Atlantic Canada Mathematics Curriculum

New Brunswick
Department of Education
Educational Programs & Services Branch

Trigonometry and 3-Space 121/122

(Implementation Edition)

CURRICULUM

2003

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Title Code (841970)

Acknowledgements

The departments of education of New Brunswick, Newfoundland and Labrador, Nova Scotia, and Prince Edward Island gratefully acknowledge the contributions of the following groups and individuals toward the development of this *Trigonometry and 3-Space 121/122* curriculum guide.

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- The Provincial Curriculum Working Group, comprising teachers and other educators in Nova Scotia, which served as lead province in drafting and revising the document.
- The teachers and other educators and stakeholders across Atlantic Canada who contributed to the development of the *Trigonometry and 3-Space 121/122* curriculum guide.

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I. Background and Rationale

A. Background

Mathematics curriculum reform in Atlantic Canada is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in an increasingly technological society. Curriculum reform has been motivated by a desire to ensure that students in Atlantic Canada benefit from world-class curriculum and instruction in mathematics as a significant part of their learning experiences.

The *Foundation for the Atlantic Canada Mathematics Curriculum* (1996) firmly establishes the *Curriculum and Evaluation Standards for School Mathematics* (1989) of the National Council of Teachers of Mathematics (NCTM) as a guiding beacon for pursuing this vision, which embraces the principles of students learning to value and become active “doers” of mathematics and advocates a curriculum which focusses on the unifying ideas of mathematical problem solving, communication, reasoning, and connections. These principles and unifying ideas are reaffirmed with the publication of NCTM’s *Principles and Standards for School Mathematics* (2000). The *Foundation for the Atlantic Canada Mathematics Curriculum* establishes a framework for the development of detailed grade-level documents describing mathematics curriculum and guiding instruction.

Mathematics curriculum development has taken place under the auspices of the Atlantic Provinces Education Foundation (APEF), an organization sponsored and managed by the governments of the four Atlantic Provinces. APEF has brought together teachers with department of education officials to co-operatively plan and execute the development of curricula in mathematics, science, language arts, and other curricular areas. Each of these curriculum efforts has been aimed at producing a program that would ultimately support the Essential Graduation Learnings (EGLs), also developed regionally. These EGLs and the contribution of the mathematics curriculum to their achievement are presented in the “Outcomes” section of the mathematics foundation document.

B. Rationale

The *Foundation for the Atlantic Canada Mathematics Curriculum* provides an overview of the philosophy and goals of the mathematics curriculum, presenting broad curriculum outcomes and addressing a variety of issues with respect to the learning and teaching of mathematics. This curriculum guide is one of several which provide greater specificity and clarity for the classroom teacher. The *Foundation for the Atlantic Canada Mathematics Curriculum* describes the mathematics curriculum in terms of a series of outcomes—General Curriculum Outcomes (GCOs), which relate to subject strands, and Key-Stage Curriculum Outcomes (KSCOs), which articulate the GCOs further for the end of grades 3, 6, 9, and 12. This guide builds on the structure introduced in the foundation document, by relating Specific Curriculum Outcomes (SCOs) to KSCOs for *Trigonometry and 3-Space 121/122*. Figure 1 further clarifies the outcome structure.

