding the limits.
   d) Graph
   e) Find the limit as $x \to \frac{2}{3}$ for each of *f, g,* and *j*. Explain the relationship among the three limits.

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C25 demonstrate an intuitive understanding of the concept of limit

C12 demonstrate an understanding for the conceptual foundations for limit, the area under a curve, rate of change, slope of a tangent line, and their applications

B6 determine and apply the derivative of a function

Elaboration—Instructional Strategies/Suggestions

C25/C12/B6 Students can now combine their understanding of rate of change and limits to understand that the instantaneous rate of change of a function at a point *(a, f (a))* can be found by combining limits with the procedure to find rate of change. If *f* is a function and , the instantaneous rate of change of *f* at that point, (*a*, *f* (*a*)), is , provided this limit exists. For example, if students are asked to find the instantaneous rate of change of at the point (1, 4), they might begin by finding *f (a + h)* for *a* = 1.

so, $\frac{f(1+h)-f(1)}{h}=\frac{7h+2h^2}{h}=7+2h; h\neq 0$

taking the limit as : slope of the tangent is

The above work results in the value of the slope of the tangent line, or the instantaneous rate of change at a particular point (1, 4).

To find the general expression for the slope of the tangent line of $f(x) = 2x^2 + 3x - 1$, students should write:

$$
\begin{aligned}
f(x+h) &= 2(x+h)^2+3(x+h)-1\\
&= 2\left(x^2+2xh+h^2+3x+3h-1\right)\\
&= 2x^2+4xh+2h^2+3x+3h-1\\
f(x+h)-f(x) &= 2x^2+4xh+2h^2+3x+3h-1-\left(2x^2+3x-1\right)\\
&= 4xh-2h^2+3h\\
&= h(4x-2h+3)\\
\frac{f(x+h)-f(x)}{h} &= \frac{h(4x-2h+3)}{h}=4x-2h+3; h\neq 0
\end{aligned}
$$

For this function, the expression for the slope of the tangent line is: $\lim_{h\to 0}(4x-2h+3)=4x+3$. Thus the instantaneous rate of change of *f* at any given point is $4x + 3$.

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Multiple Choice **(C25/C12)**

1) The slope of the tangent line to $y = 5$ at $x = 1$ is:
   a) 0 b) 1 c) 5 d) undefined
   Explain.

2) The instantaneous rate of change of the function $f(x) = 3x^2 - 5x + 2$ at $x = 2$ is:
   a) 2 b) 7 c) 1 d) $(7x - 10)$

Extension **(C25/C12)**

3) What is the instantaneous rate of change of the function $y = \frac{1}{x}$ at $x = 0$?

Pencil and Paper **(B6)**

4) Find an expression for the slope of the tangent line, then determine the slope at the given *a*-values on the curve:

   a)

   b)

   c)

   d)

   e)

$f(x) = \frac{3}{5}x - \frac{2}{3}x - 1;\ a = 2$

5) Find the equation of the tangent to the curve at the given point. Use a graphing calculator to check your answer.
   a) $y = x^2 + 4x - 1$; (1, 4)
   b) $y = 2x^3$; (1,2)

Performance **(C25/C12)**

6) A ball is dropped from a height of 100m. The distance an object in freefall travels downward is $s(t) = -4.9t^2$ in metres per second squared. The negative sign indicates the downward direction. The velocity is negative since it is directed downward as well.
   a) Find the speed of the ball after it has been falling for 2 s.
   b) When will the ball reach the ground?
   c) What is the speed of the ball as it hits the ground?

7) Throw a pebble into a still pond and watch the circular ripples it makes. What is the rate of change of the area enclosed by a circular ripple with respect to the radius when the radius of the circle is 5 m?

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C7 demonstrate an understanding for slope functions and their connection to differentiation

C14 analyse relations, functions, and their graphs

C5 use tables and graphs as tools to interpret expressions

Elaboration—Instructional Strategies/Suggestions

C7/C14/C5 Because the slope function (the function defined as the slope of the tangent line) is derived from a function *f,* mathematicians refer to the slope function as the derived function or the derivative of *f.*

If $y = f(x)$ describes some function of x, then (read "y prime") or

describes the instantaneous growth rate, that is, the **derivative,** of the function *f.* The simplest notation is the prime notation, however, it is often preferable to use

$\frac{dy}{dx}$ when it is important to clearly specify the independent variable with respect to which the growth rate is being calculated. For example, when students are asked to differentiate $f(x) = x^2$, or to find the derivative of x^2, they would write: if

$f(x) = x^2$, then , or .

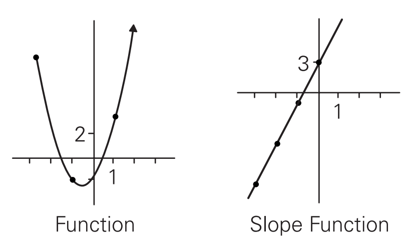
Beginning with a visual representation is helpful for students to understand that the slope function (the derivative) tells them the rate of change at any given point on that function. In the activity in the next column students will begin with the graph of a function. They will measure and find approximate values for the slope of the graph at many points, and use this to draw the graph of the slope function.

The slope function is important because it represents the rate of change of the original function. In the case of the mountain (in the activity) it tells students how steep it is at different points—the rate of change of height with respect to horizontal distance. In the case of the runners it tells students how fast the runners are going at different times—the rate of change of displacement with respect to time.

To draw a slope function graph:

Given the function $f(x) = 2x^2 + 3x - 1$, for , determine the slope of the tangent line at several points: (from p. 76):

so,



$$f(x) = 2x^2 + 3x - 1$$

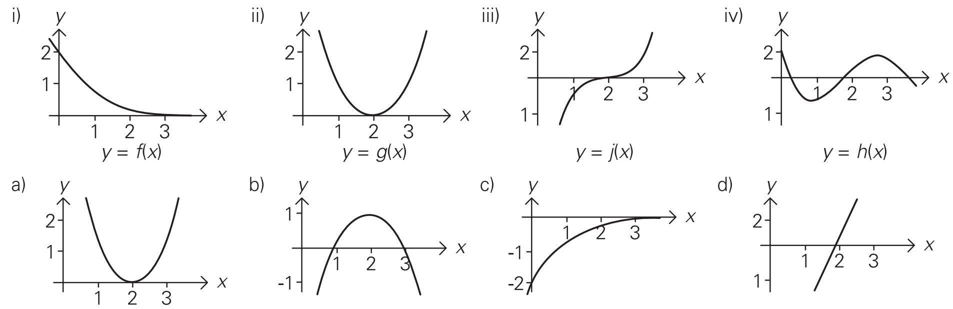
... continued

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Paper and Pencil (C7/C14/C5)

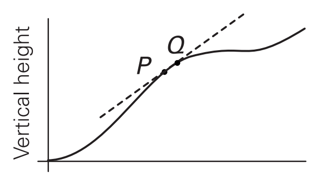
1) In the first row are the graphs of four functions. In the second row, in a different order, are graphs of their slope functions, labelled (a) to (d). Match each function with the correct slope function. Indicate where the slope is positive, where it is negative, and where it is zero.



Activity (C7/C14/C5)

2) Students can apply their ideas about finding the slope of a curve to the contour of a mountain, to find out how steep it is.

   The graph shows the profile of a mountain. The point *O* is the foot. The slope of the mountain keeps changing.



Horizontal distance (m) from the foot (point 0) (mountain profile)

   a) The graph looks like a picture of the mountain. Be careful not to confuse pictures with graphs. Ask students to explain what the curve represents.
   b) Look at the point on the slope marked *P*. Point *Q* has been chosen very close to *P*, and the line *PQ* drawn. Ask students to measure the slope of *PQ* (use a ruler and trigonometry.)
   c) Measure the *x*-coordinate of *P* (the horizontal distance *OR*).
   d) Mark a series of other points on the curve, roughly equally spaced along the *x*-axis, and repeat the last two steps for each point.
   e) Establish a table, like to one below, and enter the results in it.

x-coordinate of point	slope of the curve at this point

   f) The table gives the values of a new function—the function whose value is the steepness of the mountain at any point.

   Ask students to draw a graph of this function. They will need to choose a fairly large scale for the *y*-axis. Students may need to measure the slope at a few more points.

... continued

Suggested Resources

Barnes, M. *Investigating Change: An Introduction to Calculus for Australian Schools.* Carlton Victoria: Curriculum Corporation, 1993.

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

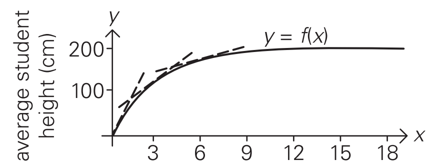
C7 demonstrate an understanding for slope functions and their connection to differentiation

C12 demonstrate an understanding for the conceptual foundations for limit, the area under a curve, rate of change, slope of a tangent line, and their applications

Elaboration—Instructional Strategies/Suggestions

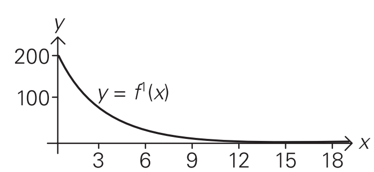
C7/C12 When drawing slope function graphs from the graph of an original function, students have to measure slopes carefully at many different places on the original graph. Clearly, they need to find more efficient ways of finding slope functions. Later on in this course they will discover a rule for finding the slope function if they know the original polynomial function.

In many cases, however, students are given a graph, but do not know the equation for it (in data collection, for example). It is useful for students to be able to sketch a rough graph of the slope function without doing a lot of calculation. They should estimate slopes by sketching tangent lines. For example, have students look at this graph, then



- ❐ Working in pairs ask students to describe the graph to their partner. Have students pay special attention to the slopes. Ask them to describe how the slope changes in moving from left to right along the curve. They may find it easier to think of the graph as the displacement-time graph of a moving object. Then, ask them to comment on the velocity of the object.

Students should be able to work out the general shape of the slope function without making accurate measurements. It might sound something like this ... "the slope function has a value of about 200 at $x = 0$. The slope decreases quickly at first, and then changes slowly. The slope of the tangent reaches zero when $x = 15$, and is zero from then onwards. When this information is plotted, the slope function will look something like this:"



As presented on earlier pages, there is a convention in mathematics to call the slope function (or derivative) to show its relationship to the original function, $f(x)$.

Developing a Function Toolkit

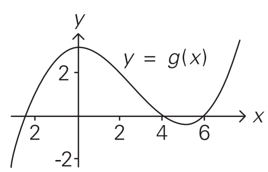
Worthwhile Tasks for Instruction and/or Assessment

... continued

Same Activity **(C7/C14/C5)**

g) The profile of the mountain defines a function which could be called the altitude function of the mountain. The new function that has been derived from this is the slope function (the derivative). It shows the rate of change of the altitude function at any point.

Ask students to work out what the graph of the slope function of $g(x)$ would look like (see (h)–(i) below).



the slope function graph

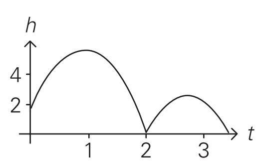
h) Ask students to begin by studying the slope function graph carefully, looking at it from left to right. Ask them to describe to their partner how its slope changes. Where is the slope positive? Where is it negative? Where does it change from positive to negative, or from negative to positive?

i) Working together, ask students to sketch a graph of the slope function.

Performance **(C7/C12)**

4) The graph below shows *h*, the height of an object above ground in metres as a function of *t*, the time in seconds. Describe the motion of the object.

   i) What sort of an object do you think it was?
   ii) Draw a graph showing the velocity as a function of time.



Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C14 analyse relations, functions and their graphs

C15 determine the equations of a polynomial and rational functions

C5 use tables and graphs as tools to interpret expressions

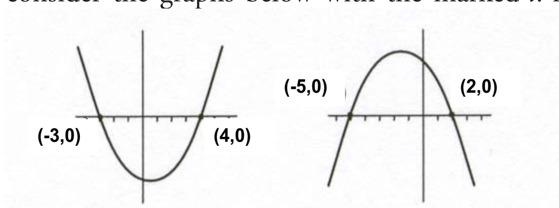
Elaboration—Instructional Strategies/Suggestions

C10/C14 Students have been evaluating and sketching slopes of tangent lines to quadratic functions. Now, they will examine the roots of quadratic and other polynomial functions.

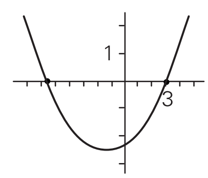
❒ Ask students to graph $0.5n^2 - 1.5n - 23 = y$. Have students read the graph to state the solution when $y = 0$. Students should be able to read the x-intercepts as approximately –5.5 and 8.5. These values are called the zeros of the function, the x-intercepts of the graph, and the roots of the equation $0.5n^2 - 1.5n - 23 = 0$.

C15/C5 In this section students will learn to write the equations of the polynomials when they know the roots. See the examples below. Conversely, they will learn about the factored form of the polynomial equation and how it relates to the roots of the equation and the zeros of the graph.

❒ Have students consider the graphs below with the marked x-intercepts

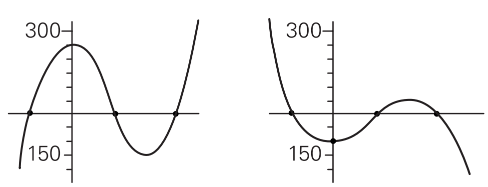


C18/C11/C8/C12 It should be no surprise to the students that the y-coordinate is zero at each x-intercept. Students should use this information and the zero-factor property to find x-intercepts. For example, suppose $y = -1.4\ (x - 5.6)(x + 3.1)$. Have students look at the graph and the equation and name the x-intercepts. They should respond by saying that the x-intercepts will be the two values in the equation that make $y = 0$.



5.6 works since $-1.4(5.6 - 5.6)(5.6 + 3.1) = 0$
–3.1 works because $-1.4(-3.1 - 5.6)(-3.1 + 3.1) = 0$

❒ Ask students to write an equation for the graph below with x-intercepts at –2.5, 7.5 and 3.2. Students should use the factored form of a polynomial ($y = a\ (x - r_1)(x - r_2)(x - r_3)$ where r represents the root(s)). Sometimes students forget the impact of the value a. Students should note that the roots are unaffected by a vertical stretch. They would record $y = a\ (x + 2.5)(x - 7.5)(x - 3.2)$. They should understand that there are many graphs with these roots, and that to fix it to one equation they need to determine a stretch value a. They should experiment to see what effect each choice will have on the graph. For example, the graph on the left below has an a-value of 4 while the graph on the right has $a = -1.5$.

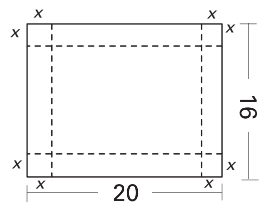


Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Activity (C15/C5)

1) Students will need graph paper and scissors for this activity. Have students ...
   a) Cut several 16-unit-by-20 unit rectangles.
   b) Construct open-top boxes with *all* possible integer *x*-value dimension. (Construct the boxes by cutting a square from each corner and folding up the sides).



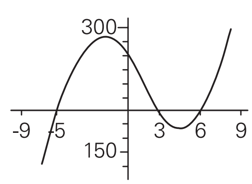
   c) Record the dimensions of each box and calculate the box's volume. Prepare a table showing the *x*-values and volumes of the boxes.
   d) Use sequence of differences to find the degree of the relation. Write an equation that gives the volume *V* in terms of *x*, the side length of the removed square.
   e) Graph the equation and determine the *x*-intercepts. (There are three). Call these three values r_1, r_2, r_3. If boxes are made using these values as *x*, what will they look like?
   f) Graph the equation $y=(x-r_1)(x-r_2)(x-r_3)$, as well as your first equation. What are the similarities and differences? How can you alter the second equation to make the graphs of the equations identical?

Performance (C5/C10/C14)

2) a) Find the *x*-intercept and *y*-intercept of $y = 2.5\,(x - 7.5)(x + 2.5)(x - 3.2)$.
   b) Write the equation in general form.
   c) Graph the general form to make certain your work is correct.

3) Find the *x*-intercept(s) and *y*-intercept for the graph of each equation, without actually graphing. If the equation represents a parabola, find the vertex. Check each answer by graphing.
   a) $y = -0.25(x + 1.5)(x + 6)$ c) $y = -2(x - 3)(x + 2)(x + 5)$
   b) $y = 3(x - 4)(x - 4)$ d) $y = 5(x + 3)(x + 3)(x - 3)$

4) a) Write the equation of a polynomial of least degree that contains the *x*-intercepts pictured.
   b) Adjust the leading factor *a* so that it contains the *y*-intercept (0,180).
   c) Write the equation that contains points that are exactly 100 units up from the graph pictured.
   d) Write the equation that contains points that are exactly 4 units left of the points pictured.



5) Given $f(x) = x + 2$, $g(x) = x - 3$, and $h(x) = 2x - 1$
   a) Graph *f* and graph *g*, then graph $f(x)\cdot g(x)$. How does the graph of $f(x) \cdot g(x)$ compare to the graphs of *f* and *g*?
   b) Graph $p(x) = f(x) \cdot g(x) \cdot h(x)$, and describe the graph that results.

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C3 model real-world phenomena using polynomial functions and rational functions7

C15 determine the equations of polynomial and rational functions

C14 analyse relations, functions and their graphs

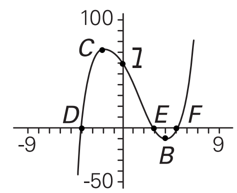
C10 analyse and solve polynomial, rational, irrational, absolute value and trigonometric equations

Elaboration—Instructional Strategies/Suggestions

C3/C15 Traditionally, polynomials with degree 3 or more are called higher degree polynomials. The graph of a polynomial function with real coefficients has a *y*-intercept, possibly one or more *x*-intercepts, and other features like turning points (local maxima or minima). In this section students will discover how to make the connections between the equations of polynomials and their graphs, in order to predict where different features will occur.

$y = 5$	constant polynomial (zero degree)
$y = 5x + 2$	linear polynomial (1st degree)
$y = 5x^2 + 2x - 1$	quadratic polynomial (2nd degree)
$y = 5x^3 - x + 3$	cubic polynomial (3rd degree)
$y = 5x^4 + 2x^3 - x^2 - 4$	quartic polynomial (4th degree)

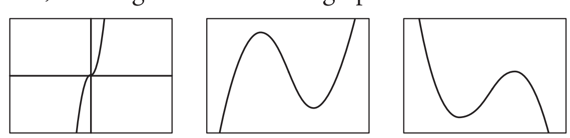
C14/C10 To the right is the graph of $y = a(x - 3)(x - 5)(x + 4)$ with $a = 1$.



Notice that points *D*, *E*, and *F* are the three required *x*-intercepts and point *I* is the *y*-intercept (0, 60). Ask students to explain how they know that the *y*-intercept is 60.

Point *B* on this cubic function is a local minimum point because it is the lowest point in its immediate region of *x*-values. Point *C* is a local maximum point because it is the highest point in its immediate region of *x*-values. Quadratic relationships have either a maximum or a minimum, cubic relationships often have both. Students should have the opportunity to experiment with different *a*-values to see the effect on the graph. Students should also explore what happens on the graph when factors are rasied to various powers. For example, how does the given function change if $(x - 3)$ is squared, or how does it change if $(x + 4)$ is cubed, or both?

C10 Students should be able to identify the degree of any polynomial graph by looking at its shape. For example any third-degree polynomial has essentially one of the shapes shown below. The first graph is $y = x^3$. It seems to have only one root (0), however it actually has a triple root of 0. Also, it has no local maximum or minimum. The next two graphs are of the general cubic equation $y = ax^3 + bx^2 + cx + d$. In the middle graph, *a* is positive, but negative in the last graph.



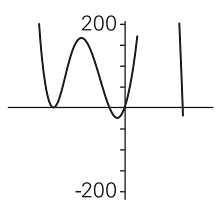
C10/C14 Ask students to graph several polynomial equations in order to explore how the graphs are the same and how they are different. For example, when students graph $y = x^3 - x^2 - 16x - 20$, they will see only two *x*-intercepts. When factored, the polynomial can be expressed as $y = (x + 2)^2(x - 5)$. This leads to three roots for the equation {–2, –2, 5}. The –2 root is a double root and causes the graph (as do all even numbers of roots) to be tangent to the *x*-axis rather than pass through the *x*-axis as with functions with odd numbers of roots. The result is that the function values do not change signs on either side of the zero. Students might conjecture that this phenomenon occurs with all zeros that come from factors with even exponents.

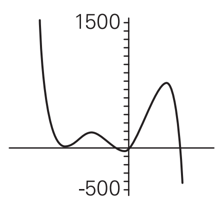
Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C15/C14/C10**)

1) a) Write a linear equation with x-intercept at (4,0).
   b) Write a quadratic equation with its only x-intercept at (4,0).
   c) Write a cubic equation with its only x-intercept at (4,0).
   d) Write a quadratic function which is tangent to the x-axis at –2.
   e) Write the equation of the cubic function whose three zeros are –1, 2, and 5, and its y-intercept is –5.
2) The graph for $y = 2(x-3)(x-5)(x+4)^2$ has zeros at 3, 5 and –4 because they are only possible values of x that make $y = 0$. This is a 4th-degree polynomial, but it has only three different x-intercepts. Graph each equation and name a graphing window that provides a complete graph. Complete graphs display all of the relevant features (local/absolute maximums and minimums and intercepts).
   a) $y = 2(x-3)(x-5)(x+4)^2$
   b) $y = 2(x-3)^2(x-5)(x+4)$
   c) $y = 2(x-3)(x-5)(x+4)$
   d) $y = 2(x-3)^2(x-5)(x+4)^2$
   e) Describe a connection between the exponent on a factor and what happens at the x-intercept of its corresponding graph.
3) Both graphs below are of the same polynomial function. The one on the left is not a complete graph, but shows the x-intercepts better.





   a) How many x-intercepts are there?
   b) What is the smallest possible degree for this polynomial?
   c) Write an equation for the graph pictured which includes the points (0,0), (–5,0), (4,0), (–1,0), and (1, 216).
4) True or false: Every polynomial equation has at least one real solution. Explain your answer. Modify the statement, if necessary, to make it true.

Journal (**C15/C14/C10**)

5) Given $y = ax^3 + bx^2 + cx + d$, with $a < 0$, and a triple root (three equal roots). Describe the general appearance of the graph in detail. How would the graph's appearance change if instead of three equal roots, there were two equal roots and one other?

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

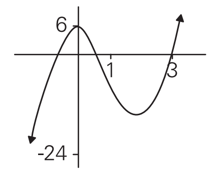
C14 analyse relations, functions, and their graphs

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C20 factor polynomal expressions

Elaboration—Instructional Strategies/Suggestions

C14/C10/C20 To the right is the graph of $y = 6x^3 - 17x^2 - 5x + 6$. How can students find the zeros when the x-intercepts are not obvious?