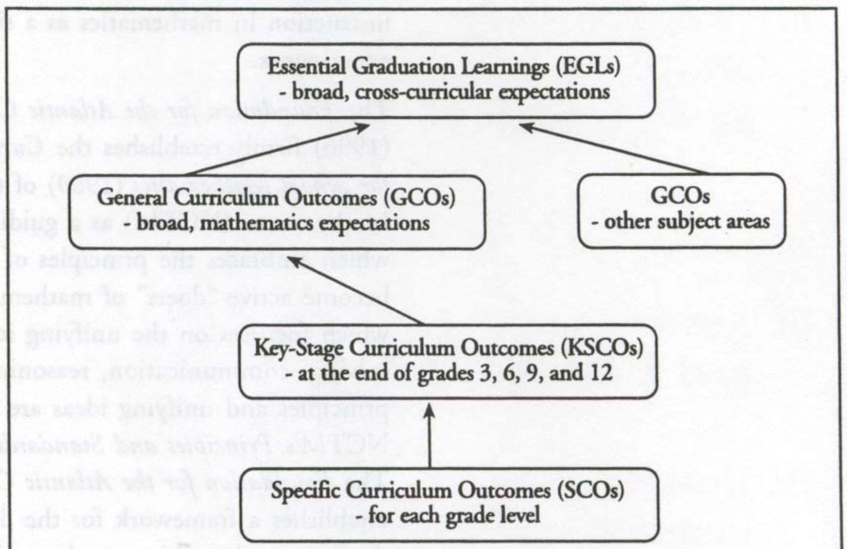


Figure 1: Outcome Framework

This mathematics guide is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice, including the following: (i) mathematics learning is an active and constructive process; (ii) learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates; (iii) learning is most likely when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and nurtures positive attitudes and sustained effort; (iv) learning is most effective when standards of expectation are made clear and assessment and feedback are ongoing; and (v) learners benefit, both socially and intellectually, from a variety of learning experiences, both independent and in collaboration with others.

II. Program Design and Components

A. Program Organization

As indicated previously, the mathematics curriculum is designed to support the Atlantic Canada Essential Graduation Learnings (EGLs). The curriculum is designed to significantly contribute to students meeting each of the six EGLs, with the communication and problem-solving EGLs relating particularly well with the curriculum's unifying ideas. (See the "Outcomes" section of the *Foundation for the Atlantic Canada Mathematics Curriculum*.) The foundation document then goes on to present student outcomes at key stages of the student's school experience.

This curriculum guide presents specific curriculum outcomes for *Trigonometry and 3-Space 121/122*. As illustrated in Figure 2, these outcomes represent the step-by-step means by which students work toward accomplishing the key-stage curriculum outcomes, the general curriculum outcomes, and, ultimately, the essential graduation learnings.

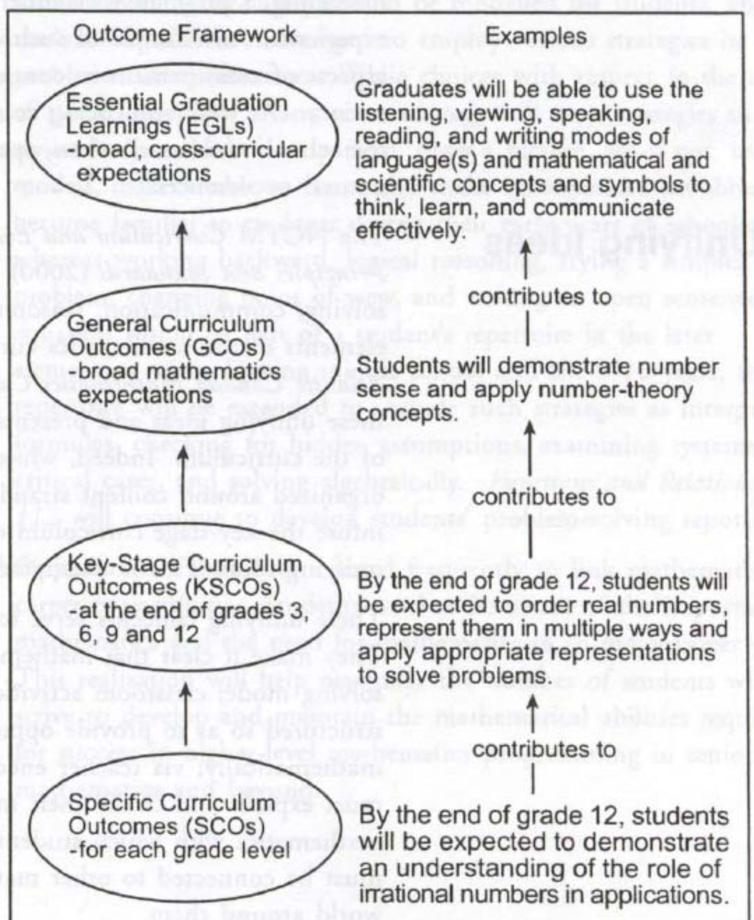


Figure 2: Examples of Outcomes

It is important to emphasize that the initial presentation of the specific curriculum outcomes for this course (pp. 17-31) follows the outcome structure established in the *Foundation for the Atlantic Canada Mathematics Curriculum* and does not represent a natural teaching sequence. In *Trigonometry and 3-Spaces 121/122*, however, a suggested teaching order for specific curriculum outcomes has been given within a sequence of four topics or units (i.e., The Algebra of 3-Space; Trigonometric Functions; Trigonometric Equations and Identities; and Trigonometry-Further Topics). While the units are presented with a specific teaching sequence in mind, some flexibility exists as to the ordering of units within the course. It is expected that teachers will make individual decisions as to what sequence of topics will best suit their classes. In most instances, this will occur in consultation with fellow staff members, department heads, and/or district level personnel.

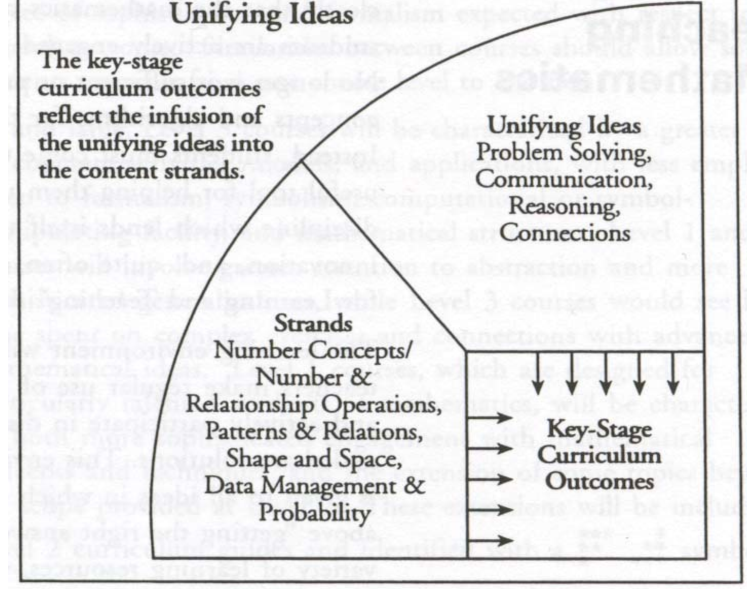
Decisions on sequencing will depend on a number of factors, including the nature and interests of the students themselves. For instance, what might serve well as a "kickoff" topic for one group of students might be less effective in that role with a second group. Another consideration with respect to sequencing will be co-ordinating the mathematics program with other aspects of the students' school experience. An example of such co-ordination would be studying aspects of measurement in connection with appropriate topics in science. As well, sequencing could be influenced by events outside of the school, such as elections, special community celebrations, or natural occurrences.

B. Unifying Ideas

The NCTM *Curriculum and Evaluation Standards* (1989) and *Principles and Standards* (2000) establish mathematical problem solving, communication, reasoning, and connections as central elements of the mathematics curriculum. The *Foundation for the Atlantic Canada Mathematics Curriculum* (pp. 7-11) further emphasizes these unifying ideas and presents them as being integral to all aspects of the curriculum. Indeed, while the general curriculum outcomes are organized around content strands, every opportunity has been taken to infuse the key-stage curriculum outcomes with one or more of the unifying ideas. This is illustrated in Figure 3.

These unifying concepts serve to link the content to methodology. They make it clear that mathematics is to be taught in a problem-solving mode; classroom activities and student assignments must be structured so as to provide opportunities for students to communicate mathematically; via teacher encouragement and questioning, students must explain and clarify their mathematical reasoning; and mathematics with which students are involved on a day-to-day basis must be connected to other mathematics, other disciplines, and/or the world around them.