One way would be to use the graphing calculator and the various procedures and menus (CALC, TABLE, SOLVE).

Factoring polynomial expressions to find roots for corresponding functions is often not possible, and using a formula works if a formula is available. Traditional paper and pencil methods for finding the zeros are tedious and take much time, however, students may increase their understanding of relationships between roots and x-intercepts by using some pencil and paper methods. For example, from the graph of $y = 6x^3 - 17x^2 - 5x + 6$, students can see three zeros: one at $x = 3$, and the other two, less obvious, around and . So the only root known with some confidence is +3. Another possibility is to examine how the rational root theorem can be used to determine that +3 is a root. Students could determine roots solely by using the remainder theorem. When factoring is not possible, using long division allows students a way to find other factors. In this case, a quadratic factor, which in turn might itself factor.

Using long division:

Since the remainder is zero, the divisor $(x - 3)$ is a factor (remainder theorem), and the quotient $6x^2 + x - 2$ is another factor. If the remainder was not zero, then the divisor is not a factor. When the quotient is factorable students should try to factor it. In this case, the quotient does factor to be: $6x^2 + x - 2 = (3x + 2)(2x - 1)$.

From these factors the other two roots are confirmed to be $\frac{1}{2}$ and . If the quadratic does not factor then the remaining two roots (found by using the quadratic formula) must be irrational or non-real.

Synthetic division is an optional way to divide polynomials *. Dividing $6x^3 - 17x^2 - 5x + 6$ by one of its factors $(x - 3)$ using synthetic division:

* See next two-page spread to understand why synthetic division works ...

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C10/C14/C20**)

1) a) How many zeros does $y = x^4 + 3x^3 - 11x^2 - 3x + 10$ have?
   b) Name the x-intercept(s).
   c) Name the y-intercept.
   d) Write the polynomial in factored form.
   e) Determine all the roots.
   f) Melvin changed the equation to read $y = x^4 + 3x^3 - 2x + 2$ and claims that this quartic has only 2 roots. Is he correct? Explain.
2) Find the x-intercept(s) for each equation to 3 decimal places:
   a) $y = x^5 - x^4 + 16x + 16$
   b) $y = 2x^3 + 15x^2 + 6x - 6$
   c) $y = 0.2(x - 12)^5 - 6(x - 12)^3 - (x - 12)^2$
   d) $y = 2x^4 + 2x^3 - 14x^3 - 9x - 12$
3) a) Given the function $f(x) = -5x^3 - 2x^2 + 3x - 2$. Estimate an x-intercept from the graph. Explain how this estimation can be used to make a larger statement about the roots of the cubic function f.
   b) Given the function $g(x) = -5x^3 - 15x^2 + 10x + 30$. Estimate an x-intercept from the graph. Explain how this estimation can be used to make a larger statement about the roots of the cubic function g.
   c) Make up a question like (a) and/or (b) above, but having a response than either of these two has.
4) Explain how the remainder theorem can be used to determine if $\sqrt{2}$ is a root of an equation.

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C27 represent complex numbers in a variety of ways

C14 analyse relations, functions and their graphs

Elaboration—Instructional Strategies/Suggestions

C10 When solving polynomial equations, students should either factor them if possible (see the next two-page spread), use paper and pencil methods as described on the previous two-page spread, or use technology to find the zeros. Most times the zeros will be represented as decimals or decimal approximations. To get the most precise estimate for the zeros, students might use the "zero" command in the CALC menu or the TABLE feature where they can better determine the x-value that causes the y-value to be zero. (The TABLE does not work as well when there is a double root, since there will be no sign change when the roots are equal).

C27/C14 Consider the function $f(x) = x^3 - 4x^2 + 9x - 36$. On the graph, students will only see one x-intercept at (4, 0). But the polynomial is cubic, thus must have three roots. The other two must be complex roots. Students should determine these complex roots by solving the equation $x^3 - 4x^2 + 9x - 36 = 0$ by factoring, by grouping, or by division. The other two roots are $\pm 3i$. Students should know then, that the three factors of the given polynomial are $(x - 4)(x - 3i)(x + 3i)$.

C10 When solving polynomial equations and inequalities, students may need to find roots. On the previous two-page spread, synthetic division was suggested as a alternate way to perform long division. From where does this procedure originate?

Begin with a polynomial to evaluate: $f(x) = 6x^3 - 17x^2 - 5x + 6$. First factor out the x^2 from the first two terms, then factor out . The polynomial is now said to be in nested form.

Suppose one wants to find $f(3)$, then one would substitute into the nested form …

$6\times3=18,\ 18-17=1$, then , then , so $f(3) = 0$, thus 3 must be a root of $f(x)$.

The "multiply by x then add the next coefficient" is repeated, so is easy to program or record:

The student "brings down" the coefficient 6, multiplies it by x, or in this case 3, and writes the answer 18 under the –17, then adds to –17 and writes the answer 1 under the 18, beside the 6:

Now the student multiplies the 1 by 3 and records below the –5, then adds to the –5 and gets –2 which is recorded below the 3, beside the 1. The process is repeated until the final number (zero in this case) under the –6. This number represents $f(3)$, or the remainder in a division process.

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance (C10/C27/C14)

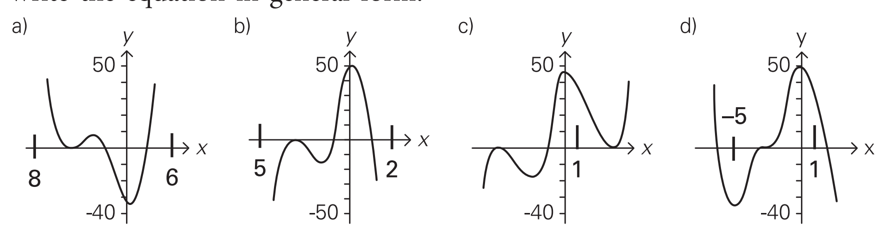
1) For what values of x does $g(x) = x^3 - 1$ have negative values?

2) For what values of x does $h(x) = x^3 + 3x^2 - 4x - 1$ have positive values?

3) a) Given $f(m) = 3m^3 + m^2$ and $g(m) = 9m + 3$, for what values of m is $f(m) = g(m)$?
   b) Given $h(x) = x^4$ and $f(x) = 8x^2 - 7$, for what values of x is $h(x) = f(x)$?
   c) Given $f(x) = x^4 - 15$ and $g(x) = 2x^2$, determine the values of x for which $f(f(x)) = g(x)$?

4)`True or false: Every polynomial equation has at least one real solution. Explain your answer. Modify the statement, if necessary, to make it true.

5) In each graph below, *the leading coefficient* is 1, or –1. Using the graphing calculator, determine the approximate roots, then write an equation in factored form that will produce each graph.
   a) Explain anything that you learned.
   b) Write the equation in general form.



6) a) Graph $y = 2x^2 + 4$.
   b) From the graph state the roots of the corresponding equation $2x^2 + 4 = 0$ for $x \in R$.
   c) Find any imaginary roots.

7) a) Explain, using graphs and word, how you know that a polynomial function has real roots and/or complex roots.
   b) Create a polynomial function that has 3 real roots and 2 imaginary roots.
   c) Determine a polynomial function whose roots are $\pm 2i$, and 5.

8) "*X*-intercepts are always roots of the equation, but roots are not always *x*-intercepts." Discuss.

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C20 factor polynomial expressions

Elaboration—Instructional Strategies/Suggestions

C20 As students work with higher degree polynomials there will be times when they will need to factor these expressions, especially when trying to find roots.

Students have already learned to factor using common factoring, trinomial factoring, and difference of two squares factoring.

Two types of factoring should be explored at this time: factoring by grouping and factoring the sum or difference of two cubes.

Grouping:

Sometimes grouping a polynomial expression will allow it to be factored: For example, while exploring the graph of the function $f(x) = x^3 - 2x^2 + 4x - 8$, students would be able to see that one factor was $(x - 2)$. Factoring $x^3 - 2x^2 + 4x - 8$ by grouping would enable students to see this factor and any others.

This is an example of grouping two and two. Given $x^2 + 4x + 4 - 16y^2$, students should factor by grouping three and one, then using difference of squares. Given $x^2 - 2x + 1 - y^2 + 4y - 4$, students should factor by grouping three and three.

Sum and difference of cubes:

When two cubes are being added or subtracted the expressions representing these can be factored:

For example, factor $a^3 + 8$

Knowing that a is the cube root of a^3 and 2 is the cube root of 8, then $a^3 + 8$ can be expressed as $(a)^3 + 2^3$. Call a the 1st term and 2 the 2nd. $(a + 2)$ is the first factor. The second factor can be obtained by long division or synthetic division to be $a^2 - 2a + 4$. However, through repetition and exploring patterns (see the activity on the next page) students should discover the second factor, and state its form - "square first term (a) to be first term in second factor, square second term (2) to be the third term in the second factor, and then subtract product of the fist and second term ($2a$) to be the middle term of the second factor.

Another example, factor $x^3 - 125 = (x)^3 - (5)^3$

First factor is $(x - 5)$. The second factor is made up of three positive terms; the first being the square of the x; the second, add the product of the 5 and the x; the third, the square of the $(+5) \rightarrow (x^2 + 5x + 25)$.

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Activity (**C20**)

1) Ask students to follow the directions and record their answers carefully:
   a) Given the first factor and a difference of cubes expression, use division to find the second factor:
      i) $a^3 - 125$; $(a - 5)$ ii) $x^3 - 8$; $(x$ æ $2)$ iii) $27 - y^3$; $(3 - y)$
   b) Ask students to examine their two factors for each of the above and make a conjecture about how the second factor can be obtained from the first.
   c) Change all the negative signs to positive in the three parts in (a) above, then follow the directions for (a) and (b) above.
   d) Use your conjectures to state the two factors for
      i) $b^3 + 64$ ii) $c^3 - 1$ iii) $8x^3 + 1$ iv) $27y^3 - 8$
   e) In the above answers explain how you determined the first factor. Describe by writing words how you obtained the second factor from the first.
   f) Using the above pattern, if $(x^2 + 6)$ is the first factor, what is the second? Using your two factors what must be the expression?

Paper and Pencil (**C20**)

1) Factor each of the following:
   a) $ax - 2ay - 6by + 3bx$ k) $x^6 - y^3$
   b) $x^3 - 2x^2 + 4x - 8$ l) $8a^3 + 27b^3$
   c) $3y(2x + 5) - 4x(2x + 5)$ m) $(a - 4)^2 - (b + 3)^2$
   $5x^{\frac{2}{3}} = 10$ $\frac{1}{2}y^3 = 5y^2 - 3y + 30$
   d) $y^2 - 2y - 8$ n) $a^2 + 2ab + b^2 + ac + bc$
   e) $2x^2 + 8xy + 16y^2$ o) $x^2 - y^2 - z^2 + 2yz$
   f) $3y^2 - y - 10$ p) $(m + n)^2 + 2(m + n) + 1$
   g) $36 - 4a^2$ q) $\frac{1}{a^2} - \frac{3}{a} = 10$
   h) $x^4 - 8x^2 + 16$ r) $(x - y)^2 + 6(x - y) - 16$
   i) $32x^2 - 18y^2$ s) $b^2 - (2c + d)^2$
   j) $m^3n^3 - 27$ t) $m^2 - n^2 - 2np - p^2$

3) Solve the following equations.
   a) $99a^2 = 27a$ b) $49b^2 - 7b^3 = 0$
   c) $2x^3 - 5x^2 + 6x = 15$ d)

Journal

4) Explain how you can tell by inspection whether this polynomial can be

   factored:

Suggested Resources

Developing a Function Toolkit

Outcomes

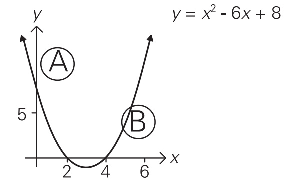
SCO: In this ourse, students will be expected to

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C11 analyse and solve polynomial, rational, irrational, and absolute value inequalities

Elaboration—Instructional Strategies/Suggestions

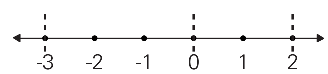
C10/C11 Students should explore several methods used to solve polynomial inequalities. It should not be the intent of any final assessment to test the use of any of these specifically. Let the student select the method he/she is most comfortable with.



Method 1: In the graph to the right, the equation $y = x^2 - 6x + 8$ divides the *x-y* plane into two regions. The inequality $y > x^2 - 6x + 8$ describes region *A* and the inequality $y < x^2 - 6x + 8$ describes region *B*. The inequality statement $x^2 - 6x + 8 > 0$ indicates the *x*-values that cause the points $y = x^2 - 6x + 8$ to be positive or to lie above the *x*-axis. Without sketching a graph, students would note that the graph would open upwards, and that the *x*-values left of the root 2, and right of the root 4 would result in *y*-values that are positive. This can be written in **interval notation** where the round parentheses, indicating the intervals, suggest non-inclusion of the end-values, and the indicates the union of the two sets of points.

Method 2: Have students use a number-line approach to solve , or . Since a polynomial function can change from a positive value to a negative value, or vice-versa, only at its zeros, then solving an inequality can be simplified by first calculating the roots of the corresponding equation, then determining the sign of some value for the function. For example, they would find the zeros of the equation by factoring:

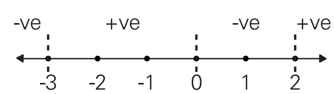
$x(x + 3)(x - 2) = 0$, thus



$x = 0, -3$ or 2

Now, have students draw a number line, marking the roots on it. Test a convenient value in any interval for its sign. For example, test $x = 1$.

$(1)^3 + (1)^2 - 6(1) = 1 + 1 - 6 = -4$



The sign indicates that the interval containing $x = 1$ has negative *y*-values. The fact that the intervals were formed by non-repeated roots indicates that the *y*-values change signs as they move from one interval to another and leads to the solution . The use of square brackets in this notation indicates the inclusion of –3, 0, and 2 in the solution set. Note that the end parenthesis indicates that infinity cannot be inclusive. When the roots are equal (repeate), the signs do not alternate across regions.

Method 3: An *optional* method may be used with quadratics such as $2x^2 + 3x + 10 < 12$. In this case, since this inequality simplifies to $2x^2 + 3x - 2 < 0$, values for x substituted in the two linear factors $(2x - 1)$ and $(x + 2)$ must have a negative product. This means that one factor must be negative **and** the other positive:

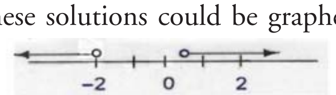
$2x - 1 > 0$ **and** $x + 2 < 0$ or $2x - 1 < 0$ **and** $x + 2 > 0$

Because only one of the above possibilities leads to a solution, students will conclude that the second leads to the solution . If, however, students were asked to solve $2x^2 + 3x - 2 > 0$, then both factors would be positive or, both negative:

$2x - 1 > 0$ and $x + 2 > 0$ or $2x - 1 < 0$ and $x + 2 < 0$

From the first pair, and $x > -2$. Since both must be true, then . Similarly, from the second pair $x < -2$ and , so, $x < -2$ and the final conclusion:

$-2 > x > \frac{1}{2}, x \in R$, or $x \in (-\infty, -2) \cup \left(\frac{1}{2}, \infty\right)$. These solutions could be graphed on a number line.



Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

Performance (C10/C11)

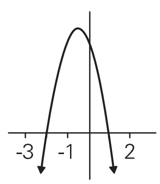
1) Solve, using any method you choose
   a) $(x - 2)(x + 3) < 0$
   b)
   c) $t^2 - t - 12 > 0$
   d) $t^2 - 6t + 13 > 0$
   e)
   f) $t^4 - < 1$
   g)
   h)
   i)
   j)

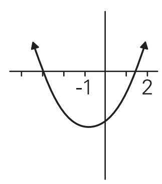
2) A rectangular building is to be built on a 30 m by 50 m rectangular lot in such a way that there is a path x metres wide surrounding the building. The building can occupy up to 60% of the lot's area. What is the range of possible integral values for the width of the path?

3) a) Given $f(m) = 3m^3 + m^2$ and $g(m) = 9m + 3$, for what values of m is $f(m) < g(m)$?
   b) Given $h(x) = x^4$ and $f(x) = 8x^2 - 7$, for what values of x is $h(x) > f(x)$?
   c) Given $f(x) = x^4 - 15$ and $g(x) = 2x^2$, determine the values of x for which
   ?

4) From the following graphs, state the x-values that result in the functions being positive:

[illegible] $x^2 - 10$





Journal (C10/C11)

5) Find all of the possible values of p for which $x^2 - px + p = 0$.
6) Which methods for solving polynomial inequalities do you prefer? Why? Explain how it works.

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C3 model real-world phenomena using polynomial functions and rational functions

C15 determine the equations of polynomial and rational functions

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration—Instructional Strategies/Suggestions

C3/C15/C10 Given certain information, students should be able to determine a polynomial function for a problem and solve the related equation. For example:

> At 130 m below sea level, Derth Valley is the lowest land in Eastern Canada. An eagle flies through Derth Valley looking for prey. During this time, its height above sea level, *h*, in metres, is recorded as ordered pairs every 15 seconds. The following are some of the recorded data: (0, 0), (2, 96), (6, 112), (12, –64), (20, 20). Assuming that a cubic equation best describes this data, determine the times at which the eagle is at sea level.

Students should use the model for a cubic function:

> $h(t) = at^3 + bt^2 + ct + d$ where *h* is the height of the eagle in metres above sealeve and *t* is the time intervals (15 seconds in each interval). Students should substitute each set of coordinates into this equation to form a system of 4 equations:

$(t,\ h)$:	$h(t) = at^3 + bt^2 + ct + d$	
(0, 0):	$0 = 0(a) + 0(b) + 0(c) + d$	①
(2,96):	$96 = 8a + 4b + 2c + d$	
(6, 112):	$112 = 216a + 36b + 6c + d$	
(12, –64):	$-64 = 1728a + 144b + 12c + d$	

From equation ①, they will see that $d = 0$. The three other equations form a system that could be solved using matrices and technology, giving the following results , and thus the equation $0.3\&t^3 - 10t^2 + 66.6\&t$. (Note: Using cubic regression leads to the equation $h\ (t) = 0.34t^3 - 10.14t^2$ +. $67.214t$. Students might want to discuss the difference and why it has occurred. Perhaps, not using the coordinate (12, –64) in the system makes the slight difference.) Solving this equation for $h = 0$ (using technology) leads to *t*-values of 0, and approximately 10 and 20, that represent in which 15 second intervals that the eagle was at sea level. The times at which the eagle could have been at sea level are time 0 seconds, 2.5 minutes, and 5 minutes. Other questions that students should be encouraged to ask, and to answer:

> How much time does it take the eagle to reach a maximum height, and what is that maximum height?
> How much time does it take the eagle to reach a minimum height, and what is that minimum height?
> After reaching the minimum level, how long does it take the eagle to reach a height of 10 metres?

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance (C3C15/C10)

1) An 18-wheeler truck is stopped on an horizontal bridge exerting a downward force. As a result, the bridge sags according to the equation $y = x^4 + 35x^3 - 300x^2$, where x is the distance in metres from one end of the bridge, and y is the distance, in hundredths of a millimetre, from the horizontal to the bridge.
   a) What is an appropriate domain for x?
   b) Find the zeros of this function and explain their meaning.
   c) Find the maximum and minimum values for this function. Explain their meaning.

2) Aeroplanes have fuel ratings. The lower the rating, the less fuel it burns. The less fuel it burns, the more efficiently the plane operates. An aircraft designer is interested in knowing the efficiency E, of a passenger plane at various speeds. Examine the following data:
   a) Determine a relationship that describes efficiency in terms of speed.
   b) Predict the efficiency rating of the plane at 600 km/h; 125 km/h
   c) Would the efficiency rating ever equal zero while the plane is moving? Explain.

Speed in 100 km/h	Effiencieny rating (E)
1	53.5
1.5	51.0
2	50.0
2.5	50.6
3	52.0
3.5	53.0
4	54.0
4.5	53.0

3) Let $S(n)$ be the sum of the cubes of the integers from 0 through n.
   a) Find $S(0)$, $S(1)$, $S(2)$, $S(3)$, and $S(4)$. What do you know about all these numbers?
   b) $S(n)$ is a quartic function of n. Find the equation expressing $S(n)$ in terms of n using ordered pairs from part (a). Express the function with integral coefficients.
   c) How can you conclude that $S(n)$ is always a perfect square?
   d) Prove with mathematical induction that this formula works for all integers $n \geq 0$.

Journal (C3/C15/C10)

4) Explore the intersection of the graphs of cubic and quartic functions to determine the maximum number of regions in a plane that can be created. Explain why you think you have found the maximum number of regions.
5) If r is a root of $x^3 - 39x + 70 = 0$, explain why $(r + 4)$ is a root of $x^3 - 12x^2 + 9x + 162 = 0$.
6) Cubic function graphs always have a point of inflection where the graph stops curving in one direction and begins to curve in the other. A mathematician once said that given the general cubic $y = ax^3 + bx^2 + cx + d$, the x-coordinate for the point of inflection can be found using $x = \frac{-b}{3a}$. Do you think this is always true? Explain.

Suggested Resources

Murdock, Jerald et al. *Advanced Algebra Through Data Exploration*. Berkley, CA: Key Curriculum Press, 1998.

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C14 analyse relations, functions, and their graphs

C5 use tables and graphs as tools to interpret expressions

B6 determine and apply the derivative of a function

Elaboration—Instructional Strategies/Suggestions

C14/C5/B6 The slope of the tangent line is useful, both in determining minimum and maximum points and in accurately sketching curves. To sketch polynomial curves students need to find the zeros, the local maximum and minimum, and the y-intercept.

Students are asked to sketch $y = x^3 - 9x$ and label the local minimum and maximum points. Students have discovered that the slope of the tangent line was zero at the local maximum and minimum points on a curve. To find the x-value for these points, students should set the first derivative equal to zero and solve for the x-value(s):

To get the first derivative using first principles:

$$f(x) = x^3 - 9x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 9x - 9h - \left(x^3 - 9x\right)}{h}$$

$$f'(x) = \lim_{h \to 0} \left(3x^2 + 3xh + h^2 - 9\right)$$

$$f'(x) = 3x^2 - 9$$

To get the x-values for the local maximum and minimum points, set the derivative equal to zero, since the slope of the tangent line is zero at local maximum and minimum points.

$$3x^2 - 9 = 0$$

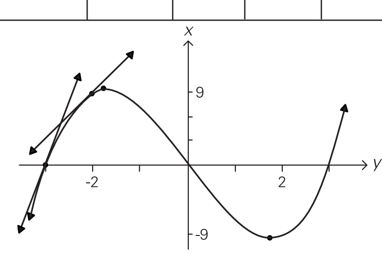
$$3x^2 = 9$$

$$x = \pm\sqrt{3}$$

Substituting these x-values into $y = x^3 - 9x$ would give the y-values of these points, and thus, the coordinates of the local maximum and minimum points $\left(-\sqrt{3}, 6\sqrt{3}\right)$ and $\left(\sqrt{3}, -6\sqrt{3}\right)$. To sketch the curve, students can plot the zeros, local maximum and minimum, and the y-intercept, in this case (0, 0).