Figure 3 - Integrating Strands and Unifying Ideas



Students will be expected to address routine and/or non-routine mathematical problems on a daily basis. Over time, numerous problem-solving strategies should be modelled for students, and students should be encouraged to employ various strategies in many problem-solving situations. While choices with respect to the timing of the introduction of any given strategy will vary, strategies such as try-and-adjust, look for a pattern, draw a picture, act it out, use models, make a table or chart, and make an organized list should all become familiar to students during their early years of schooling, whereas working backward, logical reasoning, trying a simpler problem, changing point of view, and writing an open sentence or equation would be part of a student's repertoire in the later elementary years. During middle school and the 9/10 years, this repertoire will be extended to include such strategies as interpreting formulas, checking for hidden assumptions, examining systematic or critical cases, and solving algebraically. *Trigonometry and 3-Space 121/122* will continue to develop students' problem-solving reportoires.

Opportunities should be created frequently to link mathematics and career opportunities. Students need to be aware of the importance of mathematics and the need for mathematics in so many career paths. This realization will help maximize the number of students who strive to develop and maintain the mathematical abilities required for success in higher-level mathematics programming in senior high mathematics and beyond.

C. Learning and Teaching Mathematics

The unifying ideas of the mathematics curriculum suggest quite clearly that the mathematics classroom needs to be one in which students are actively engaged each day in the doing of mathematics. No longer is it sufficient or proper to view mathematics as a set of concepts and algorithms for the teacher to transmit to students. Instead, students must come to see mathematics as a vibrant and useful tool for helping them understand their world, and as a discipline which lends itself to multiple strategies, student innovation, and, quite often, multiple solutions. (See the "Contexts for Learning and Teaching" section of the foundation document.)

The learning environment will be one in which students and teachers make regular use of manipulative materials and technology, and actively participate in discourse and conjecture, verify reasoning, and share solutions. This environment will be one in which respect is given to all ideas in which reasoning and sense making are valued above "getting the right answer." Students will have access to a variety of learning resources, will balance the acquisition of procedural skills with attaining conceptual understanding, will estimate routinely to verify the reasonableness of their work, will compute in a variety of ways while continuing to place emphasis on mental computation skills, and will engage in homework as a useful extension of their classroom experiences.

D. Meeting the Needs of All Learners

The *Foundation for the Atlantic Canada Mathematics Curriculum* stresses the need to deal successfully with a wide variety of equity and diversity issues. Not only must teachers be aware of, and adapt instruction to account for, differences in student readiness, but they must also remain aware of avoiding gender and cultural biases in their teaching. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom.

NCTM's *Principles and Standards* (2000) cites equity as a core element of its vision for mathematics education. "All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study – and support to learn – mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students"(p. 12).

At grade 12 in New Brunswick, variations in student readiness, aptitude, and post-secondary intentions are addressed in part by the provision of courses at levels 1and 2. Students at both levels will work toward achievement of the same key-stage and general curriculum outcomes, and many of the course-specific curriculum outcomes will also be the same. As well, the instructional environment and philosophy should be the same at both levels, with high expectations maintained for all students. The difference between levels will be the depth, breadth, and degree of sophistication and formalism expected with respect to each general outcome. Similarities between courses should allow some students to move from one course level to the other.

Level 1 courses, which are designed for particularly talented students of mathematics, will be characterized by both more sophisticated engagement with mathematical concepts and techniques, and the extension of some topics beyond the scope provided at Level 2. These extensions will be included in Level 2 curriculum guides and identified with a ****** ****** symbol.

Finally, within any given course at any level, teachers must understand, and design instruction to accommodate, differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Further, the practice of designing classroom activities to support a variety of learning styles must be extended to the assessment realm; such an extension implies the use of a wide variety of assessment techniques, including journal writing, portfolios, projects, presentations, and structured interviews.

E. Support Resources

This curriculum guide represents the central resource for the teacher of *Trigonometry and 3-Space 121/122*. Other resources are ancillary to it. This guide should serve as the focal point for all daily, unit, and course-long planning, as well as a reference point to determine the extent to which the instructional outcomes should be met.