Alternatively students could sketch the graph of $y = x^3 - 9x$ by creating a table with x- and y- values, and corresponding slope of the tangent line at various points. Plotting the points (x, y) and sketching the tangent lines would give them an approximation of the curve that joins the points.

x	–3	–2	–1	0	1	2	3
$y = x^3 - 9x$	0	10	8	0	–8	–10	0
Slope of the tangent to $y = x^3 - 9x$	18	3	–6	–9	–6	3	18



Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance **(C14/C5/B6)**

1) Sketch each function and clearly label all maximum and minimum points.
   a) $f(x) = x^3 + x^2$
   b) $m(x) = x^3 + 6x$

2) Show that the cubic curve $y = 2ax^3 + 3bx^2 + 6cx$, where $a, b, c \in R, a > 0$, has $6(ax^2 + bx + c)$ as the slope of the tangent to the curve at any point on the curve.

3) a) Graph each function. Label all maximum and minimum points, and the zeros, where they exist.
   i) $y = x^3 - 6x + 9x$

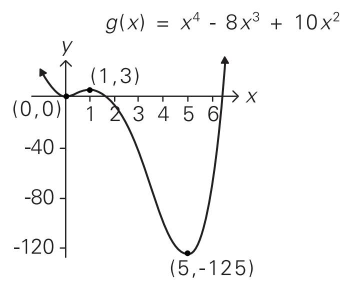
   ii)

   iii) $y = x^4 - 2x^2 - 8$

   b) For what values of x do the functions in (a) increase? decrease?

4) When an object is thrown into the air, its vertical distance, d, in metres, above the point where it is thrown is given by , where r is in metres per second (upward velocity) and t is the time in seconds since it was thrown. To dunk a basketball, Jeff must jump 1.25 m off the floor. He can jump with an initial velocity of 5.4 m/s. Can Jeff dunk the ball? Justify your answer.

d B $rt - 5t^2$
$y = \frac{1}{3}x^4 - 9x - 1$



5) From the graph it looks as if the local maximum value is located at (1, 3) and the local minimum value at (5, –125). Show how to verify these co-ordinates as local minimum and maximum in two different ways.

6) Given $g(x) = x^4 - x^2$
   a) Find the local maximum and minimum points, and sketch the graph of g.
   b) i) State the range of g .
      ii) For what values of x does g increase?
      iii) For what values of x is g positive ?
   c) Sketch the slope function of g.
   d) In what ways are parts (c) and (b) related?

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C7 demonstrate an understanding for slope functions and their connection to differentiation

C5 use tables and graphs as tools to interpret expressions

B7 derive and apply the power rule

Elaboration—Instructional Strategies/Suggestions

C7/C5/B7 Students should investigate patterns and make conjectures with respect to finding slope functions of given polynomial functions. They might begin by looking at the patterns with simple powers of x (see below). The purpose for this activity is for students to discover the power rule, first for single term polynomial functions, then for multiple terms. Once students have discovered this rule they should be able to explain what it is, what it means, and be able to apply it when they need to find the derivative of (or differentiate) a polynomial function.

❐ Have students enter each of the given functions $f(x) = x$, $f(x) = x^2$, $f(x) = x^3$, $f(x) = x^4$, one at a time. To produce the slope function graph have students enter $\frac{\left(f(x+h)-f(x)\right)}{h}$ where $h = 0.001$ at the Y_2 = prompt. (Students have used this previously.) Y_1 will graph the function, Y_2 will graph the slope function. Students should determine an equation for the slope function from the graph (or by using points from the TABLE), then enter this equation at the Y_3 prompt to display the graph of the calculated function. If this function is correct the graph of it should coincide with the slope function already displayed. The results of the above work should result in a table like the one on the left. Students should continue until they see the pattern, then express the slope function for x^n as the last entry. Eventually they should generalize that the slope function for $f(x) = x^n$ would be $f'(x) = nx^{n-1}$.

$f(x)$	slope function
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
.	
.	
.	
x^n	

Students should continue this activity with functions that have 2 or more terms. For example, graph each of the following:

$f(x) = x^2 + 3x$; $2x^3 - 5x^2$; $2x^4 - 4x^3 + 5x^2 - 2x + 3$; and so on. Each resulting function is obtained by knowing that " ... the derivatives of the sums are the sums of the derivatives."

C7 To develop their understanding of the differentiation process students should investigate

- the derivative of a constant and its meaning (the rate of change is zero since the relationship is constant)
- the effect on the derivative of multiplying a polynomial function by a constant

$$\left(2x^2, 3x^3, \frac{1}{2}x^2, \text{K}\right)$$

B7 From the above activity students should understand the power rule: given $f(x) = x^n$, then the derivative would be , also given $g(x) = ax^n + bx^{n-1}$, the derivative would be $g'(x) = anx^{n-1} + b(n-1)x^{n-2}$.

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

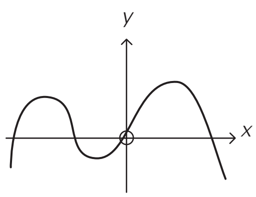
Paper and Pencil (B7)

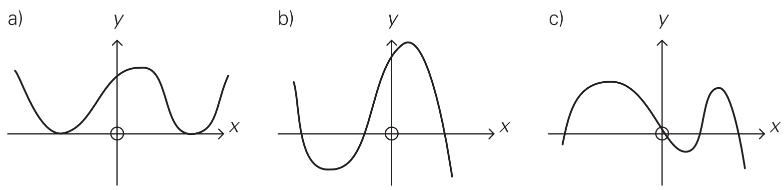
1) Find the derivative of $f'(x)$.

a) $f(x) = 5(x^2 + 1)$ b) $f(x) = x^4 - 3x^2 + x$
c) $f(x) = 4x^3 + x^2$ d) $f(x) = x^2 - 6x^2$
e) $f(x) = (x - 4)(x + 4)$ f) $f(x) = (x + x)(x^4 + x - 5)$

Objective (C7/C5)

2) This is the graph of $f(x)$. Which graph below the given function is the graph of the derivative of $f(x)$? Explain.





d) Neither (a), (b), nor (c)

If (d) explain why

Journal (B7)

3) Show why the derivative of a constant function is the zero function.

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C7 demonstrate an understanding for slope functions and their connection to differentiation

Elaboration—Instructional Strategies/Suggestions

C7 The term differentiation comes from the fact that students are finding the difference in the *y*-values when they are finding the slope of the tangent line. The resulting expression that represents the slope of the tangent line is an expression for the instantaneous rate of change at any point on the curve and is called the derivative of the function because it is found through the process of differentiation. Students should be able to produce calculations like those that follow and/or interpret these calculations using "rate of change" when they see them:

if $f(x) = x^3 - 3x^2 - 9x + 6$, then

so, $f'(1) = 3 - 6 - 9 = -12$

Students would interpret this to mean that the slope of the tangent line at $x = 1$ of the function $f(x) = x^3 - 3x^2 - 9x + 6$ is –12. The slope of the tangent line represents the rate at which the phenomenon is changing at the point $x = 1$ on the curve. If $f(x)$ represents the relationship between volume and time, then the volume is decreasing at the rate of 12 units^3 per unit time.

Students can use technology (as learned in a previous course) to see the tangent line, and to find the slope of the tangent line at any given point. Follow the procedure:

- enter the function on the function screen (Y=), and press GRAPH
- select the DRAW mode by pressing 2nd DRAW
- select 5:Tangent(
- the graph screen reappears with the trace cursor waiting for the *x*-value to be inputted
- tell the cursor to draw the tangent line at $x = 2$ by typing 2, then hit ENTER

At the bottom of the screen the student should see the equation of the tangent line at that point, and be able to determine the slope of the tangent by reading the coefficient of the *x*-term.

If the student wants to use the calculator to only evaluate the derivative of the function for the slope of the tangent line at any given point, then the student instead could select the CALC menu and 6: $\frac{dy}{dx}$. This returns the student to the graph where they type 2. At the bottom of the screen they would see the instantaneous rate of change of the function at that point on the function.

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Paper/Pencil (B7/C7)

1) Find the derivatives of the following functions:

a)

b)

c)

d) $j(x) = x^4 - x^3 + x^2 + x - 12$

e)

f)

g)

h) $w(r) = \frac{1}{5}r^5 + \frac{1}{3}r^3$

i)

2) Find the slope of the tangent line to the graph of $y = x^3 - 5x^2 + 4$ at the point where $x = 2$.

3) Find the instantaneous rate of change of the graph of $y = t^4 + 3t^3 - 4t^2 + 5t - 6$ at the point where $t = -1$.

4) For each of the functions that follow, find all the coordinates of the points where the slope of the graph is zero; is one.

a) $y = x^2 - 4$

b) $z = r^2 + 2r + 1$

c) $x = t^3 - 12t$

d) [illegible]

5) Given for functions *f* and *g*.

a) Find two distinct functions that satisfy this equality.

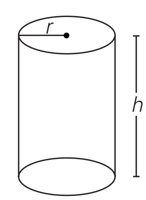
b) In general, what can you say about the shapes of *f* and *g* if ?

6) For each function below find the intervals of increase and decrease. Use the derivative test to find the local maxima and minima. Sketch a rough graph. Check each answer with a graphing calculator.

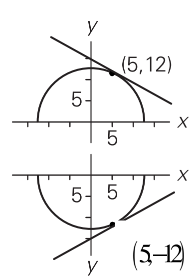
a) $f(x) = 5x + 10 - x^2$

b) $f(x) = 2x^3 - 5x + 1$

7) The surface area of a closed right circular cylinder of radius *r* and height *h* is given by the formula . Find the rate of change of *A* with respect to *r* for a fixed height *h*.



8) There are two points on the graph of $x^2 + y^2 = 169$ where $x = 5$, one above the *x*-axis (where) and one below the *x*-axis (where $y = -(169 - x^2)^{\frac{1}{2}}$). Find the equation of each of the tangents at $x = 5$.



Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C3 model real-world phenomena using polynomial functions and rational functions

B7 derive and apply the power rule

C14 analyse relations, functions, and their graphs

C12 demonstrate an understanding for the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of the tangent line and their applications

Elaboration—Instructional Strategies/Suggestions

C3/B7/C14/C12 Now that students have developed more efficient ways for differentiating any polynomial, they should recall why derivatives are required in the first place. In previous study students found that the rate of change of a function at any given point is given by the slope of its graph at that point. The slope function is another name for the derivative. To find the rate of change of a function, students can find its derivative.

Problems dealing with motion are an important application of rates of change. The velocity of a moving object at any time is given by the rate of change (the derivative) of its displacement function.

The following examples will provide practice and give an idea of some of the ways derivatives can be used.

For example,

> If a brick or some other heavy object falls from a height, how can its velocity (directed speed) be determined when it gets near the ground?

To answer this, students first need a mathematical model for the situation. Students should be comfortable calling “rate of change” velocity. By experiment it has been found (or it could be found) that a falling object will cover a distance of approximately 4.9 t^2 metres in t seconds. If d metres is the displacement of the object above the ground level after t seconds (with the positive direction chosen to be upwards) and the object’s displacement at the start (the height from which it falls) is called d_o, then,

$d = d(t) = d_o - 4.9t^2$.

A brick is dropped from high up on a building under construction. How fast is the brick falling after 3 seconds.

The rate of change (velocity) is given by the derivative, $d'(t)$. Since

Substitute $t = 3$ to find that the velocity after 3 seconds is –29.4 m/s. Ask students to interpret the fact that the speed is 29.4 m/s. (Ans-

Speed is absolute value of velocity.)

... continued

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance **(C3/B7/C14/C12)**

1) If an object thrown vertically upwards from ground level with velocity u m/s, reaches a height of s m after t seconds, then $s = ut - 4.9t^2$. If the projectile is thrown from a height of s_0 metres instead of from ground level, the model is changed to $s = s_0 + ut - 4.9t^2$.
   a) A ball is thrown upwards from ground level with a velocity of 20 m/s. Use the model given the motion of projectiles to find expression for its velocity t seconds later.
   b) When is the velocity positive? How do you interpret this?
   c) When is the velocity negative? How do you interpret this?
   d) When is the velocity equal to zero? What is the ball doing at the moment when the velocity is zero?
   e) When does the ball reach its greatest height, and what is the greatest height?
   f) When does the ball hit the ground again, and what is its velocity at the instant just before it does so?

2) The distance a toy car travels is represented in metres by $s(t) = 0.1t$, where t is the time in seconds since the car began its trip.
   a) How fast is the car travelling?
   b) What is the slope of the tangent to the graph of the distance function?
   c) What can you conclude, in general, about the velocity of an object travelling in a straight line, if the distance function is also a straight line?

$s(t) = \frac{1}{t^2}, t = \frac{1}{2}$

Extension **(C3/B7/C14/C12)**

3) An object moves in a straight line according to the distance functions given. Find the velocity and acceleration of the object at the time given. Note that acceleration is the rate of change of the velocity with respect to time, so $a(t) = v't$.
   a) $s(t) = -4.9t^2 + 3t - 1;\ t = 2$

   b)

Suggested Resources

Developing a Function Toolkit

Outcomes

SCO: In this ourse, students will be expected to

C3 model real-world phenomena polynomial functions and rational functions

B7 derive and apply the power rule

C14 analyse relations, functions, and their graphs

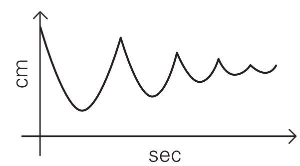
C12 demonstrate an understanding for the conceptual foundations of limit, the area under a curve, the rate of change, and the slope of the tangent line, and their applications

Elaboration—Instructional Strategies/Suggestions

... continued

C3/B7/C14/C12 Knowing that the instantaneous rate of change is the first derivative, students can apply the derivative in problems concerning motion, population explosion, economics, finance, and many other related topics.

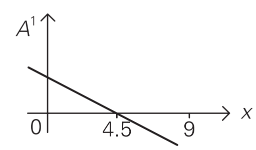
❐ The graph shows the displacement from a tabletop for a mass bouncing up and down on a spring. Time is recorded in seconds.



a) Create a suitable scale and values for each axis.
b) Sketch the graph of velocity.
c) Determine locations where the instantaneous rate of change is great, and explain how you know.
d) For what values for time is the displacement a relative maximum or minimum. Explain how you can tell.

One of the many uses of differentiation is to find the "best" way of doing something—for example, the way that is the quickest, highest or lowest, cheapest. Differentiation also enables students to determine which uses the least amount of materials, or makes the biggest profit. This usually means finding the greatest or least value of some function. These are often referred to as "max/min problems." See the activity that follows:

❐ Greta has decided to fence off a rectangular section of her garden to grow vegetables, using 18 metres of wire fence she has been saving. How can she do this so as to enclose the greatest amount of ground? Sadie asks, "Did Greta find the formula $A(x) = x(9 - x) = 9x - x^2$ for the area of the garden? Did she also notice that the domain is the set of x-values such that $0 < x < 9$?"



Pat comments that when she graphed this function that it was a parabola with a single highest point, and at this point the . When $A'(x) = 0$, then $x = 4.5$. The graph of the derivative is positive when $x < 4.5$ and negative when $x > 4.5$. John says that this means that A is increasing up to the point where $x = 4.5$ and from there on it is decreasing. Thus $x = 4.5$ produces the largest value of A. Lynn concludes that Greta will get the largest rectangular area for her garden if she makes a square garden with a side measure of 4.5 m. Her garden will then have an area of $A(4.5)\, \mathrm{B}\, 20.3\ \mathrm{m}^2$. Have students comment on the above thoughts.

Developing a Function Toolkit

Worthwhile Tasks for Instruction and/or Assessment

Performance (C3/B7/C14/C12)

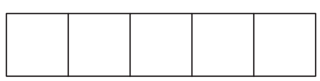
1) A ball is thrown up in the air. Its height (s metres) above the ground level is a function of the time (t seconds) since it was thrown, given approximately by the formula $s = 30t - 5t^2 + 1$.
   a) Find the derivative of this function.
   b) For what value of t is the derivative s' equal to zero? For what values of t is s increasing? For what values is s decreasing?
   c) What is the greatest height the ball reaches?
   d) Check your answer by graphing $s(t)$.

2) People planning a children's playground want to save on the cost of fencing it by locating the playground against a wall of a building so that one side will not need to be fenced. If the playground has to be a rectangle 400.0 m^2 in area, what should be its dimensions if they want to use the least amount of fencing?

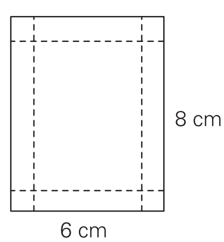
3) Some airlines have restrictions on the size of items of luggage that passengers are allowed to take with them. One has a rule that the sum of the length, width and height of any piece of baggage must be less than 158.0 cm. A passenger wants to get a case of the maximum allowable volume. If the length and width are to be equal, what should be the dimensions? If the length is to be twice the width, what should be the dimensions? Calculate the volume.

$a \times b$

4) Five identical kennels for five small dogs are to be constructed as shown. If 30 m of fencing is available, what dimensions of the pens maximize the total area?



5) Four squares of the same size are cut from each corner of an 8.0 cm × 6.0 cm piece of cardboard.
   a) How do you cut the squares to maximize the volume of the resulting box?
   b) Generalize for a rectangle of dimensions .



Suggested Resources

Unit 3
Developing a Function Toolkit
Part II
(25-30 hours)

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C19 investigate and interpret combinations and compositions of functions

C5 use tables and graphs as tools to interpret expressions

C3 model real-world phenomena using polynomial functions and rational functions

Elaboration – Instructional Strategies/Suggestions

C19 Rational functions are closely related to polynomial functions, in fact they

are the quotient of two polynomials , for $q(x) \neq 0$. The denominator

polynomial must be of degree 1 or more.

C19/C5 Given $h(x) = (x^2 - 2x - 8)$, and the composition $f(x) = g(h(x))$

results in the rational function .

Certain rational functions have special characteristics that are important in applications. Although the domain of $h(x)$ contains all real numbers, because it is the denominator of $f(x)$ its zeros must be eliminated from the domain of $f(x)$. So, the domain consists of all real numbers except –2 and 4. Visualize the parabola h by seeing the transformational form of h: $h + 9 = (x - 1)^2$, with vertex (1, –9). Plotting the reciprocals of the y-coordinates from this graph will produce the graph for $f(x)$.

C3 A rational function can describe a relationship in which the y-values decrease but remain positive as the x-values increase infinitely. This is called indirect variation, e.g., the variables are said to vary indirectly or inversely. This

relationship can be described by a rational function of the form .

For example, in the world of food packageing, a company wants to minimize the materials used to construct a can that holds 1 litre of their product. Re-expressing the surface area formula (which has two variables) as a function of the radius gives

.

Students could use the "3:minimum" feature found in the CALC menu and the graph to locate a minimum value between $r = 5$ and 6.

Vol $= \pi r^2 h$

$1000 = \pi r^2 h$, so

$\frac{1000}{r} = \pi r h$

SA $= 2\pi r^2 h$

substituting ...

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Activity (C3/C5)

1) While travelling in a car at night headlights of an oncoming vehicle vary in intensity. As the vehicle gets close and closer the brightness of the headlights increases rapidly. This happens because light waves tend to spread out as they move away from their source. As a result, intensity decreases quickly as the distance from the light source increases. In this activity a relationship between the distance from a light source and the intensity will be observed and recorded.
   a) Use a calculator or computer based laboratory and an appropriate program and probes to record changes in light intensity as the sensor is moved away from the light source at a constant speed.
   b) The data collected can be modeled with a power relation of the form $y = ax^b$. Find values for a and b. (Find the approximate light intensity when $x = 1$). Record what a equals.
   c) Move the cursor to any other data point and record the x- and y-values (to 4 decimal points). Use the values for a, and x, and y in $y = ax^b$. Now, the only variable is b.
   d) Rewrite the equation equal to zero, and solve for b. (Use the "solve" feature on the TI-83). Record the value for b to the nearest hundredth.
   e) Substitute the values for a and b into the equation $y = ax^b$ and check the graph of this equation against the curve from the data.
   f) Explain why this equation represents a rational function.
   g) Perform a pwrReg to compare the equation obtained there to your equation.

Performance (C3/C5)

2) The illumination from a light source varies inversely as the square of the distance from the source. The illumination is 4.5 cd (candela) at a distance of 2.10 m.
   a) What is the illumination if the distance is decreased to 1.20 m?
   b) State the function that represents this situation. Why is this function considered to be rational?
   c) What is the illumination if the distance is changed to 4.25 m?
   d) At what distance would the illumination be 0.80 cd?

Journal (C19)

3) a) Create 5 different rational functions using combinations and/or compositions.
   b) Explain how you know that the resulting functions in (a) are rational functions. Use sketches of graphs to help in your explanation.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

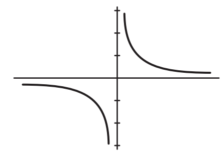
C14 analyse relations, functions, and their graphs

C16 analyse the effect of parameter changes on the graphs of functions and express the changes using transformations

C6 demonstrate an understanding for asymptotic behaviour

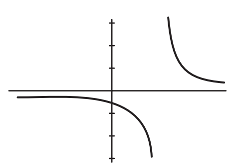
Elaboration – Instructional Strategies/Suggestions

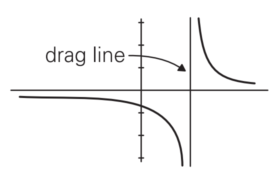
C14 The parent or model rational function $f(x)=\frac{1}{x}$ demonstrates features that are characteristic of more complicated examples (see the discussion on the previous 2-page spread). Students should graph this function and verify the discussion that follows. The graph is made up of two pieces. One part occurs where x is negative, and the other where x is positive. Note that there is no value for the function when $x = 0$.



C14/C16/C6 Have students use the TABLE features of their technology or complete a table for x-values close to zero. In the example above, a vertical line at $x = 0$ is called a vertical asymptote because the function approaches infinity as x-values approach 0 from the right, and the function approaches – as x-values approach 0 from the left. Consider x-values at the extreme ends of the axis. As x approaches extreme values at the left and right, the graph approaches the x-axis. The horizontal line $y = 0$ is called the horizontal asymptote because the function approaches it as takes on very large values. This asymptote is a global or end behaviour of the function. In general, the end behaviour of a function is the behaviour of the function for x-values that are large in absolute value.

C16 If is considered the model, then are typical examples of transformed rational functions. Students should note the symmetry (and pattern in the numbers) in the table for the model function. For example when $x = 1$, $y = 1$. When $x = 2$, $y=\frac{1}{2}$, and when $y = 2$. Students should also note the point in the first quadrant closest to the origin (1, 1) and the effect that stretches have on this point. For example, given a horizontal stretch of 3 the point moves to (3,1). Ask students where it moves for a vertical stretch of 3. This pattern of movement will be helpful when sketching or visualizing graphs. Students should know what happens to a function when x is replaced with $(x – 4)$. [The image graph has the same shape and size but is 4 units to the right of the model graph]. Both graphs below are of the function .





Graphing calculators often show the vertical asymptote, or what seems to be the vertical asymptote. However, it is not. Instead, it is called a vertical-line drag. The reason this happens is because the screen display is made up of a matrix of pixels (62 by 94). Some of the pixels are selected (darkened on the screen) with very small values of x as the x-values are calculated every few hundredths of the way as they approach the asymptote. This can be avoided by using a “friendly” screen (window settings as multiples of 4.7 and –4.7).

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

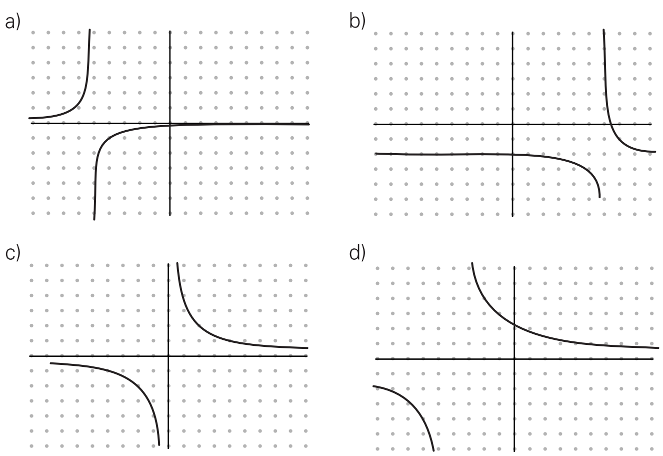
Pencil and Paper (C14/C16/C6)

1) Use as the model. Sketch a graph and write an equation for each transformation of $f(x)$:
   a) vertical translation of 2
   b) horizontal translation of 3
   c) horizontal stretch of 3, vertical translation of 1, and horizontal translation of –2.
   d) vertical stretch of one half, and a horizontal stretch of 2.

2) Write altered forms of the equation so that the graph of has the specified characteristics:
   a) A horizontal asymptote at $y = 2$ and a vertical asymptote at $x = 1$.
   b) A horizontal asymptote at $y = -4$ and a vertical asymptote at $x = 2$, and a vertical stretch of 4.

3) Write a rational equation to describe each graph. Assume each grid mark represents one unit.

$\frac{1}{2}$ $f(x) = \frac{1}{x}$ $\frac{1}{x}$



4) Students have already considered , and its transformations. They know that has both horizontal and vertical asymptotes. If $g(x) = x$, ask students to predict the shape, then graph $f(x) + g(x)$.

5) Explain why a vertical stretch of could also be called a horizontal stretch of . Is this true for a quadratic curve? Is this true for all other functions that you have studied?

6) Which transformation or transformations affect the location of the two asymptotes for the function ? Explain why this is so.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions and their graphs

C16 analyse the effect of parameter changes on the graphs of functions and express the changes using transformations

C5 use tables and graphs as tools to interpret expressions

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration – Instructional Strategies/Suggestions

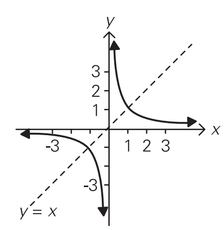
C14/C16 Students should graph and analyse as they did $f(x) = \frac{1}{x}$ and compare the two functions. (See activity 2 in the next column.) The analysis in that activity would result in statements similar to those that follow for this function:

- domain: or 2, range:
- asymptotes: $y = 3$, $x = -1$, and $x = 2$.
- zero:
- increases for $x \in (0.5, 2) \cup (2, \infty)$; decreases for $x \in (-\infty, -1) \cup (-1, 0.5)$
- positive for $x \in \left(-\infty \frac{3-\sqrt{93}}{6}, -1\right) \cup \left(2, \frac{3 \pm \sqrt{93}}{6}, \infty\right)$;
- negative for

C14/C5/C10 When sketching or analysing a graph of a function it is useful to look for the following information: domain; range; any lines or points of symmetry; zeros; y-intercept; and asymptotes, if they exist.

When students analyse and/or sketch the graph of they should record:

- Domain: ; range:
- For non-zero values for x and y, the equation is equivalent to $xy = 1$. Interchanging x and y produces an equivalent equation (the function is its own inverse). Therefore, $f(x) = f^{-1}(x)$ and the graph is symmetric with respect to the line $y = x$.
- There is no zero and no y-intercept.
- The graph of $f(x)$ approaches infinitely closely both the positive and negative x-axis and the positive and negative y-axis. Thus, the x- and y-axes are asymptotes of the graph. Symmetry with respect to the line $y = x$ can be used when sketching to obtain additional points in the first quadrant.



- Then, symmetry can be used with respect to the origin to locate points in the third quadrant. This graph which is composed of two branches, each of which approaches asymptotes, is called a hyperbola.
- This function decreases over two intervals, and $x \in (0, \infty)$.
- It is positive for {x: $0 < x <$ }, or , and negative for {x: $-$ $< x < 0$}, or .

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Activity (C16/C12/C18/C19)

1) Use technology to draw the graph for $y=\frac{1}{x^2}$ and sketch it. Examine the table of values. Write an analysis of $f(x)=\frac{1}{x^2}$ that includes:
   a) the domain and range
   b) the behaviour of y as approaches
   c) the equations of any asymptotes
   d) any zeros
   e) the interval for which f increases, decreases

2) Consider the function .
   a) Rewrite it in transformational form and ask students to describe in words the transformation of . State the mapping rule.
   b) What does each transformation do to the graph of $y=\frac{1}{x^2}$? Based on this, sketch the graph.
   c) Using the graph, evaluate

$x \to \pm\infty$ $x \to a$ $y = 2 - \frac{1}{x^2}$ $\frac{1}{x^2 2x+1}$

   i) $\lim_{x\to -1} f(x)$ ii) $\lim_{x\to 0} f(x)$ iii) $\lim_{x\to \infty} f(x)$

Pencil and Paper/Performance (C16/C12/C18/C19)

3) For each function
   a) Describe it as a transformation of $y=\frac{1}{x^2}$ and state the mapping rule.
   b) Sketch the graph.
   c) State the domain; range; the equations of any asymptotes; the intervals increase, and decrease.
   d) Calculate the zeros. State the intervals for which the function is positive, and those that are negative.

   i) $y=\frac{1}{x^2}$ ii) $y=\frac{1}{(x-3)^2}$ iii) $y=\frac{2}{(x-1)^2}$ iv) $y=\frac{3x^2-1}{x^2}$

   e) Find the limit, if any, for each of the above as $x \to 0$; as ; as , where $x = a$ is the vertical asymptote.

4) The amount of heat received from a radiator varies inversely as the square of the distance from it. Declan stands 4 m from the radiator and Fergus stands 1 m from it. How much more heat does Fergus receive than Declan? Predict how much heat Tavish, who stands 6 m from the radiator, will receive.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C3 model real-world phenomena using polynomial functions and rational functions

C15 determine the equations of polynomial and rational functions

C5 use tables and graphs as tools to interpret expressions

Elaboration – Instructional Strategies/Suggestions

C3/C15 In a lab experiment students find that for a given mass of oxygen, its volume is 4 L when the pressure is 4 Mpa (megapascals). Using these values for V and P, students find that a constant C = 16 and describes this relationship.

Use appropriate variables and technology to graph this relation. For this experiment, only the portion of the graph in the first quadrant applies. Have students explain why.

The equation is an example of a rational function. Other examples are

and . Ask students to write their own definition of a rational function, stating any restrictions that must apply. (See the definition on p. 108.) Is a polynomial function a rational function? Why, or why not?

C15/C5 Students should be able to determine the equations of rational functions indirectly from tables and directly from graphs using technology. The equations will be modeled with the power equation $y = ax^b$ where $b < 0$ (see the activity on p. 109).

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Performance (C3/C15/C5)

1) The amount of carbon dioxide gas (CO_2) that can be dissolved in water decreases as the temperature increases. (For example, open a bottle of warm soda pop and the gas escapes immediately.) We say that the amount of CO_2 which can be dissolved in a liquid varies **inversely** as the temperature, T.

   The results of an experiment where CO_2 is dissolved in a quantity of water are as follows:

Temperature (°C)	Amount of CO_2 (*g*)
3	2.37
6	1.19
10	0.71
20	0.36
25	0.28

   a) Sketch a graph of the relationship.
   b) Could this experiment have resulted in continuous data?
   c) Describe the relationship in words. Explain what is happening on the graph as the temperature continues to increase, and as it gets closer to 0°.
   d) Multiply pairs of corresponding values and examine their products. Describe the pattern. Generalize the pattern using a formula.
   e) Obtain the answer to the following in at least two different ways:

      i) What amount of CO_2 would be expected at a temperature of 17°?
      ii) What temperature would you expect for the liquid if the amount of CO_2 was 3 g?

2) The slope at which the road surface around a highway curve should be banked varies inversely as the radius of the curve. A curve with radius 400 m requires a slope of 0.035. Find the slope of the bank required on a curve with radius 600m.
   a) State an equation that represents this relationship.
   b) Represent this relationship with a graph.
   c) Do you think there would ever be a road with slope of 1? Explain using your graph.

3) When the tension in a wire is kept constant, the number of vibrations per second varies inversely as its length. A wire vibrates at 520 vib/sec when it is 1.2 m long. How many vibrations per second if the wire was 0.8 m long? How long should the wire be in order to vibrate at 416 vib/sec ?

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions and their graphs

C6 demonstrate an understanding for asymptotic behaviour

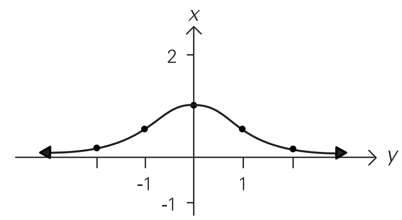
C23 explore and describe the connections among continuity, limits, and functions

C25 demonstrate an intuitive understanding of the concept of limit

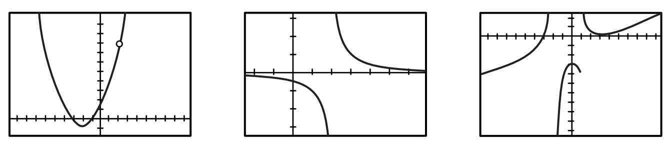
Elaboration – Instructional Strategies/Suggestions

C14/C23 Rational functions create interesting and very different kinds of graphs than those students have studied previously. Although it is difficult to find real-world applications for some of these functions, their graphs are fun to explore and may give students new insights into algebraic concepts. Students may have thought that all rational functions are discontinuous, however when the denominator has only non-real roots, the graph will be continuous, for example,

$$y = \frac{1}{x^2 + 1} .$$

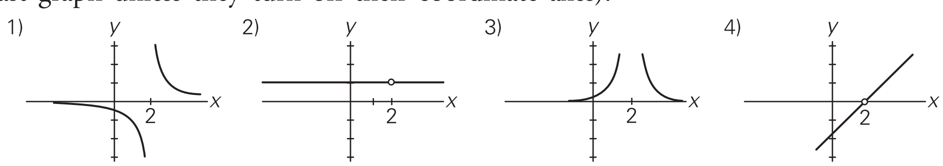


When the denominator, a polynomial function, equals zero for at least one x-value, the function will be undefined (discontinuous) at that *x*-value causing two or more regions on the plane. Sometimes it will be difficult to see the different parts of the graph because they may be separated only by a missing point (a hole in the graph). Other times two parts will look very similar—one part may look like a reflection or rotation of another part. Or, there may be multiple parts that each look totally different. These graphs display discontinuity as they approach certain values.



C6/C23/C25 Students should explore local and end (or global) behaviour of rational functions, so that expressions in the equation make it possible for the student to predict some of the features of a rational function's graph. By definition, a rational function can always be written as a quotient of two polynomials, and the polynomial in the denominator must contain the variable. When examining a rational function it is often helpful to look at the factors of the numerator and denominator.

Find a match between each graph and rational function listed below. Use a "friendly" window as you graph and trace each function. Describe the unusual occurrences at and near $x = 2$ and try to explain the equation feature that makes the graph look like it does. (Students may not actually see the hole pictured in the last graph unless they turn off their coordinate axes).



a) $y = \frac{1}{(x-2)^2}$ b) $y = \frac{1}{x-2}$ c) $y = \frac{(x-2)^2}{x-2}$ d) $y = \frac{x-2}{x-2}$

Graphs 1 and 3 show asymptotic behaviour at $x = 2$ and are discontinuous at $x = 2$. Graphs 2 and 4 are said to have point discontinuity at $x = 2$. The limit for x as it approaches 2 for graphs 2 and 4 can be found. For graph 3, there is an infinite limit and for graph 1 no limit exists.

... continued

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Activity (C14/C6/C23/C25)

1) a) Use technology to draw a graph of each function in the given order and copy it onto graph paper:

i) ii)

iii) iv)

b) Examine your graphs from above. Find the horizontal asymptote in each one. What are the equations of the asymptotes? Is this line an asymptote in the middle of the graph, or only at the extreme left and extreme right of the graph?

c) Graph $p(x)=\dfrac{2x}{x^2-3x+4}$ and . Do these functions have the same horizontal asymptotes as the graphs in question (a)? For each function, compare the degree of the numerator to that of the denominator. Make up some other functions with the same property and graph them. What do you notice about the horizontal asymptotes? Conjecture about horizontal asymptotes of a function when the degree of its numerator is less than the degree of its denominator.

[illegible]

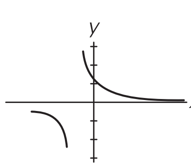
d) i) Graph . Does there appear to be a horizontal asymptote? What is its equation?

ii) Repeat your steps in (i) for .

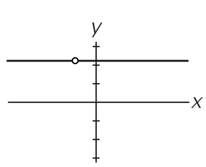
iii) For both functions, compare the degree of the numerator to that of the denominator. Make up some similar functions and graph them. Do you see a pattern? Conjecture about the horizontal asymptotes of a function when the degree of the numerator equals the degree of the denominator. (Activity continued on p. 123)

Pencil and Paper (C14/C6/C23/C25)

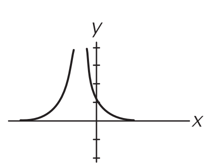
2) Name a rational function equation for each graph below and write a few sentences that explain the appearance of the graph with respect to asymptotes and holes:



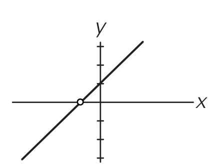
Graph A



Graph B



Graph C



Graph D

... continued

Suggested Resources

McKillop, David et al. *Pre-Calculus Mathematics Two*, Toronto: Oxford University Press, 1993.

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions, and their graphs

C6 demonstrate an understanding for asymptotic behaviour

C23 explore and describe the connections among continuity, limits, and functions

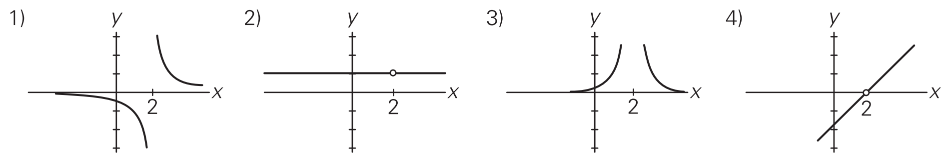
C25 demonstrate an intuitive understanding of the concept of limit

B1 describe the relationship between arithmetic operations and operations on rational algebraic expressions and equations

Elaboration – Instructional Strategies/Suggestions

... continued

C14/C6/C23/C25/B1 Another way to determine horizontal asymptotes is to examine limits of the function as *x*-values approach extreme values. At the same time students should be expected to make connections between asymptotes, holes, limits and discontinuities. For example, looking at the four functions given on the previous two-page spread:



$y = \frac{1}{x-2}$ $y = \frac{x-2}{x-2}$ $y = \frac{1}{(x-2)^2}$ $y = \frac{(x-2)^2}{x-2}$

To determine horizontal asymptotes students should find the limits of each of the functions as . Students might recall this procedure from the Sequences and Series unit. Their work for numbers 3) and 1) above might lool like:

3)

1)

Since both limits are zero, students should conclude that both functions have horizontal asymptotes at $y = 0$.

Students should notice that all four graphs seem to be discontinuous at $x = 2$, although the screens on their graphing calculators may not show the discontinuity unless the student is using a "friendly" screen. Finding the left- and right-hand limits of each of the functions might help students determine discontinuity. The following should be the findings:

- In graph 1, since *x*-values less than 2 result in negative values for *y*, and *x*-values greater than 2 will result in positive values, the left- and right-hand limits approach different values, . The limit does not exist.
- In graph 2, since x-values less than 2 approach 1, and greater than 2 also approach 1, the left- and right-hand limits both approach 1. The limit can be found (1).
- In graph 3, since *x*-values less than or greater than 2 both will, when squared, be positive, the left- and right-hand limits approach . This is called an infinite limit.
- In graph 4, since *x*-values less than 2 approach 0, and greater than 2 also approach 0, the left- and right-hand limits both approach 0. The limit can be found (0).

Students must understand that a function is discontinuous at a point when the function is not defined at that point.

... continued

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

... continued

Pencil and Paper (**C14/C6/C23/C25/B1**)

3) Determine, if possible, the horizontal asymptotes.

a)

b) $y = \dfrac{9x^4 - 7x^2 - 19}{3x^4 + 7x + 9}$

c) $y = \dfrac{17x^2 + 14x - 5}{x^3 + 2x - 6}$

d) $y = \dfrac{3x^4 = 2x + 4}{18x^4 + x^3 + 44}$

e)

f) $y = \dfrac{3x^3 + 4x - 9}{6x^3 + 7x^2 + 13}$

4) Which of the following have point discontinuity?

a)

b) $y = \dfrac{x^2 + 5x + 6}{x - 2}$

c) $y = \dfrac{x^2 + 4x - 5}{x - 1}$

5) Evaluate, if possible:

[illegible]

a) $\lim\limits_{n \to \infty} \left(\dfrac{5n + 1}{3n} \right)$

b)

c)

d)

e)

f)

g)

If not, use a technology-produced graph or table to show why there is no limit.

6) Determine the limits, if possible for the following:

a)

b) $\lim\limits_{x \to 4} \dfrac{x^2 - 2x - 8}{x - 4}$

c) $\lim\limits_{x \to 2} \dfrac{2x^2 - 5x - 3}{x - 3}$

d)

e) $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{x^2 + 3x}$

Journal (**C14/C6/C23/C25**)

7) Describe in a paragraph what it means when a function is discontinuous. Diagrams may be used.

8) Kevin declared that he could find all the equations of all the asymptotes of any rational function using limits. Do you think he is correct? Explain your thinking, using examples and pictures.

... continued

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions, and their graphs

C6 demonstrate an understanding for asymptotic behaviour

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

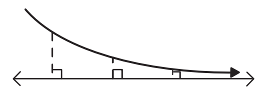
Elaboration – Instructional Strategies/Suggestions

... continued

C14/C6 Students should investigate the asymptotic behaviour of rational functions. Their investigations should include:

- vertical asymptotes
- horizontal asymptotes (see p. 117 parts (b) and (c) for a sample activity)
- and oblique asymptotes (see the top of p. 123 for a nice introductory activity to oblique asymptotes)

Students have encountered asymptotes with exponential and logarithmic functions, tangent functions in trigonometry, and with rational functions already explored. Have students explain in their own words what an asymptote is. How does their definition compare to this one:



> An asymptote is a line such that the distance to the line from a moving point on the curve approaches zero as the point moves an infinite distance from the origin.

Note: While the curve does not intersect the horizontal asymptote as it moves an infinite distance from the origin, it might intersect this asymptote in another region.

C13/C18/C19 Students should find that, for rational functions:

- Vertical asymptotes can be determined from the restrictions on the variable which in turn can be determined from the factors in the denominator. When a factor of the denominator has no duplicate in the numerator, it causes a vertical asymptote at the zero of the factor.
- When the same factor appears in the denominator and numerator a hole will appear in the graph instead of an oblique or vertical asymptote. The location of this hole is called the point of discontinuity).
- Zeros occur at values that result when the numerator is set equal to zero (determined by factors in the numerator).
- Function values (graph regions) alternate between positive and negative across asymptotes and zeros. However if there is a double asymptote (two equal asymptotes) or a double root (two equal roots) function values do not alternate. (See graph 3 for a double asymptote on p. 118). (This observation will help students in solving rational inequalities (see p. 130)).

... continued

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

... continued

Performance **(C14/C6/C10)**

9) Find the equations of the asymptotes, and the location of the holes, and zeros, if any exist.

a) $y=\frac{3x^2-7x+2}{x-2}$

b) $y=\frac{5x^3+2x+1}{x-4}$

c) $y=\frac{3(x-2)^2(x+3)}{2x(x-1)^2}$

d) $y=\frac{x^3+2x-5x-6}{x^2(x^2+x)}$

10) Determine the behaviour of y as the x-values get very large and/or very small.

a) $y=\frac{x}{1+x^2}$

b)

$f(x)=\frac{x^3-x2x^4x^2-x^3x+1x3^2-5x-2}{(x-1)(x-x^2 2)+f}$

c) $y=\frac{2(x-2)}{(x-1)(x-3)}$

11) Graph the function and find the limit, if it exists: $\lim_{x\to 2}\frac{x^4-x^3+x^2-5x-2}{x^2-4}$

Journal **(C14)**

12) Describe how you would explain to a friend on the telephone how you can tell which intervals of a given rational function are positive and which are negative, using asymptotes and zeros only.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions and their graphs

C6 demonstrate an understanding for asymptotic behaviour

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C5 use tables and graphs as tools to interpret expressions

Elaboration – Instructional Strategies/Suggestions

... continued

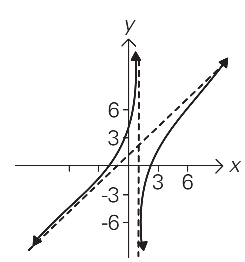
C14/C6/C10

- From their investigation of asymptotic behaviour concerning rational functions, students should be able to conjecture and verify the following . It should not be the intent that students will memorize and regurgitate these conjectures, but use them when appropriate as part of their study and analysis of rational functions).
  1) When the degree of the numerator is less than the degree of the demoninator, the *x*-axis is the horizontal asymptote.
  2) When the degree of the numerator is the same as the degree of the denominator, the horizontal asymptote is a constant where the value is equal to the ratio of the leading coefficients.
  3) When the degree of the numerator is one more than the degree of the denominator there will be a linear oblique asymptote for which the equation is obtained by finding the quotient of the numerator and denominator and investigating its limit as *x* approaches infinity.
  4) Optionally, when the degree of the numerator is more than one degree more than the degree of the denominator there will be an asymptote for which the shape and equation is determined by the polynomial quotient (without remainder).

C14/C6/C10/C5 To help students better understand the behaviour of rational function graphs and what causes the graphs to bend the way they do, students should graph some by hand. The ability to locate asymptotes and zeros that separate the *x*-axis into specific intervals, and the ability to recognize symmetries, provide the tools students need to graph rational functions. Now they are ready to consolidate their work with a procedure they can use to graph all rational functions.

- If possible, factor the numerator and denominator completely to identify common factors.
- Determine any horizontal or oblique asymptotes and sketch them.
- Determine any vertical asymptotes and sketch them.
- Determine any zeros and plot them.
- In each interval between the zeros and asymptotes, find a function value and plot the point. Include a large value of $|x|$.
- Sketch the curve.
- For example, to graph $y=\dfrac{(x^2-4)}{(x-1)}$:

since the degree of the numerator is one greater than the denominator, students should use division to find the oblique asymptote ($y = x + 1$). The factors in the numerator locate zeros at ± 2. The factor in the denominator locates a vertical asymptote at $x = 1$. Evaluate a few *x*-values in different regions to sketch the graph.



Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Activity **(C14/C6/C5)** continued from p. 117

1) e) i) Graph . Does there appear to be an horizontal asymptote? Explain why.

ii) Graph $y = x - 2$. Describe your observations about the graph of $y = x - 2$ with respect to $f(x)$.

iii) Graph and $y = 2x - 9$ on the same axes. What appears to be true?

iv) A linear asymptote that is neither horizontal nor vertical is called oblique. Explain whether $f(x)$ and $g(x)$ both have oblique asymptotes.

f) i) Use division to rewrite $f(x)$ and $g(x)$ as a quotient plus a remainder.
ii) What value does $f(x)$ approach as $x \pm \infty$? What about $g(x)$?

g) Make a conjecture with respect to the degrees of the numerator and denominator as to when an oblique asymptote occurs.

h) Test your conjecture by graphing . Make sure your window settings are "friendly" values. Explain what is happening with $h(x)$. How should you alter your conjecture?

[illegible]

Performance **(C14/C6/C5)**

2) For each function: State its domain and range; calculate the zeros, state the intervals for which it increases and those for which it decreases; state the intervals for which it is positive and for which it is negative; sketch the graph.

a) $y = \dfrac{x}{x-2}$

b) $y = \dfrac{2x^2 + x}{x}$

c)

d) $g(x) = \dfrac{x+1}{x^2 + 3x + 2}$

3) a) Use the same set of axes to sketch and .

b) Describe the change in the value of $f(x)$ in the interval from –6 to 6.
c) What are the coordinates of the points of intersection of the graphs?
d) Explain what is meant when it is said that these intersection points provide the solution to the original system.

Journal **(C14/C6/C10)**

4) In his discussions about asymptotes, my friend David said, "... while a curve does not intersect the asymptote as it moves an infinite distance from the origin, it might intersect the asymptote in another region." Explain what you think David means. Create a rational function that exemplifies this.

5) Create a rational function in which the curve approaches a horizontal asymptote at extreme values but crosses the same asymptote in another region. Write a note to a friend to explain how to determine where the curve crosses the asymptote.

Suggested Resources

McKillop, David et al., *Pre-Calculus Mathematics Two*, Toronto: Oxford University Press, 1993.

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

B1 describe the relationships between arithmetic operations and operations on rational algebraic expressions and equations

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration – Instructional Strategies/Suggestions

B1/C10 In previous study students have had some instruction and opportunity to operate on rational algebraic expressions and equations. This would have occurred just before the work on trigonometric identities so that students would have the skills and strategies needed in working with trigonometric expressions and equations.

As this unit develops there may be times when students need to combine and operate on rational algebraic expressions and functions. To strengthen their manipulation skills students should re-examine procedures for operations on rational expressions.

Teachers might begin by asking students to consider the expressions $(3x + 7)$, $(x^2 - x - 2)$, and $(x - 2)$. Ask students about the restrictions on x. Then have them consider the rational expressions $\frac{(3x+7)}{(x^2-x-2)}$ and . Ask them if there are any restrictions on x now, and why. Have them give their own definition of a rational expression and compare it with this one: A rational expression is an expression in fractional form that has polynomial terms in the numerator and denominator. Students should be reminded that in order to add or subtract rational expressions they should use the same methods they used when adding and subtracting rational numbers (e.g., begin by re-expressing the rational expressions as equivalent fractions with common denominators, then combine, factor, and simplify).

Have them study examples like the following:

- Factor any denominators, if possible.
- Determine the LCM.
- Express each fraction as an equivalent fraction with the common denominator (multiply by 1.)
- Combine the fractions.
- Simplify as required.
- State the restrictions.

... continued

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper **(B1/C10)**

1) Perform the indicated operations. State any restrictions on each variable.

a) $\frac{2x-3}{5}-\frac{x-1}{10}+\frac{3(2-x)}{15}$

b)

c)

d) $\frac{a-b}{c-d}-\frac{b-a}{d-c}$

e) $\frac{5x-6y}{3x-2y}+\frac{x-4y}{2y-3x}$

f) $\frac{6}{x^2-7yx+12}+\frac{5x+9}{x^2-2x-3}$

g)

h)

2) Given $y_1=\frac{2}{1-3(x+2)}$ and , find:

a) $y_1 + y_2$

b) $y_2 + y_1$

c) $y_1 - y_2$

d) $y_2 - y_1$

Does order make a difference with addition and subtraction of functions?

3) If $x > y > 0$, which is greater, $\frac{2x^2}{x^2-y^2}$ or ?

[illegible]

4) The sum of two positive numbers is 24. Show that the sum of their reciprocals can never be less than .

5) If $xy = 30$, and x and y are both positive, find the minimum value of .

... continued

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

B1 describe the relationships between arithmetic operations and operations on rational algebraic expressions and equations

Elaboration – Instructional Strategies/Suggestions

... continued

B1 To multiply or divide rational expressions, use the same methods used to multiply or divide rational numbers.

Example 1: Multiply

Solution:

$$\frac{x^2+5x+6}{x^2-x-20}\times\frac{x^2+3x-4}{x^2+x-2}$$
Factor the numerators and denominators.

$$=\frac{(x+2)(x+3)}{(x+4)(x-5)}\times\frac{(x-1)(x+4)}{(x-1)(x+2)}$$
Look for common factors in the numerators and denominators.

$$=\frac{(x+2)}{(x+2)}\times\frac{(x+4)}{(x+4)}\times\frac{(x-1)}{(x-1)}\times\frac{(x+3)}{(x-5)}$$
This can be rearranged as shown, where each of the first three expression has a value of 1.

$$=1\times1\times1\times\frac{(x+3)}{(x-5)}$$
Simplify.

$$=\frac{(x+3)}{(x-5)}$$
It is clear from the answer that $x\neq5$. Are there other values that x cannot have? Explain why.

Simplifying Complex Rational Expressions:

A fraction with a fraction in its numerator or denominator, or both, is a complex fraction. Each of the following are examples of complex rational expressions.

$$\frac{x-\frac{1}{x}}{x-\frac{1}{x^2}},\quad \frac{\frac{4}{x}-x}{x+\frac{x}{2}},\quad \frac{x-\frac{2}{3}}{6}$$

Example 2: Simplify . State any restrictions.

Solution:

$$=\frac{a}{a+1}\div\frac{a}{a-1}=\frac{a}{a+1}\times\frac{a-1}{a}$$

$$=\frac{a-1}{a+1};\ a\neq-1,0,1$$

Ask students why in addition to –1, 1 and 0 are included in the list of restrictions.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Performance (**B1**)

For questions 6–10, perform the indicated operations. State the restrictions on any variable.

6)

7) a) b) $\frac{x^2-6x+9}{x^2-7x+12} \cdot \frac{x^3-4x^2+9x-36}{x^4-81}$

8) $(mx-nx) \div \frac{m^2-n^2}{4x}$

9) $\frac{1+\frac{1}{a}}{1-+\frac{1}{a^2}}$

10)

11) If and , then find $r(x)=f\left(\frac{1}{x}\right)+g\left(\frac{1}{x}\right)$.

$x+2+\frac{4}{x-3}$

12) Given , find:

a) c)

b) d) $f\left(\sqrt{3}\right)$

13) If $f(x)=\frac{3-2x}{2-3x}$ then show that .

14) If , find .

15) If , find the value, in terms of *c*, of:

.

16) Transform the expression as indicated:

a) to b) to tan *B*

17) Verify the following identities:

a) $\frac{1-3\cos x-4\cos^2 x}{\sin^2 x}=\frac{1-4\cos x}{1-\cos x}$ b) $\frac{1+\sin x+\cos x}{1-\sin x+\cos x}=\frac{1+\sin x}{\cos x}$

18) Create an identity involving rational expressions, then trade it with your partner to verify.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

B1 describe the relationships between arithmetic operations and operations on rational algebraic expressions and equations

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration – Instructional Strategies/Suggestions

B1/C10 Students will need to solve rational equations for various reasons which include finding zeros of rational functions (which in turn will assist in solving rational inequalities - see the next two-page spread), finding intersection points with other functions or asymptotes, and solving problems involving rational equations and/or inequalities.

Example 1: Melodie posed the following question to two of her friends:

What value for the variable would make $\frac{1}{3x-5}$ and the same?

Henri's solution:

use reciprocals

$$3x-5=\frac{7}{3}$$
$$3(3x-5)=7$$

Michelle's solution

$$\frac{1}{3x-5}=\frac{3}{7}$$

multiply by the LCD

Each went on to solve for *x*, getting , and were able to answer the question.

Both of the above methods of attack look fairly simple with this example. To simplify expressions, students may use either method above to re-express the equation into a familiar form without fractions, and then solve the resulting equation by methods previously learned.

Example 2: A more complex example follows. Solve for *x*: ,

students might simplify the left side into one expression:

$$\frac{(x+2)(x-6)+(x-2)(x-3)}{(x-3)(x-6)}=2$$
$$\frac{x^2-4x-12+x^2-5x+6}{(x-3)(x-6)}=2$$
$$\frac{2x^2-9x-6}{(x-3)(x-6)}=2$$
$$\vdots$$

Students could have used a different strategy by multiplying each term in the given equation by the LCD. This would result in clearing all the fractions and having only to solve the resuling polynomial equation $(x-6)(x+2)+(x-3)(x-2)=2(x-3)(x-6)$.

and continue to solve for *x*.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

1) Solve the following equations

   a)

   b) $\frac{4x+3}{2x-3}=\frac{6x+5}{3x-2}$

   c)

2) a) The total resistance R_T in ohms (Ω), of three parallel resistors in a circuit is found using the formula . Find R_T for three parallel resistors with resistances 12 Ω, 15 , and 10 .

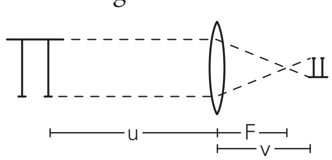
   b) The formula gives the load power of a circuit. What is the load power for the circuit in (a)?

3) A problem from the Rhind Papyrus of Egypt, about 1650 BC, reads as follows: "A quantity and its $\frac{2}{3}$ are added together, and of the sum is added. Then of this sum is 10. What is the quantity?"

4) The sum of and its reciprocal is 10. Find *x*.

5) In photography the relationship between the constant focal length of a lens (*F*), the distance between the lens and the film (*v*), and the distance between the object and the lens (*u*) is given by .

   a) Suppose the focal length *F* is 60 mm. Solve for v in terms of *u*.



   b) Graph the function $v = g(u)$ over the interval [60 mm, 60 000 mm].

   c) "Depth of field" refers to the range of distances in which an object remains in focus. Use the graph to help you explain why the depth of the field is broad when you focus on objects that are very far from the lens but is narrow for objects close to the lens.

6) An open topped box with a square base and rectangular sides is to hold 250 cm^3. If sides are double thickness and the bottom is triple thickness, what size box will use the least amount of material?

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

B1 describe the relationships between arithmetic operations and operations on rational algebraic expressions and equations

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C11 analyse and solve polynomial, rational, irrational, and absolute value inequalities

Elaboration – Instructional Strategies/Suggestions

B1/C10/C11 Students have already, in their analysis of rational functions, been determining the regions of the graph where the function is positive and where it is negative. They should understand that the function values change from positive to negative (or vice versa) as the curve passes through single zeros and as it approaches a single vertical asymptote from the right or left. Note, however, the exception to this is when there occurs a double zero (two or an even number of equal roots) or a double asymptote (two or an even number of equal asymptotes). For these, the positive and negative regions do not alternate.

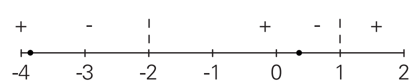
There are algebraic methods that students could use to calculate the intervals in which a function is positive or negative.

For example: to solve students would:

- simplify the inequality so that it is greater than (or equal to) or less than (or equal to) zero
- calculate the zeros (noting any equal zeros) and plot them on a number line
- calculate the vertical asymptotes (noting any double asymptotes) and plot them on the number line.
- test a domain value to determine a function value in one interval, then alternate across the zeros and asymptotes as appropriate.

Possible solution:

Students would then solve the polynomial equation for the numerator to find the zeros (0.386, 0), and (–3.886, 0), and plot them on a number line, along with the vertical asymptotes (x = 1, and x = –2). Testing $x = 0$, gives $\frac{3}{2}$, a positive value. Students can now mark the other regions either positive or negative by alternating appropriately across the zeros and asymptotes.



The most efficient way to solve this inequality is to examine the graph produced on the graphing calculator, and use its features to determine the zeros. Students can then see the positive and negative regions, and write the intervals to solve the inequality:

When calculators are being used this way it is important that students and teachers discuss the recording of the solution, and what minimum information must be shown.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Performance (B1/C10/C11)

1) For each function below:
   a) Describe it as a transformation of $f(x)=\frac{1}{x}$ and state the mapping rule.
   b) Sketch the graph.
   c) State the domain and range, the equation of the asymptotes, and the intervals in which it increases or decreases.
   d) Calculate the zeros. State the intervals for which the function is positive and/or negative.

   i)   iv)

   ii)   v)

   iii) $y=\frac{1}{1-x}$

2) Solve the following inequalities.

   a)   d) $\frac{12}{x-3}<x+1$

   b) $2x-3\geq\frac{-4x}{x-1}$ e)

   c) $\frac{2x-3}{x-2}>0$

[illegible]

3) Given: $f(x)=y=\frac{2}{x+1}-3$
   a) Describe $f(x)$ as a composition of transformations of $y=\frac{1}{x}$.
   b) Sketch the graph of $f(x)$.
   c) State its domain, range, and the equations of any asymptotes.
   d) Evaluate its zeros.
   e) For what interval is the function positive? negative?
   f) For what interval does f increase? decrease?
   g) Find the equation of .
   h) Evaluate $f(-1)$, $f(0.5)$, $f^{-1}\left(\frac{2}{3}\right)$.
   i) For what value of x is $f(x)=5$?
   j) Without graphing, describe the graph of , then graph it.
   k) Use your graph to solve $\frac{2}{x+1}-3\leq 1$.
   l) Find the intersections of $f(x)$ and $h(x) = x-3$.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

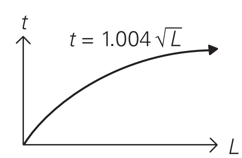
C2 model problem situations with combinations and compositions of functions

C14 analyse relations, functions, and their graphs

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration – Instructional Strategies/Suggestions

C2/C14/C10 By studying irrational functions, students will get a better insight into the nature of extraneous roots. An irrational function is a function in which the independent variable in its simplest form is located under a radical sign (thus, sometimes named radical function). Students will examine the effect on other families of functions when the square root of the function is found. A common application of the irrational function is the relationship between the length of a pendulum and its period. For example:

> The time, *t*, it takes a pendulum to swing back and forth once, called its period, relates directly to the square root of its length, *L*. This can be represented in equation form by $t = k\sqrt{L}$.
>
> Suppose that while conducting an experiment, students find that a pendulum 0.5 m long has a period of 0.71 s. These values can be substituted into the equation above to find *k*.
>
> $$0.71 = k\sqrt{0.5} - k = \frac{0.71}{\sqrt{0.5}} = 1.004$$
>
> Therefore, the function that describes the relationship between a length of a pendulum, in metres, and its period, in seconds, is
>
> $$t = 1.004\sqrt{L} \text{ or } t = 1.004L^{\frac{1}{2}}.$$
>
> 
>
>
> Students should notice that (0,0) is the endpoint of the graph of this function. This should be reasonable to them, since pendulums do not have negative lengths. Furthermore, students will recall square roots of negative numbers are undefined in the set of real numbers.

The variable of a polynomial function has a whole number power, and the variable of a rational function has an integral power. In comparison, is an irrational function, because its variable, *L*, has a non-integral constant power. Students will only study irrational functions with the index restricted to 2 (square root functions).

... continued

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Performance (C2/C14/C10)

1) The famous seventeenth century astronomer Johannes Kepler discovered that the time a planet takes to revolve once around the sun relates directly to the planet's average distance from the sun (in millions of kilometres). His discovery, expressed as an equation, is $t = f(d) = kd^{\frac{3}{2}}$.
   a) Use the fact that it takes Earth one year to revolve around the sun at an average distance of 150 000 000 km to determine the value of the constant *k*. Then write the equation for *f (d)*.
   b) Use *f(d)* to find how long it takes, in Earth years, for Mercury to revolve once around the sun, if you know its average distance from the sun is 58 000 000km. Check an encyclopaedia or almanac to see whether you are correct.
   c) Is this an irrational function? Does it have a restricted domain?

2) The velocity which a satellite must sustain in order to remain in orbit varies inversely as the square root of its distance from the centre of the earth. The radius of the earth is 6400 km. At a distance 1000 km above the surface of the earth, the orbiting velocity is 7.34 km/s. Find the orbiting velocity 2000 km above the surface of the earth.

3) The speed at which a parachutist strikes the ground varies directly as the square root of the mass of the jumper and inversely as the radius of the parachute. A jumper with mass 80 kg hits the ground at 7 m/s when the radius of the parachute is 3.5 m.
   a) Find the formula describing the relationship.
   b) Sketch a graph of the situation and explain how it can be used to answer (c).
   c) Find the falling speed for a jumper with mass 100 kg and a parachute with radius 4.0 m.

... continued

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions, and their graphs

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration – Instructional Strategies/Suggestions

... continued

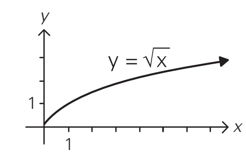
C14/C10 Irrational functions have restricted domains when they involve even roots. This is because the even roots of negative numbers are undefined in the set of real numbers. Thus, the domain of $f(x)=\sqrt{x+1}$ is restricted to

. Also, any values of the independent variable that lead to

division by zero would not be in the domain. For instance, would

have $x = 0$ excluded from its domain because $f(0)=\frac{1}{0}$ is undefined. Also, x cannot be negative since square roots of negative numbers are not defined in the real number family. Therefore .

The graph of is shown.

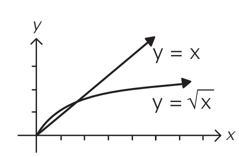


The connection to the inverse of the function $y = x^2$ should be made. Notice that the graph is increasing but the rate of increase is diminishing (the value of the derivative is decreasing). Another important feature of this graph is the downward curvature. This is called *concave down*; in contrast, the shape of the parabola $y = x^2$ is concave up.

The steepness of the graph of $f(x)=\sqrt{x}$ can be compared with that of linear, quadratic, and cubic graphs by determining for each function the x-value that is paired with a y-value of 64. For $f(x) = x^3$, the ordered pair is (4, 64). For $f(x) = x^2$, the ordered pair is (8, 64). For $f(x) = x$, the ordered pair is (64, 64). For $f(x)=\sqrt{x}$, the ordered pair is (4096, 64). Notice that the cubic function attained the value of 64 when $x = 4$, whereas the square root function doesn't attain this value until $x = 4096$.

It is also interesting to study the steepness of the graph of $f(x)=\sqrt{x}$ in another way.

When $0<x<1, \sqrt{x}>x$ and the graph is above the line $y = x$.



Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

... continued

Performance **(C14/C10)**

4) Given: .

   a) Describe *f* as a composition of two functions.
   b) State the domain and range of *f (x)*.
   c) Find .
   d) Calculate the zeros of *f(x)*.
   e) Sketch its graph.
   f) For what values of *x* is *f* positive? Explain.
   g) Would you describe *f* as an increasing or a decreasing function? Explain.
   h) Find the equation of the inverse of *f*. Is it a function? Explain.
   i) Explain the connections between the graph of *f* and the solution to the equation $2\sqrt{x}-1=5$.
   j) If *g* (*x*) = 5*x* + 2, and , describe the effect of *h* (*g* (*x*)) on the graph of *g* (*x*), and *g* (*h* (*x*)) on the graph of *g* (*x*).

5) Ask students to work in pairs. One student creates an equation for a function, the other must predict its shape and sketch. Then the first student must predict the shape of the square root of that function. The second will record it using notation and check using technology. Take turns starting. The first couple of functions might be:

   a) $f(x)=-2x^2-5$

   b)

Journal **(C11/C18)**

6) Explain why does not have a restricted domain, while does. Are they irrational functions? Explain

7) Compare the domain and range of , for with the domain and range of , for .

8) Given:

   a) Write Y_3 and Y_4 and list the restrictions on each function.
   b) Explain the differences between the restrictions of Y_4 and Y_3.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C8 explore and describe the connections between quadratic equations and their inverses

C14 analyse relations, functions, and their graphs

C5 use tables and graphs as tools to interpret expressions

C16 analyse the effect of parameter changes on the graphs of functions and express the changes using transformations

Elaboration – Instructional Strategies/Suggestions

In a previous course students learned about inverse relations. They learned that when the graph of a function is reflected across the line $y = x$ the resulting graph is the inverse of the given function. The x and y values of the function and its inverse are interchanged.

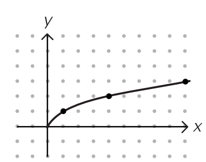
C8/C14/C5 Students have studied the quadratic equation $y = x^2$. Have them reflect this over the line $y = x$, and have them write the equation of the image.

Students should remember that when a graph is reflected across $y = x$ the image is the inverse of the given function. The inverse of $y = x^2$ is not a function because for each x other than 0, there are two y-values. The inverse equation can be found by interchanging y and x: $y = x^2 \rightarrow x = y^2$. Some students will take $x = y^2$, solve for y and try to graph this using their graphing calculators. Many will graph
and expect to see a sideways parabola. Have them explain what is happening here. The inverse equation $y^2 = x$ can also be written as $y = \pm\sqrt{x}$ where $y = \sqrt{x}$ is the equation of the upper branch of its graph and $y = -\sqrt{x}$ is the equation of the lower branch.

C16 If a quadratic function is already graphed, its inverse relation can easily be graphed by plotting the ordered pairs with x and y values interchanged. This can be done by hand, or by using the list feature on the graphing calculator. (The lists can be assigned to either axis). If students are graphing the inverse relation given its equation, they could use transformational concepts and the model $y^2 = x$. For example, $(y-3)^2 = \frac{1}{2}(x+2)$ would be a parabola opening to the right with vertex at (-2, 3) and a horizontal stretch of 2. In general, the

function is the result of transforming

by a VS of a, HS of c, VT of k, and HT of h.



The square root graph is another parent or model function that can be used to illustrate transformations. Both the domain and range of $f(x) = \sqrt{x}$ are real numbers that are zero or greater. If you trace the graph, there are no function values for y unless x is a least 0. Trace to show that $\sqrt{3}$ is about 1.732 and that is about 2.828. Describe how you would use the graph to find . What happens when you try to trace for values of $x < 0$? What is ?

C16/C8/C14/C5 Have students explore the transformations of $y^2 = x$. Have students interplay between graphs, equations, words, and mapping rules.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (C8/C14/C5/C16)

1) Find the equation of the inverse of each of the following:

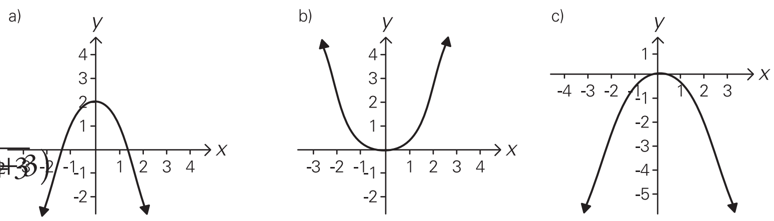
   a)

   b) $f(x) = 2(x^2 + 3)$

   c)

   d)

2) For each of the following graphs:
   a) Draw its inverse.
   b) Describe the domain and range of the function and its inverse.
   c) Explain why the inverse is or is not a function. If not, how would you restrict its domain to make it a function?
   d) State the equation for the inverse



3) Consider the parent or model function $f(x) = \sqrt{x}$.
   a) Name three pairs of integer coordinates that are on the graph of $f(x+4) - 2$.
   b) Write in *y*= form and graph it.
   c) Write in *y*= form and graph it.
   d) Write the equation for the image and sketch the graph given the mapping .
   e) Describe in words the transformations and the mapping rule in (d).

4) a) Explain why it could be said: “All polynomial functions are continuous but only some rational and irrational functions are continuous.”
   b) What relationships, if any, exist among: , and ?

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C19 investigate and interpret combinations and composition of functions

C14 analyse relations, functions and their graphs

C5 use tables and graphs as tools to interpret expressions

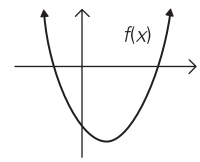
C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

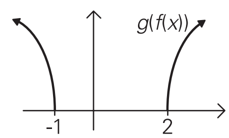
C11 analyse and solve polynomial, rational, irrational, and absolute value inequalities

Elaboration – Instructional Strategies/Suggestions

C14/C5/C19 Students should understand that irrational functions can be obtained from other functions through composition of functions.

❐ Have students graph using a graphing calculator. Have students examine its slope. What is happening to the slope in the interval ? (Note: Square brackets used to indicate inclusion of –1 and 2 in the set of points.) Now give students the function $g(x)=x^{\frac{1}{2}}$ and ask students to graph the composition $g(f(x))$. Students should be able to state that this graph has a break from $x = -1$ to $x = 2$. It is said to be discontinuous for $-1 < x < 2$. Have them explain why. Have students compare this graph to that of $f(x)$. Have them explain why the curves of $g(f(x))$ are concave downwards.





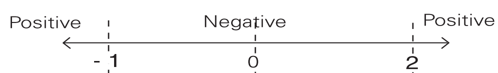
C10/C11 Students should see that algebra skills can be used to find intervals of discontinuity. For example, for the function , they could set up the inequality:

$x^2 - x - 2 < 0$

$(x + 1)(x - 2) < 0$

$x = -1$ and $x = 2$ will produce values of 0.

Test the intervals for negative values:



Therefore, $-1 < x < 2$ satisfies the inequality and indicates the interval of discontinuity for the function. Also, the domain for the function is

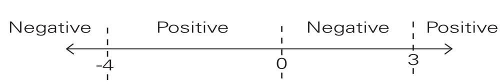
.

Algebra can be used to answer other questions regarding irrational functions.

For example, ask students what is the domain of ? They should begin with $x^3 + x^2 - 12x \geq 0$, then factor to get .

The zero values of the polynomial are $x = -4$, $x = 0$, and $x = 3$.

Test the intervals for positive values.



Therefore, the domain can be written or , or written

.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Journal **(C19/C14/C5)**

1) Given , have students explain in words, without graphing, how the graph would compare to the graph of . Then, check it using technology. Have students talk about restrictions on the domain, and a comparison in the shape of the curved part(s).

   a) For , describe its domain, determine a few ordered pairs, and sketch its graph.
   b) Explain why *g* could be described as discontinuous at some point or on some interval.

Pencil and Paper/Technology

3) Given

   a) Determine any restrictions on its domain.
   b) Determine a few ordered pairs, plot them, and sketch the graph of this function.
   c) State its domain and its range.
   d) Describe how the shape of its graph compares to the graph of $y = \frac{4x - x^2}{10 - 7x + x^2}$. Explain how it is different and why.

[illegible]

Pencil and Paper **(C18/C19)**

4) Solve:

   a)

   b) $2x - 3 > \frac{-4x}{x - 1}$

5) Determine the values of *x* for which is less than 4.

6) If , then for what values of *x* is $h(x) > 3$?

7) Explain the relationship between solving this inequality and the graph of $y = \frac{2x}{x - 3}$.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C2 model problem situations with combinations and compositions of functions

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration – Instructional Strategies/Suggestions

C2/C10 A general approach to solving an irrational equation is to isolate the radical on one side of the equation, then eliminate it by raising each side of the equation to an appropriate power.

For example, grandfather clocks have pendulums with 1 *s* periods. How long must their pendulums be so that the function shows the relationship between a pendulum's period and its length?

$1=1.004\sqrt{L}$	Let t = 1 for the period of the pendulum on the grandfather clock.
$\frac{1}{1.004}=\sqrt{L}$	Isolate the radical by dividing both sides by 1.004.
$0.996 \text{ B } \sqrt{L}$	
$0.996^2 \text{ B } \left(\sqrt{L}\right)^2$	Eliminate the radical by squaring both sides.
$0.992 \text{ B } L$	

Therefore, grandfather clocks must have pendulums that are 0.992 m long to have periods of 1 *s*.

Another example: Solve $\sqrt{1-x}+3=7$.

Solution:

Subtract 3 from both sides to isolate the square root.

Eliminate the square root.

Checking: $\sqrt{1-(-15)}=\sqrt{16}=4$.

... continued

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

Performance (**C10**)

1) a) Draw a graph of .
   b) Use a ruler parallel to the x-axis through (0, 3) to draw a line that intersects the graph of f. What is the x-coordinate of this point of intersection?
   c) Your answer in (b) is the solution to what irrational equation? Algebraically solve this equation to check.
   d) Solve an equation to find any zeros of f.

2) Given .
   a) Are there any restrictions on the domain of f? Explain.
   b) What is the value of the y-intercept, if there is any?
   c) Find the x-intercept. Check it algebraically.
   d) Find all values of x for which $f(x) = -5$; $f(x) = -1$; $f(x) = -9$. Check these algebraically.
   e) Determine the minimum value for $f(x)$.

3) Richard was asked to solve . For his first step, he squared each term to remove the radical: . Explain what Richard did wrong and why he can't do it. What should he have done?

[illegible]

Performance (**C2/C10**)

4) The area of any egg shell is directly proportional to the mass of the egg raised to the $\frac{2}{3}$ exponent . If a normal 58 gram chicken egg has a shell area of 26 square centimetres, then
   a) find the shell area of an emu egg with a mass of 1.42 kilograms, and
   b) if the area of a shell is 250 square centimetres, find the mass to the nearest gram.

5) The period of a pendulum varies directly with the square root of the length of the pendulum. After gathering data, it is found that a pendulum 28 cm long swings with a period of 52 seconds.
   a) Write an equation expressing period in terms of length.
   b) Most grandfather clocks have a pendulum that has a period of 1 second, how long would the pendulum be?
   c) The new high school in Riverswift has a pendulum hanging in its lobby (from top to bottom) with a period of 7 seconds. How high is the school lobby?

... continued

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

Elaboration – Instructional Strategies/Suggestions

... continued

C10 Sometimes it is not possible to solve the equation with radicals by isolating the radical in one step:

Subtract 1 to isolate the radical on the left. Square to eliminate the radical on the left. Simplify, then isolate the other radical.

Eliminate this radical by squaring.

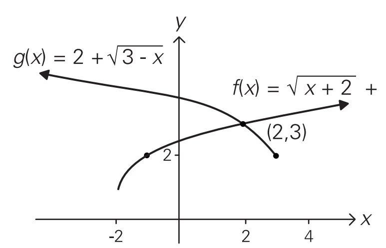
Solve the resultant quadratic equation by factoring.

Students should always check that their final solutions satisfy the original equation and that they are meaningful for any real situations being modeled.

Given $\sqrt{x+2}+1=2+\sqrt{3-x}$, check for $x = -1$.

Check $x = 2$ yields $3 = 3$:

Therefore, $x = -1$ is an extraneous root. Some students might solve the equation at the top of the page by graphing. If they do, they will find that $x = -1$ is an extraneous solution, since the only intersection is at $x = 2$. In the case of irrational equations, an extraneous root often happens because the distinction between $\sqrt{x}$ and is lost when they are squared.



Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

... continued

Performance **(C10)**

6) Given $f(x) = x - \sqrt{x+5}$.

a) Calculate any zeros of *f*.
b) State the domain of *f* and use a graphing calculator to determine its range.
c) What is its *y*-intercept?
d) If $f(x) = 1$, what is *x*?

7) a) Algebraically, find the point(s) of intersection of $y = \sqrt{x+2}$ and
.

b) Solve this system:

c) Algebraically, determine whether the straight line $x - 3y + 4 = 0$ intersects the function $f(x) = \sqrt{x^2 - 16}$, and if so, where.

d) Graphically determine whether and have any point(s) in common.

[illegible] 8) Solve for *t*: .

a) Given . For what value of *x* is $g(x) = 3$?

b) Given . For what value of *x* is $h(x) = 0$?

c) Given and . For what value of *x* is $j(x) = k(x)$?

d) Solve .

e) Solve .

Performance **(C2/C10)**

10) Given: $f(x) = \dfrac{3x}{(x+2)^2}$, $g(x) = \sqrt{2x}$, and $h(x) = \dfrac{1}{\sqrt{x}}$

a) Does $f(a+b) = f(a) + f(b)$?
b) Examine the graphs of $g(x)$ and $h(x)$. Predict and sketch the graph of $g(x) + h(x)$. Check using your graphing calculator.
c) Calculate $g(f(x)) = 0.75$.
d) Which has the greater maximum value, $g(f(x))$ or $f(g(x))$?
e) Solve $h(g(x)) = g(x)$.

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C2 model problem situations with combinations and compositions of functions

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C14 analyse relations, functions and their graphs

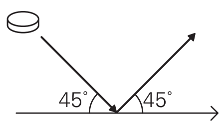
C5 use tables and graphs as tools to interpret expressions

C16 analyse the effect of parameter changes on the graphs of functions and express the changes using transformations

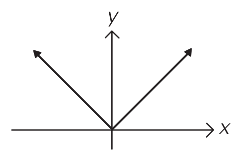
Elaboration – Instructional Strategies/Suggestions

C2/C10/C14/C5 Students have already been introduced to absolute value functions as functions that behave similarly to quadratic functions when transformations are performed on them. Students will now take a closer look at the absolute value function and relate it to piece-wise functions. They will also analyse the graphs and the effects of combinations and compositions of this function with others that they have studied.

Suppose a student hit a hockey puck against the boards at a 45° angle. The puck would be reflected at the angle shown.



If you were to model this reflection on a coordinate system, with the *x*-axis as the mirror, and the origin as the point of reflection, its graph would be this V-shape.



Ask students to determine what equations describe this graph. Have them first look at the section to the left of the origin. What equation describes this straight line? Now look at the section to the right of the origin. What equation describes this straight line? Therefore, this graph is described by a different expression for different parts of its domain.

Mathematicians call this special function the absolute value function (introduced in *Mathematics 10*) and denote it $f(x)=|x|$. The absolute value is a distance and it produces only non-negative values. If the distance formula is used to find the distance from any point on the *x*-axis, (*x*, 0), to the origin, students would obtain

. Consider the relation (*x*-coordinate of point on axis, distance to origin), from which students would have ordered pairs like . . ., (–2, 2), (–1, 1), (0, 0), (1, 1), (2, 2), ..., Have them graph these ordered pairs and join them with lines. They should understand why this is the absolute value function.

Therefore, another definition for $f(x)=|x|$ is . Students can use either definition to calculate the absolute value of any number. For example, to find the value of $|-3|$:

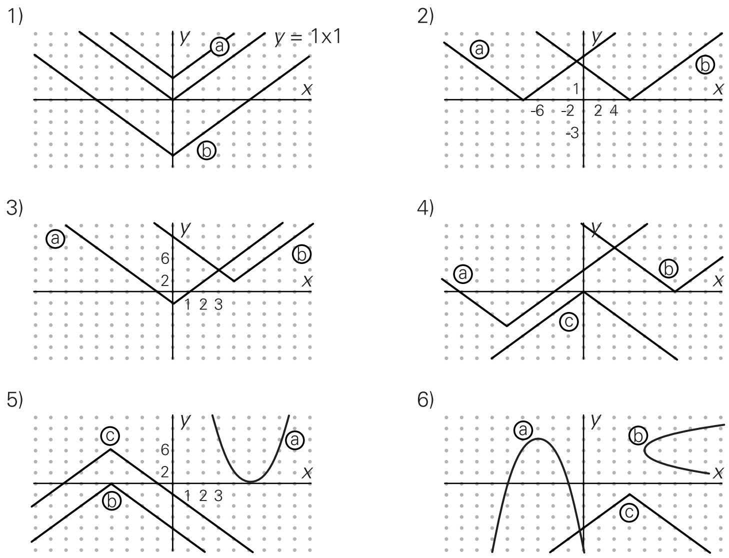
- because –3 is 3 units from the origin, then .
- because –3 < 0, it is on the $y = -x$ section of the graph.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Performance **(C14/C16)**

1) For graphs 1–3, write the equation of each graph in piece-wise form. For graphs 4–6, write the equation of each function.



[illegible]

2) Consider the line given by . Suppose . Then *g* (*f* (*x*)) will be absolute value of the inner linear function *f (x)*. Predict and then describe what *g* (*f* (*x*)) would look like. Check it on your calculator.

3) Suppose and .

   a) Determine the expression for each function composition.
      i) *f* (*g* (*x*))
      ii) *g* (*f* (*x*))
   b) In words describe the effect that *f* (*x*) has on the graph of *g* (*x*) in the composition *f* (*g* (*x*)).
   c) In words describe the effect that *g* (*x*) has on the graph of *f* (*x*) in the composition *f* (*g* (*x*)).

Performance **(C10)**

4) Express each as an absolute value:

   a)

   b) $y = 3\sqrt{x^2 - 4x + 4}$

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C2 model problem situations with combinations and compositions of functions

C19 investigate and interpret combinations and compositions of functions

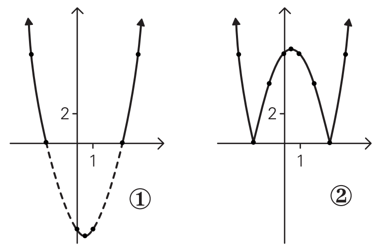
C14 analyse relations, functions, and their graphs

C5 use tables and graphs as tools to interpret expressions

Elaboration – Instructional Strategies/Suggestions

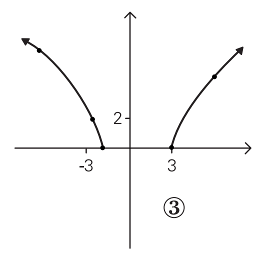
C2/C19/C14/C5 Students will analyse the behaviour of graphs of where $f(x)$ may be linear, quadratic (polynomial), rational, sinusoidal, or exponential as examples of composition of functions.

For example, if $f(x) = x^2 - x - 6$ and then . When students graph , they should note that, by definition, all the *y*-values must be positive, thus the points where the graph of ① $y = x^2 - x - 6$ is negative will be reflected across the *x*-axis into the positive region ②.



$f(x) = x^2 - x - 6$ $\quad g(f(x)) = |x^2 - x - 6|$

Also, given , students should compare the graphs of $g(f(x))$ with ③, $h(f(x))$, and be able to explain why there is a difference, and what that difference is. In other words, students should be able to compare or contrast the effects on the graph of a function by taking its square root, and by taking its absolute value. In particular they should note that by taking its square root any negative values of *y* disappear from the graph, but when taking the absolute value those negative values of *y* become positive and appear as a reflection in the *x*-axis of their corresponding negative values. Similarly, they should discuss what happens to the positive values.



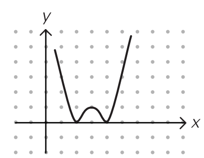
$h(f(x)) = \sqrt{x^2 - x - 6}$

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

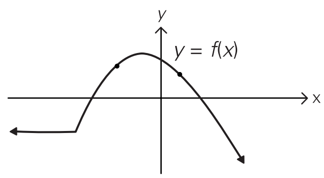
Paper and Pencil/Performance (C19/C14/C5)

1) Given $f(x)=|x|, g(x)=x^2-2x-24, h(x)=\sqrt{x}$
   a) Predict and sketch your prediction for
      i) $f \circ g(x)$
      ii) $h \circ g(x)$
   b) Check your prediction using technology.
   c) Compare and contrast (i) and (ii) above and explain their differences.



2) a) Write the equation in y= form for the graph at the right.
   b) Create two functions f and g so that the graph shown is $f(g(x))$.

3) Given: $y = f(x)$ as shown, sketch
   a) $y=|f(x)|$
   b) $y=\sqrt{f(x)}$



Extension (C19/C14/C5)

4) Explain using words and graphs why $|x^2-2x-3|=y$ looks different than $x^2-2|x|-3=y$.

Performance (C2)

5) Assume that the side of a billiard table is the x-axis, and the cue-ball bumps the side at an angle of 30°.
   a) Sketch the graph of the path of the ball and its reflected image.
   b) Ask students to express this relation as an equation in three ways: piece-wise, square root, and absolute value. How do they relate to ?

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C10 analyse and solve polynomial, rational, irrational, absolute value, and trigonometric equations

C11 solve polynomial, rational, irrational, and absolute value inequalities

Elaboration – Instructional Strategies/Suggestions

C10 Students should understand that there are two numbers for every absolute greater than 0. For instance, if then could be –6 or 6, because both are 6 units from the origin. For this reason, it is important for students to consider both positive and negative numbers when they solve equations involving absolute values. For example, when solving for t, students must consider the two cases:

1) If , then , and students will solve $t - 3 = 5$, getting $t = 8$.
2) If $t - 3 < 0$, then and students will solve $-t + 3 = 5$ or $t - 3 = -5$, getting $t = -2$.

Graphically, this means that (–2, 5) and (8, 5) are both points on the graph of . Checking , and .

When solving , students would restate as . Using the above procedure they would then restate as two equations: $3x - 1 = 2$, and $3x - 1 = -2$. This will lead to two possible solutions, $x = 1$ and . Students should check all their answers for extraneous possibilities. For example, in this case, the graph of opens upward and intersects the x-axis in 2 places, thus it has two real roots. Checking algebraically:

Similary, 1 checks out so $-\frac{1}{3}$ and 1 are both roots.

Students might also solve using the fact that . So using the above example: $|t-3|=5$ would become , and then $(t - 3)^2 = 25$ or .

C11 When students solve absolute value inequalities, the roots should be plotted, a region tested to see if the inequality is true or false, and then (as with other inequalities) the regions (separated by zeros) alternate, unless any zero occurs an even number of times. Students can also use an algebraic approach. For example, students should write the absolute value as an interval. Students should then operate on all three parts to isolate the variable, thus finding the solution interval. In this example, the solution is *and* . The solution to the inequality leads to the solution *or* .

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Performance (C10/C11)

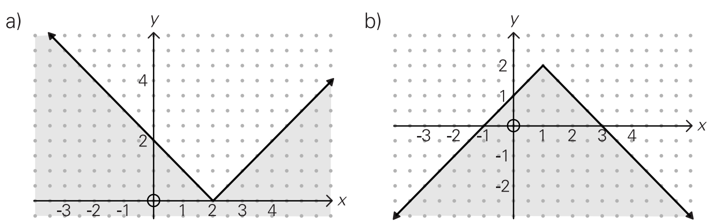
1) Assume the x-axis is a mirror and a beam of light shines at the point (–3, 0) from a point in the fourth quadrant. The reflected beam makes a 45° angle with the x-axis.
   a) Sketch the graph of this beam and its reflected image.
   b) You can express this relation as an equation in three ways: the union of two linear equations, a square root, and an absolute value. Write all three equations. How do they relate to $y = |x|$?
   c) If the angle is changed to 60°, how is the absolute value equation affected?
   d) If the angle is 30° with the y-axis, but is pointed at (0,5) from the third quadrant, what would be the absolute value equation of the light beam.
   e) Find the area of the polygonal region determined by the intersection of and y = 2.

2) a) Graph $f(x) = 2x^2 + 5$ and $g(x) = 2x^2 + 5x - 3$.
   b) Predict what the graphs of and will be like. Check your prediction by graphing them.
   c) What is the connection between the graph of and ?
   d) Algebraically, solve the equation in (c) and check your solution on a graphing calculator.

[illegible] $|3x \; 2| = \frac{3}{2}x + 7$

3) An architect wants to draw a circle with an area of 60 cm^2 having an error of no greater that 2%. How accurately must the radius of the circle be measured? (Hint: Use an absolute value inequality in which the variability of the area is a function of the radius.)

4) State the inequalities shown here



Pencil and Paper (C10)

5) Ben was asked to solve . He began his solution with "... if $x > 0$, then , so ." Is this the correct first step? Explain.

6) Solve the following:
   a)
   b)
   c)

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions, and their graphs

C5 use tables and graphs as tools to interpret expressions

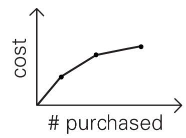
C23 explore and describe the connections among continuity, limits, and functions

C25 demonstrate an intuitive understanding of the concept of limit

Elaboration – Instructional Strategies/Suggestions

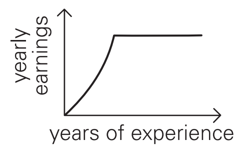
C14/C5/C23/C25 The exploration of the function can serve as an introduction to piece-wise functions. Applications of piece-wise functions should be explored. In addition to this, a brief discussion of limits should occur such that students will recognize the conditions for which a limit exists.

e.g., purchasing a product from a distributor



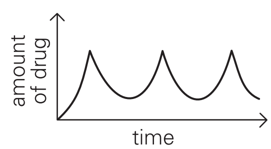
e.g., earnings with a salary cap

$$I = \begin{cases} 1000x^2 + 30000, 0 \le x \le 10 \\ 130000, x \ge 10 \end{cases}$$



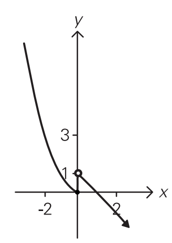
e.g., drugs in system; taking medication every 5 hours.

$$D = \begin{cases} 2^x - 1, 0 \le x \le 3 \\ -\left|5\sin\left(.5\left(x-3\right)\right)\right| + 8, 3 \le x \le 15 \\ 1.5^{-x+21} - 1, x > 15 \end{cases}$$



When asked to determine the point(s) of discontinuity for $f(x)\begin{cases} x^2, x \le 0 \\ 1-x, x > 0 \end{cases}$

students should graph to see that the function is continuous at every point, except at $x = 0$. Since $f(x)$ is defined according to whether x is to the left or right of zero, students should consider the left and right-hand limits separately:



$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 = 0^2 = 0$$
$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1-x) = 1 - 0 = 1$$

Students would conclude that since , does not exist. When the limit does not exist the function is discontinuous at that point, e.g., at the point $x = 0$.

For a function to be continuous at any point $x = a$,

- the function must be defined at the point $x = a$
- the left- and right-hand limits must both exist and be equal for the function at the point $x = a$.
- the limit as $x \to a$ must be equal to the $f(a)$, or .

Developing a Function Toolkit II

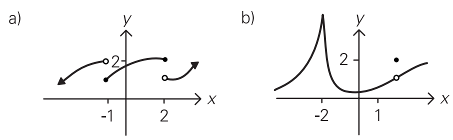
Worthwhile Tasks for Instruction and/or Assessment

Performance (C14/C5/C23/C25)

1) a) Given these **piecewise functions**, sketch each one.

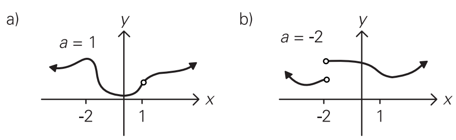
i) $f(x)\begin{cases} 2x^2+1, x>0 \\ 0, x=0 \\ \frac{1}{3}x-1, x<0 \end{cases}$ ii) $g(x)\begin{cases} -\frac{1}{2}x^2-3x, x\geq 2 \\ |3x-1|, x<2 \end{cases}$

b) Indicate at which real number(s) each function below is discontinuous. Explain your decisions.

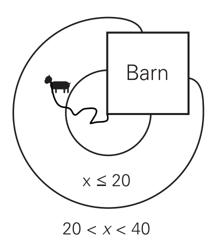


2) For each value *a*, define a value *f* (*a*), so that *f* will be continuous at *a*. If this is not possible, explain why.

$0 \le x \le 40$ $f(x) = x^2 + 3x, a = -2 - 1$ $f(x) = \frac{\quad}{x+2}, a = 2$



3) A goat is kept on a rope tied to one corner of a $35\text{m}\times 35\text{m}$ barn surrounded by grass. The goat eats the grass it can reach. Viewed from above, a patterns forms where the goat has eaten. This pattern takes one shape when the rope is 20 m or less and another when the rope is between 20 m and 40 m.



a) Write the area, *A*, of the grass the goat can reach as a piecewise defined function A(*x*), where *x* is the length of the rope in metres, and .

b) Is this function continuous at *x* = 10? At *x* = 22?

Journal (C14/C5/C23/C25)

4) Use your own words to explain whether each function is continuous at the given value for *a*.

a)

b)

c)

Suggested Resources

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C13 extend the understanding of exponential growth and decay through multiple contexts

Elaboration – Instructional Strategies/Suggestions

C13 Students should revisit exponential and logarithmic functions through applications and to quickly review and have the opportunity to apply the laws and properties.

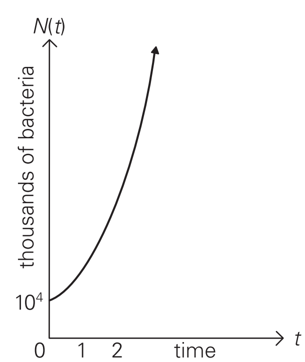
Students should be introduced to the difference between algebraic functions and **transcendental functions**. Not all functions defined in the set of real numbers are algebraic. In an exponential or a logarithmic function, the dependent variable is not determined directly by algebraic operations on the independent variable. For example, the dependent variable y in $y = 2^{2x}$ is determined by operating on the independent variable x and then using that as an exponent and raising 2 to that value. In a logarithmic function the independent variable is operated on, and then the logarithm of the value will determine the dependent value. The independent variable is called the argument of the logarithm. Students should note that trigonometric functions are also transcendental functions, and be able to explain why.

Have students consider the exponential function $N(t) = 10^4 \times 2^t$, which describes how many bacteria are present in a pure nutrient solution as a function of time, t, in hours elapsed.

$N(0) = 10^4$ is the initial quantity of the bacteria

$N(2) = 40\ 000$ is the amount of bacteria after 2 h.

To express time as a function of the number of bacteria present, students can use logarithms to transform the equation.



This logarithmic equation is not the inverse of $N(t)$; it is another way to express the same relationship, just as $y = 2x + 1$ and $x = \frac{y-1}{2}$ both describe the same linear relationship.

From their previous work, students will recall that exponential functions are defined for all real numbers. The above model describes the growth of a particular bacteria as a function of time. As such, its domain does not include negative numbers.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

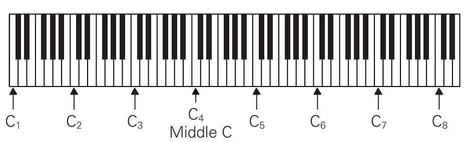
Performance (C13)

1) The decay of the radioactive element actinium is described by the function $S = S_0 \times 2^{0.05t}$ where S is the quantity of substance present at time t, in years, and S_0 is the initial quantity.
   a) What part of the initial quantity is left after 10 years?
   b) After how many years will $S = \frac{1}{2}S_0$? This is the half-life of actinium.
   c) Let S_0 = 100. Sketch a graph of S against t.
   d) Express t as a function of S.

2) The intensity of sound, D, measured in decibels (dB) is given by $D = 10\log\left(\frac{I}{10^{-16}}\right)$ where I is the power of the sound in watts per square centimetre (W/cm^2) and 10^{-16} W/cm^2.
   a) Find the number of decibels of a 10^{-13}W/cm^2 whisper.
   b) Find the number of decibels in a normal conversation of W/cm^2.
   c) Find the number of W/cm^2 of the orchestra section seated in front of the brass, measured at 107 dB.
   d) How many times more powerful is a sound of 47 dB than a sound of 42 dB?

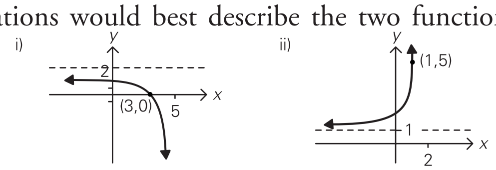
3.16×10^{-10} $g(x) = |x|$

3) The relative frequencies of the C-notes on a grand piano (C_1, C_2, C_3, C_4, C_5, C_6, C_7, and C_8) are pictured on the right. These consecutive notes are one octave apart, which means the frequency will double from one C-note to the next.



   a) If the frequency of middle C, (C_4) is 261.6 cycles per second, and the frequency of C_5 is 523.2 cycles per second, find the frequencies of the other C-notes.
   b) Even though this is a discrete function, you can model it using a continuous explicit function. Do so. Write a function that can be used to generate these notes.
   c) Sketch the general shape of a grand piano and describe how its shape is related to the curve in the graph.

4) a) What equations would best describe the two functions pictured by these graphs?



   b) State the inverse of these equations and draw their graphs.

5) Given $f(x) = -\log_3(2x-1)+5$.
   a) Describe f as a composition of transformations of $y = \log_3 x$.
   b) State its domain, range, and the equation of f^{-1}.
   c) If , predict what the graph of $g(f(x))$ would look like. Check your prediction using a graphing calculator.

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions, and their graphs

C5 use tables and graphs as tools to interpret expressions

C12 demonstrate an understanding for the conceptual foundations of limit, the area under the curve, the rate of change, and the slope of the tangent line, and their applications

B6 determine and apply the derivative of a function

A2 determine, describe, and apply the value for "*e*"

C13 extend the understanding of exponential growth and decay through multiple contexts

Elaboration – Instructional Strategies/Suggestions

C14/C5/C12/B6/C13 As students have seen, exponential functions are used to model a great many situations that involve both growth and decay. If they want to use these models to find how fast these things are changing, students need to know how to find derivatives of exponential functions.

❑ Have students explore the graphs of $y = 2^x$ and $y = 3^x$. Have them consider and discuss the slope of the tangent lines drawn to these curves at various points. Have them respond to the following:
- is the slope ever zero?
- when is the slope positive, when is it negative?
- does the slope get larger or smaller as x increases?
- sketch a rough graph of the slope function for both $y = 2^x$ and $y = 3^x$
- check these using graphing technology (enter $y_1 = 2^x$ and

Have students make conjectures about the derivative of $y = 2^x$. At first, they might conjecture that the derivative of 2^x is 2^x but they should refine this to something like 0.7×2^x. Have students test this using $y = 3^x$. They should be able to generalize

that if $y = b^x$, then . They should verify this conjecture using values like

4, 10, and 1.5 for b. In each case, they would have to estimate the value of k.

B6/A2 Students can use properties of exponents to justify the conjecture that the derivative of b^x is a constant multiple of b^x. Have them rewrite the approximate slope function of b^x in a different form.

$$\frac{b^{x+h}-b}{h}=\frac{b^x\bullet b^h-b^x}{h}$$ using the exponent rule $b^{x+h}=b^x$ C b^h

$$=b^x\bullet\frac{b^h-1}{h}$$ taking out the common factor b^x

The second factor in this expression does not contain x. This suggests that the slope function of b^x is $b^x \times k$, where k is a constant. They can calculate an approximate

value of k by substituting 0.01 for h. This tells them that .

When $b = 2$, k is roughly 0.7 and when $b = 3$, k is roughly 1.1. Some students will see that it looks as if there should be a value of b between 2 and 3 for which $k = 1$. They might conclude that for this base b, the function b^x and its derivative will be the same.

Have students continue their search to find the value of b for which the graphs of b^x and its slope function coincide.

... continued

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

Activity **(C14/C12/A2/C13)**

1) a) Calculate the value of $ 4000 in Canada Savings Bonds, at 8% interest compounded annually, after 5 years (sometimes written 5*a*). Use the formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ to calculate the value, *A*, of money invested, *P*, at an annual interest rate, *r*, compounded *n* times annually for *t* years. (Enter *r* into the formula as a decimal.) What would the bonds be worth if the interest were compounded monthly?

   b) One bank offers this bargain: “100% interest paid annually, compounded as often as you want during the year.” Predict how much money you would have at the end of the year if you invested $1. Now use the formula to calculate how much you would have if the interest were compounded monthly, weekly, daily, and hourly.

   c) Judging from your results in part 2, what do you think is the most money you could earn, regardless of how often the interest is compounded?

   d) Evaluate $\left(1+\frac{1}{n}\right)^{n}$ as $n \to \infty$, to a number of decimal places. (You might want to use the TABLE feature on your graphing calculator.)

   e) Explain why you would use the formula $A = Pe^{rt}$ to calculate continuously compounded interest.

   f) Find the value of $100 000 compounded continuously for ten years, at 6% interest rate.

2) Show that $(1 + x)^5$ fits the generalization for $(1 + x)^n$, which is:

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \ldots$$

   a) Substitute $\frac{1}{k}$ for *x*, and *k* for *n*, in the binomial expansion.

   b) As *k* becomes very large, what happens to the value of $\frac{1}{k}$? If it becomes very small, simplify your expression from (a).

   c) Evaluate the first 10 terms of the simplified expression you obtained in (b).

   d) Compare the value you obtained in (c) to the Euler number. To how many decimal places do you agree?

Performance **(C13)**

3) A manufacturer of lawn mowers determines that he will use this function to quote the unit wholesale price for his most popular product: $P(x) = 600 - 0.5e^{0.004x}$, where *x* is the number of lawn mowers ordered.

   a) What will the price per lawn mower be if a company orders 1000?

   b) One company orders 500 lawn mowers while another company orders 1500. What will be the difference in unit price of these two orders?

   c) If the unit price quoted is $280.47, how many units were ordered?

Suggested Resources

Barnes, M. *Investigating Change: An Introduction to Calculus for Australian Schools.* Carlton Victoria: Curriculum Corporation, 1993.

Developing a Function Toolkit II

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions, and their graphs

C5 use tables and graphs as tools to interpret expressions

C12 demonstrate an understanding for the conceptual foundations of limit, the area under the curve, the rate of change, and the slope of the tangent line, and their applications

B6 determine and apply the derivative of a function

A2 determine, describe, and apply the value for "*e*"

C13 extend an understanding for exponential growth and decay through multiple contexts

Elaboration – Instructional Strategies/Suggestions

... continued

C14/C5/C12/B6/A2/C13 After a few trials, students will discover that the "special number", the base for which the function and its derivative are identical, is about 2.7.

This number is an important mathematical constant which has been given the name of *e*. The methods students have been using to find *e* are not very precise, so you cannot expect your answer to be accurate to more than one decimal place.

Like , *e* is an irrational number, approximately equal to 2.71828 ...

B6/A2 If *e* is the base of an exponential function, differentiation is remarkably simple. Here is the new differentiation rule:

If $y = e^x$, then .

If *e* is the base of an exponential function then $\log_e$ (written *ln*) is the corresponding logarithmic expression. It is read as 'natural log'. Using growth and decay formulas like the functions $y = Pe^{kx}$ and $y = P\ln(kx)$ provides the opportunity to use the special base *e*, sometimes called Euler's number.

C13/A2 Exponential functions are often used to describe how things grow. Doing activity (1) on the previous page will help students understand how $A = Pe^{rt}$ (another form of $y = Pe^{kx}$) can be used to calculate the growth of money compounded continuously. The following problem exemplifies how *e* and *ln* could be used:

Problem: If I invest $1000 when my first child is born, and let it sit earning interest at the rate of 9% per year, compounded continuously, when will my child have $10 000 ?

Solution:

So, it would take about 25.5 years.

Students should be aware of the connection between sequences described by recursion formulas and exponential functions. For example, if given $t_1 = 100$,

, have students generate the sequence, and write the exponential function that would generate the sequence.

Developing a Function Toolkit II

Worthwhile Tasks for Instruction and/or Assessment

... continued

Performance **(C14/C5/A2)**

4) a) Graph $f(x) = e^x$.
   b) State its domain, range, and the equation of its inverse.
   c) Explain why f has no zero.
   d) What point does it share with all other exponential functions of the form $y = a^x$? Why do they all share this point?
   e) Visualize and describe the relationship between $g(x) = e^{x+1} - 3$ and f.

5) a) Given $f(x) = |x|, g(x) = \sqrt{x}$ and $h(x) = \ln x$. Predict what the graph of each of the following functions will look like. Check your predictions on a graphing calculator. State the domain and range for each function.
   i) $f(h(x))$ ii) $h(f(x))$
   iii) $g(h(x))$ iv) $h(g(x))$ Hint for (iv): $ln\sqrt{x} = \frac{1}{2} ln x$.
   b) Explain why the domain and range of $ln\ x$ were affected by each composition.

6) a) Prove that $\ln_b a = \frac{1}{\ln_a b}$.
   b) Prove that $a^{\ln b} = b^{\ln a}$.

7) a) Ask students why they would use the formula $A = Pe^{rt}$ to calculate interest that compounds continuously.
   b) Find the value of \$12 000 compounded continuously for 4 years at 7%.
   c) Create a problem where students would have to determine the value of P using the above formula.
   d) Create a problem where students would have to determine the value of r, or t, using the above formula.

8) Greasy John created this formula $L = r^{-1} \times ln\ (rRC^{-1} + 1)$ to predict the lasting time for the oil discovered on his property. C is the current consumption, R is the size of the resource, and r is the rate of consumption. Current consumption is about 12×10^6 barrels per year, and the remaining supply is about 1572×10^9 barrels.
   a) If the current consumption is growing at a rate of 3%, how much longer will the supply last?
   b) To make the supply last for 40 years, what must be the growth rate?

Journal **(B6/C12)**

9) A common mistake is to write the derivative of e^x as xe^{x-1}.
   a) Explain why this is wrong. What confusion do you think might lead people to do this?
   b) Sketch a graph of the function xe^{x-1} and compare it with the graph of e^x.
   c) What is the equation of the tangent line to $y = e^x$ at $x = 5$? Check using technology?

Suggested Resources

Unit 4
Complex Numbers
(10-15 hours)

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

A6 explain the connections between real and complex numbers

C27 represent complex numbers in a variety of ways

Elaboration—Instructional Strategies/Suggestions

A6/C27 Complex numbers were first encountered by students in their study of the roots of polynomial functions, in particular, quadratics, when exploring the nature of the roots. When examining graphs students noted that if there was no *x*-intercept on the graph, the solution of the quadratic equation included the square root of a negative number. For example, the graph of $y = x^2 + 9$ does not intersect the *x*-axis, which implies no real roots. However, when solving the equation the solution results in . Students then interpret this as a pair of roots that are not real, but imaginary.

An imaginary number is defined as a square root of a negative real number. For example, is an imaginary number. The imaginary number *i* is defined as . So, . The symbol *i* (first letter of 'imaginary') means "a number you can square to get –1 for an answer."

Thus:

$$i^2 = -1$$

or

$$i = \sqrt{-1}$$

if *x* is a non-negative real number, then

$$\sqrt{-x} = i\sqrt{x}$$

A complex number is a number in the form *a* + *bi*, where *a* is the real part and the real number *b* is the coefficient of the imaginary part. The word "complex" means composed of parts. This is similar to how "complex" was used when complex rational expressions were studied.

The form *a* + *bi* is called the rectangular form of a complex number. Students may see *a* + *bi* written as an ordered pair (*a, b*), especially on calculators and computers. Later in the unit students will study complex numbers to see how they can be expressed in polar form (see p. 170). The complex numbers *a* + *bi* and *a* – *bi* are called complex **conjugates** of each other (see next two-page spread).

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Mental Math (**C27**)

1) Write in terms of i in simplest form.
   a) $\sqrt{-15}$
   b)
   c)

Pencil and Paper (**C27**)

2) Explain why the following polynomial equations have imaginary roots:
   a)
   b)

3) Represent these imaginary numbers in the form of $a + bi$.
   a)
   b)
   c)

Journal (**A6**)

[illegible] $x - 4 = 0$

4) Explain why
   a) $a + bi$, b 0, is not a real number.
   b) the number 'three' is a complex number.
   c) is not a real number.

Suggested Resources

Complex Numbers

Outcomes

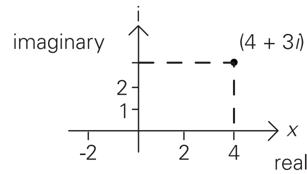
SCO: In this course, students will be expected to

C28 construct and examine graphs in the complex and polar planes

C27 represent complex numbers in a variety of ways

B9 apply operations on complex numbers both in rectangular and polar form

Elaboration—Instructional Strategies/Suggestions



C28/C27 Complex numbers can be plotted on a complex plane, called the Argand plane, or an Argand diagram. Note that the vertical axis is called the imaginary axis and the horizontal axis is the real axis. So, to plot 4 + 3*i* (sometimes noted as (4, 3)), move 4 along the horizontal axis from the origin and up 3 parallel to the vertical axis. The distance a point is from the origin is called the absolute value, or modulus. Thus on the above diagram the distance from 4 + 3*i*, or (4, 3) to the origin would be symbolized and calculated as follows:

if $z = 4 + 3i$, then its modulus, denoted can be calculated:

B9 Operations on complex numbers are analogous to those performed with linear binomials. For example, if and , find

1) 2) 3)

When adding and subtracting, the real parts and imaginary parts of the complex numbers are added and subtracted (as were the constant terms and coefficients of variables):

1) 2)

$$z_1 - z_2$$
$$= (5-3i)-(2+5i)$$
$$= 3-8i$$

When multiplying, the distributive property is used (as with the multiplication of algebraic terms).

$$z_1 \bullet z_2$$
$$= (5-3i)(2+5i)$$
$$= 10+25i-6i-15i^2$$
$$\text{remember } i^2 = -1$$
$$= 10+19i+15$$
$$= 25+19i$$

Students should be encouraged to investigate the product of complex conjugates ((2 + 3*i*) and (2 – 3*i*) are complex conjugates) to determine that their product is always a real number. For example,

$$(2+3i)(2-3i) = 4-9i^2$$
$$= 4+9$$
$$= 13$$

Students should be expected to prove this by generalizing the situation.

... continued

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**C28/C27**)

1) Graph the following numbers.
   a) $3 - 2i$ c) $4i$
   b) $-1 - 8i$ d) the sum of 7 and $\sqrt{-25}$

Pencil and Paper (**C27/B9**)

2) Perform the indicated operations given $z_1 = 4 + 8i$ and $z_2 = -3 - i$
   a) c)
   b) d)

3) a) Find the product of $a + bi$ and its complex conjugate.
   b) Explain why the answer you just got has to be real.
   c) Use the results from above to factor the sum of 2 squares $x^2 + y^2$.
   d) Factor
      i) $x^2 + 4$
      ii) $3x^2 + 5$

4) Find a quadratic equation with real-number coefficients if one root is $2 + 3i$.

Performance (**C28/C27/B9**)

5) a) Plot $3 - 2i$ and $7 + 5i$ and call them A and B respectively.
   b) Find $A + B$ and call the sum C, then plot C.
   c) Draw vectors from the origin O, to A and then to B.
   d) Imagine the vector sliding along the vector so that O maps to B. Where will A map? Explain.
   e) If slides along so that , where will B map? Explain.
   f) What conjecture can you make?
   g) Test your conjecture with $(-2 + 3i) + (1 - 5i)$.
   h) Explain how using a graph would visually show subtraction. Use diagrams in your explanation.

... continued

Suggested Resources

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

B9 apply operations on complex numbers both in rectangular and polar form

Elaboration—Instructional Strategies/Suggestions

... continued

B9 The product of complex conjugates is used in the dividing process:

(multiply by 1, using the conjugate of the denominator.)

By definition, $i = \sqrt{-1}$, so, $i^2 = -1$. Have students examine other powers of i to find a pattern:

Anytime the exponent is a multiple of 4, the power is 1, so $i^{76} = 1$. Remember from earlier study that a number is divisible by 4 if the number formed by the last two digits is divisible by 4. When the multiple is not 4, break the exponent down so that part of it is a multiple of 4:

$i^{39} = i^{36} g i^{3} = 1 g i^{3} = i^{3} = -i$,

or break it down into a base of i^2:

$i^{39} = (i^2)^{19} \cdot i = (-1)^{19} \cdot i = -1i = -i$.

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

... continued

Performance **(C27/C28/B9)**

6) Suppose and .

a) Find .

b) Find and .

c) Plot the points and on the complex plane and then connect each point to the origin (draw their position vectors). Calculate the angles made in each case between the position vectors and the horizontal axis.

d) How are the angle measures related to one another?

e) Describe in words how you would find the location of z_3z_4 given $z_3 = 2 - i$, and $z_4 = -5 + 4i$.

f) Repeat all of question (6) above, but substitute the quotient for the product z_1z_2.

7) Graph i, i^2, i^3, i^4 on the same complex plane. Describe what seems to be happening on the graph each time the power of i is increased by 1.

8) Suppose $z = 4 + 3i$.

a) Find iz.

b) Plot z and iz on the same complex plane. Connect each point to the origin.

c) Does ? Explain.

d) Explain what transformation happens to z when multiplied by i.

9) What is the relationship between the modulus of a complex number and the modulus of its conjugate? Explain.

10) Express in the form $a + bi$, where a and $b \in R$.

a) c)

b) d)

11) Given $z = \cos 2 + i \sin 2$, and $w = \cos 2 - i \sin 2$, prove the following

a) $z + w = 2 \cos 2$ c) $z^2 = \cos 22 + i \sin 2$

b) $zw = 1$ d)

12) Find the number b such that $\left|\dfrac{2-3i\sqrt{5}}{6+bi}\right| = 2$.

Suggested Resources

Complex Numbers

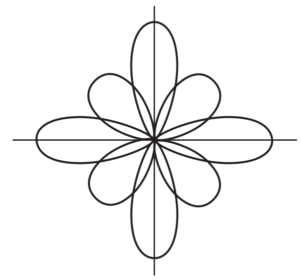
Outcomes

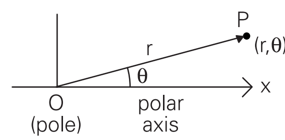
SCO: In this course, students will be expected to

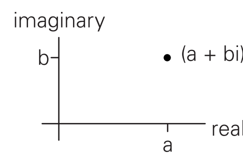
C28 construct and examine graphs in the complex and polar planes

Elaboration—Instructional Strategies/Suggestions

C28 Students should now be comfortable locating points (*a*, *b*) which are the coordinates for the complex number $a + bi$, on the Argand plane. An alternate coordinate system (the polar coordinate system) is useful for multiplication and powers of complex numbers. Students will discover that this is a coordinate system where surprisingly simple equations give very interesting graphs. Students should have an intuitive understanding of how a point in the rectangular system can be represented by a point in the polar system and that this later would help them to understand the connections between complex numbers in rectangular form ($a + bi$) and complex numbers in polar form (see p. 171).

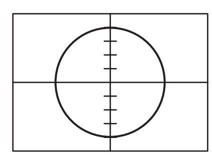


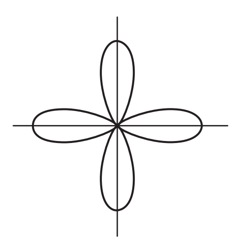




In the polar coordinate system, students should call the origin the pole *O*, and the fixed horizontal ray *OX* is called the polar axis. The position of a point *P* in the plane, given by the polar coordinates (r,θ), has a distance from the origin on the vector *OP*. This is the image of the vector on ray *OX*, where $X = r$, after a rotation of (in radians) about centre *O*.

❐ Ask students to create a table, then to graph the polar equation $r = 4$. **Solution:** This is the set of points 4 units from the origin—a circle with radius 4. There is no in this equation, so no matter what your angle measure is, the point will be 4 units from the origin.
Now, ask students to complete a table and plot the polar equation $r = 4\cos$. This time, as increases, the length *r* varies according to the calculation 4 cos . Students can use their graphing calculators, in polar mode (see the next two-page spread), and with PolarGC turned on in their Format menu (for tracing purposes).





... continued

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Performance (C28)

1) The coordinates and identify the same point. Give two additional sets of polar coordinates that also name this point.

2) a) Complete a table, like the one below, for the curve $r = 3 \sin$ for . Graph by hand, then check using a calculator.

	0°	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°
r												

60°	65°	70°	75°	80°	85°	90°	95°	100°	105°	110°	115°	...
r												

b) If instead theta was given the range , how would the table change (describe an appropriate table)? How would you complete the settings for your calculator in order to see the same graph?

3) Using a graphing calculator:

a) Graph the spiral using .

$\left(\frac{\ }{6},2\right),\left(\frac{\ }{6},-2\right)$

b) Graph the equation using . How does this compare to the original graph? Explain why this happens.

c) Graph the equation using . How does this compare to the original graph? What transformation has been performed?

d) Graph the equation using . How does this compare to the original graph? What transformation has been performed?

Suggested Resources

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

C28 construct and examine graphs in the complex and polar planes

Elaboration—Instructional Strategies/Suggestions

... continued

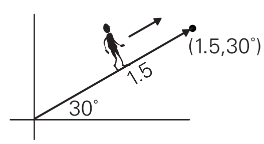
C28 If using a TI-83:

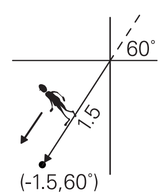
Set mode to polar, set from 0 to 360°, set step to 5 and use *x*-values from –15 to 15, Zoom 5 to square the axes.

Example 2: Have students graph the polar equation with .

Solution: Ask students to graph this equation by hand first, then check using the calculator. Give the -values, and ask them to complete the table (to see table on calculator, then set to PolarGC in Format menu), then plot the points. To plot a point given in polar coordinates, have students imagine themselves standing at the origin. Get them to stand at the origin, rotate through the angle (for example, 30° counterclockwise from the positive ray of the horizontal axis). Then have them imagine walking straight out from the origin and placing a point at distance *r*. Students should know that this distance (*r*) is called the **modulus** of the complex number. If *r* is positive, walk forward. If *r* is negative, walk backward. As increases from 0° degree to 360°, use this technique to locate points on the curve and then connect them with a smooth curve. What is the role of the coefficient 3 in the equation?

	r
0°	3
10°	2.81
20°	2.3
30°	1.5
40°	0.52
50°	–52
...	...





Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Activity **(C28)**

1) Take another good look at the equation, window, and graph for the four-petal rose graph r = 4 cos , on p. 166. Why are there four petals? Why does the graph have the rose or flower shape? Concentrate on the connection between the trace numbers displayed on your calculator and the points. This beautiful graph comes from an equation that can be generalized as . In this activity, you will investigate Rose Curves, their symmetries, and the relationship between the number of petals and the value of n.
   a) Graph the family of curves with n = 1, 2, 3, 4, 5, and 6. Write statements that describe the curves for even n and odd n.
   b) Graph the family of curves r = 3 sin n *2* with n = 1, 2, 3, 4, 5, and 6. Write statements that describe the curves for even n and odd n. How do these differ from the curves graphed in (a)?
   c) Find a way to graph a rose with only two petals. Explain why your method works.
   d) Find a connection between the polar graph and the associated function graph . Can you look at the graph of and predict the shape and number of petals in the polar graph? Explain.

2θ $r = a\cos n\theta$ $y = a\cos n\theta$

Suggested Resources

Murdock, Jerald, et al, *Advanced Algebra Through Data Exploration: A graphing Calculator Approach*, Key Curriculum Press, 1998.

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

A7 translate between polar and rectangular representations

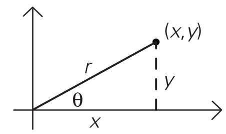
C27 represent complex numbers in a variety of ways

Elaboration—Instructional Strategies/Suggestions

A7/C27 Until now, students have used $z = a + bi$ to represent the complex number z. This is called the Cartesian or rectangular form of z. Students will now learn that z can be expressed by using its modulus and its argument . Ask students to apply the Pythagorean theorem to the figure below to find r and .

or .

Also, or .



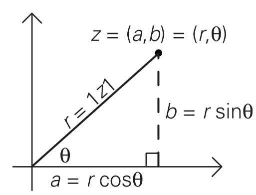
Since r represents a length (modulus), the distance $\sqrt{x^2 + y^2}$ will be r. The actual value of depends on the quadrant in which the point (x, y) is located. These two equations allow students to convert between polar and rectangular forms.

A point representing the complex number $z = a+bi$ can be given either in rectangular coordinates (a, b) or in polar coordinates .

rectangular form:

$z = a+bi$

since



polar form:

$$z = r\cos\theta + (r\sin\theta)i$$
$$= r(\cos\theta + i\sin\theta)$$

or, $z = rcis\theta$ (pronounced "r siss theta.")

For example, 3 cis 20° is the abbreviation for $3(\cos 20^{\circ} + i\sin 20^{\circ})$.

Many calculators have keys that enable students to convert from rectangular form to polar form and vice versa. For example, using the TI-83 in degree mode:

Enter a complex number in rectangular form.

On the home screen enter

To convert to polar form:

Use MATH menu, cursor to CPX.

Hit 7:Polar enter . This means 5cis36.87°.

Students are expected to do this without technology: Convert $4 + 3i$ to polar form. To get the modulus:

$$r = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$$

to get the angle measure:

C onvert 2cis 47° to rectangular form:

$$2(\cos 47^{\circ} + i\sin 47^{\circ})$$
$$= (2\cos 47^{\circ}) + (2\sin 47^{\circ})i$$
$$= 1.3639 + 1.4627i$$

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**A7/C27**)

1) Express each complex number in polar form.

   a) $1 + i$ c) -3

   b) i d) $-\sqrt{3} - i$

2) Express each complex number in rectangular form.

   a) 5cis90°   c) 4cis45°

   b)   d)

3) Express $-2 + 2i$ in polar form. Sketch a graph as part of your solution.
4) Express in polar form, using the smallest non-negative value for the argument.

   a)   d)

   b) $1 - i$ e) $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$

   c) $10\sqrt{3} - 10i$ f) $-2 + 2i$

5) Change to rectangular form

   [illegible]

   a)   d)

   b)   e)

   c)   f)

Performance (**A7/C27**)

6) Show using a graph that $-5 - 3i$ and 5.83 cis(–149°) are the same.

Journal (**A7/C27**)

7) What does the expression mean?

Suggested Resources

Complex Numbers

Outcomes

SCO: In this course, students will be expected to

B9 apply operations on complex numbers both in rectangular and polar form

B10 develop and apply DeMoivre's Theorem for powers

Elaboration—Instructional Strategies/Suggestions

B9 When students are asked to multiply complex numbers, they can either perform the operation using rectangular form as discussed on p. 162 or change to polar form and multiply in polar form.

For example, multiply (2+3*i*)(3+*i*). Students could change each factor to polar form:

then multiply $\sqrt{13}\cos 56.3° + i\sin 56.3°$ by $\sqrt{10}\cos 18.4° + i\sin 18.4°$.

By doing an introductory activity like the one on the next page, students should learn that there is a quick way to do this (i.e., multiply the *r*-values and add the arguments).

Students should also be able to convert from polar form to rectangular form. For example,

$$\sqrt{130}\text{cis}74.7° = \sqrt{130}(\cos 74.7° + i\sin 74.7°)$$
$$= \sqrt{130}(0.263873 + i(0.9645574)) \text{ B } 3 + 11i$$

B10 In general then, when squaring a complex number in polar form,

$$\left(\sqrt{a}\text{cis}\theta^2\right)$$
$$= \left(\sqrt{a}\text{cis}\theta\right)\left(\sqrt{a}\text{cis}\theta^2\right)$$
$$= \sqrt{a}^2\text{cis}(\theta + \theta)$$
$$= a\,\text{cis}\,2\theta$$

This generalizes to $(rcis\theta)^2 = r^2 cis2\theta$ and then to $(rcis\theta)^n = r^n cisn\theta$. This is known as **DeMoivre's Theorem** for powers.

Complex Numbers

Worthwhile Tasks for Instruction and/or Assessment

Activity (**B9**)

1) In this activity, students will discover a pattern involving multiplication of complex numbers written in polar form. This important discovery will allow students to easily multiply and divide complex numbers, and raise them to any power.
   a) Multiply each pair of complex numbers and write your answer in $a + bi$ form. (Remember $i^2 = -1$.)
      i) $(2 + 3i)(3 + i)$
      ii) $(1 + 4i)(3 - 2i)$
      iii) $(-1 + 2i)(3 - 4i)$
   b) Convert the first pair of numbers $(2 + 3i)$ and $(3 + i)$ to polar form. Convert the product of these numbers (found in (i) above), to polar form. Do this for each product in part (a). By examining the polar form, find a relationship between the angles of the two factors and the angle of their product. This relationship should be true for all three problems. Describe the relationship between the *r*-values (absolute values) of the factors and the *r*-value of the answer.

Pencil and Paper (**B9/B10**)

2) Perform the indicated operations. You may want to change some of the numbers to polar form but express your answer in rectangular form.

[illegible] $\cos\theta = \frac{1}{2}(A+B)$

   a)
   b)
   c)
   d) $\sqrt{6}\text{cis}36° \cdot 2\text{cis}54°$
   e) $(2 - 3i)(4 + 5i)$
   f) $4i\,(3 - 7i)(2 + 5i)^2$

3) Show that $z^3 = 1$ if $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

4) Use DeMoivre's Theorem to evaluate: (answer in rectangular form)
   a) $(-2 - i)^{12}$
   b) $(i - \sqrt{5})^{10}$
   c) $(2\text{cis}240°)^7$

Performance (**B9**)

5) If $A = \cos 2 + i \sin 2$ and $B = \cos 2 - i \sin 2$, show:
   a)
   b) $\sin\theta = \frac{1}{2i}(A - B)$
   c) $AB = 1$
   d)
   e)
   f) $A^n - B^n = 2i\sin(n\theta)$

6) If z_1 and z_2 are two complex numbers, where $z_1 + z_2 = 3 + 3i$ and $z_1 \times z_2 = -2 + 6i$, find z_1 and z_2.

Suggested Resources

Appendix A:
Assessing and Evaluating Student Learning

Assessing and Evaluating Student Learning

In recent years there have been calls for change in the practices used to assess and evaluate students' progress. Many factors have set the demands for change in motion, including the following:

- ***new expectations for mathematics education as outlined in*** **Curriculum and Evaluation Standards for School Mathematics** (NCTM 1989)

The *Curriculum Standards* provide educators with specific information about what students should be able to do in mathematics. These expectations go far beyond learning a list of mathematical facts; instead, they emphasize such competencies as creative and critical thinking, problem solving, working collaboratively, and the ability to manage one's own learning. Students are expected to be able to communicate mathematically, to solve and create problems, to use concepts to solve real-world applications, to integrate mathematics across disciplines, and to connect strands of mathematics. For the most part, assessments used in the past have not addressed these expectations. New approaches to assessment are needed if we are to address the expectations set out in *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989).

- *understanding the bonds linking teaching, learning, and assessment*

Much of our understanding of learning has been based on a theory that viewed learning as the accumulation of discrete skills. Cognitive views of learning call for an active, constructive approach in which learners gain understanding by building their own knowledge and developing connections between facts and concepts. Problem solving and reasoning become the emphases rather than the acquisition of isolated facts. Conventional testing, which includes multiple choice or having students answer questions to determine if they can recall the type of question and the procedure to be used, provides a window into only one aspect of what a student has learned. Assessments that require students to solve problems, demonstrate skills, create products, and create portfolios of work reveal more about the student's understanding and reasoning of mathematics. If students are expected to develop reasoning and problem-solving competencies, then teaching must reflect such, and in turn, assessment must reflect what is valued in teaching and learning. Feedback from assessment directly affects learning. The development of problem-solving, and higher, order thinking skills will become a realization only if assessment practices are in alignment with these expectations.

- ***limitations of the traditional methods used to determine student achievement***

Do traditional methods of assessment provide the student with information on how to improve performance? We need to develop methods of assessment that provide us with accurate information about students' academic achievement and information to guide teachers in decision making to improve both learning and teaching.

What Is Assessment?

Assessment is the systematic process of gathering information on student learning. Assessment allows teachers to communicate to students what is really valued—what is worth learning, how it should be learned, what elements of quality are considered most important, and how well students are expected to perform. In order for teachers to assess student learning in a mathematics curriculum that emphasizes applications and problem solving, they need to employ strategies that recognize the reasoning involved in the process as well as in the product. *Assessment Standards for School Mathematics* (NCTM 1995, p. 3) describes assessment practices that enable teachers to gather evidence about a student's knowledge of, ability to use, and disposition toward, mathematics and of making inferences from that evidence for a variety of purposes.

Assessment can be informal or formal. Informal assessment occurs while instruction is occurring. It is a mind-set, a daily activity that helps the teacher answer the question, Is what is taught being learned? Its primary purpose is to collect information about the instructional needs of students so that the teacher can make decisions to improve instructional strategies. For many teachers, the strategy of making annotated comments about a student's work is part of informal assessment. Assessment must do more than determine a score for the student. It should do more than portray a level of performance. It should direct teachers' communication and actions. Assessment must anticipate subsequent action.

Formal assessment requires the organization of an assessment event. In the past, mathematics teachers may have restricted these events to quizzes, tests, or exams. As the outcomes for mathematics education broaden, it becomes more obvious that these assessment methods become more limited. Some educators would argue that informal assessment provides better quality information because it is in a context that can be put to immediate use.

Why Should We Assess Student Learning?

We should assess student learning in order to

- improve instruction by identifying successful instructional strategies
- identify and address specific sources of the students' misunderstandings
- inform the students about their strengths in skills, knowledge, and learning strategies
- inform parents of their child's progress so that they can provide more effective support
- determine the level of achievement for each outcome

As an integral and ongoing part of the learning process, assessment must give each student optimal opportunity to demonstrate what he/

she knows and is able to do. It is essential, therefore, that teachers develop a repertoire of assessment strategies.

Assessment Strategies

Some assessment strategies that teachers may employ include the following.

Documenting classroom behaviours

In the past teachers have generally made observations of students' persistence, systematic working, organization, accuracy, conjecturing, modeling, creativity, and ability to communicate ideas, but often failed to document them. Certainly the ability to manage the documentation played a major part. Recording information signals to the student those behaviours that are truly valued. Teachers should focus on recording only significant events, which are those that represent a typical student's behaviour or a situation where the student demonstrates new understanding or a lack of understanding. Using a class list, teachers can expect to record comments on approximately four students per class. The use of an annotated class list allows the teacher to recognize where students are having difficulties and identify students who may be spectators in the classroom.

Using a portfolio or student journal

Having students assemble on a regular basis responses to various types of tasks is part of an effective assessment scheme. Responding to open-ended questions allows students to explore the bounds and the structure of mathematical categories. As an example, students are given a triangle in which they know two sides or an angle and a side and they are asked to find out everything they know about the triangle. This is preferable to asking students to find a particular side, because it is less prescriptive and allows students to explore the problem in many different ways and gives them the opportunity to use many different procedures and skills. Students should be monitoring their own learning by being asked to reflect and write about questions such as the following:

- What is the most interesting thing you learned in mathematics class this week?
- What do you find difficult to understand?
- How could the teacher improve mathematics instruction?
- Can you identify how the mathematics we are now studying is connected to the real world?

In the portfolio or in a journal, teachers can observe the development of the students' understanding and progress as a problem solver. Students should be doing problems that require varying lengths of time and represent both individual and group effort. What is most important is that teachers discuss with their peers what items are to be part of a meaningful portfolio, and that students also have some input into the assembling of a portfolio.

Projects and investigative reports

Students will have opportunities to do projects at various times throughout the year. For example, they may conduct a survey and do a statistical report, they may do a project by reporting on the contribution of a mathematician, or the project may involve building a complex three-dimensional shape or a set of three-dimensional shapes which relate to each other in some way. Students should also be given investigations in which they learn new mathematical concepts on their own. Excellent materials can be obtained from the National Council of Teachers of Mathematics, including the *Student Math Notes.* (These news bulletins can be downloaded from the Internet.)

Written tests, quizzes, and exams

Some critics allege that written tests are limited to assessing a student's ability to recall and replicate mathematical facts and procedures. Some educators would argue that asking students to solve contrived applications, usually within time limits, provides us with little knowledge of the students' understanding of mathematics.

How might we improve the use of written tests?

- Our challenge is to improve the nature of the questions being asked, so that we are gaining information about the students' understanding and comprehension.
- Tests must be designed so that questions being asked reflect the expectations of the outcomes being addressed.
- One way to do this is to have students construct assessment items for the test. Allowing students to contribute to the test permits them to reflect on what they were learning, and it is a most effective revision strategy.
- Teachers should reflect on the quality of the test being given to students. Are students being asked to evaluate, analyse, and synthesize information, or are they simply being asked to recall isolated facts from memory? Teachers should develop a table of specifications when planning their tests.
- In assessing students, teachers have a professional obligation to ensure that the assessment reflects those skills and behaviours that are truly valued. Good assessment goes hand-in-hand with good instruction and together they promote student achievement.