Nevertheless, other resources will be significant in the mathematics classroom. Textual and other print resources will be significant to the extent that they support the curriculum goals. Teachers will need professional resources as they seek to broaden their instructional and mathematical skills. Key among these are the NCTM publications, including the *Principles and Standards for School Mathematics, Assessment Standards for School Mathematics, Curriculum and Evaluation Standards for School Mathematics*, the *Addenda Series*, *Professional Standards for Teaching Mathematics,* and the various NCTM journals and yearbooks. As well, manipulative materials and appropriate access to technological resources (e.g., software, videos) should be available. Calculators will be an integral part of many learning activities.

F. Role of Parents

Societal change dictates that students' mathematical needs today are in many ways different than were those of their parents. These differences are manifested not only with respect to mathematical content, but also with respect to instructional approach. As a consequence, it is important that educators take every opportunity to discuss with parents changes in mathematical pedagogy and why these changes are significant. Parents who understand the reasons for changes in instruction and assessment will be better able to support their children in mathematical endeavours by fostering positive attitudes towards mathematics, stressing the importance of mathematics in their children's lives, assisting children with mathematical activities at home, and, ultimately, helping to ensure that their children become confident, independent learners of mathematics.

G. Connections Across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating learning experiences—through teacher-directed activities, group or independent exploration, and other opportune learning situations. However, it should be remembered that certain aspects of mathematics are sequential, and need to be developed in the context of structured learning experiences.

The concepts and skills developed in mathematics are applied in many other disciplines. These include science, social studies, music, technology education, art, physical education, and home economics. Efforts should be made to make connections and use examples which apply across a variety of discipline areas.

In science, for example, the concepts and skills of measurement are applied in the context of scientific investigations. Statistical concepts and skills are applied as students collect, present, and analyse data. Examples and applications of many mathematical relations and functions abound.

In social studies, knowledge of confidence intervals is valuable in intrepreting polling data, and an understanding of exponential growth is necessary to appreciate the significance of government debt and population growth. As well, students read, interpret, and construct tables, charts, and graphs in a variety of contexts such as demography.

Opportunities for mathematical connections are also plentiful in physical educaiton, many technological courses and the fine arts.

III. Assessment and Evaluation

A. Assessing Student Learning

Assessment and evaluation are integral to the process of teaching and learning. Ongoing assessment and evaluation are critical, not only with respect to clarifying student achievement and thereby motivating student performance, but also for providing a basis upon which teachers make meaningful instructional decisions. (See "Assessing and Evaluating Student Learning" in the *Foundation for the Atlantic Canada Mathematics Curriculum*.)

Characteristics of good student assessment should include the following: i) using a wide variety of assessment strategies and tools; ii) aligning assessment strategies and tools with the curriculum and instructional techniques; and iii) ensuring fairness both in application and scoring. The *Principles for Fair Student Assessment Practices for Education in Canada* elaborate good assessment practice and serve as a guide with respect to student assessment for the mathematics foundation document. (See also, Appendix A, "Assessing and Evaluating Student Learning.")

B. Program Assessment

Program assessment will serve to provide information to educators as to the relative success of the mathematics curriculum and its implementation. It will address such questions as the following: Are students meeting the curriculum outcomes? Is the curriculum being equitably applied across the region? Does the curriculum reflect a proper balance between procedural knowledge and conceptual understanding? Is technology fulfilling a proper role?

IV. Designing an Instructional Plan

It is important to develop an instructional plan for the duration of the course. Without such a plan, it is easy to run out of time before all aspects of the curriculum have been addressed. A plan for instruction that is comprehensive enough to cover all outcomes and topics will help to highlight the need for time management.

It is often advisable to use pre-testing to determine what students have retained from previous grades relative to a given topic or set of outcomes. In some cases, pre-testing may also identify students who have already acquired skills relevant to the current course. Pre-testing is often most useful when it occurs one to two weeks prior to the start of a a topic or set of outcomes. When the pre-test is done early enough and exposes deficiencies in prerequisite knowledge/ skills for individual students, sufficient time is available to address these deficiencies prior to the start of the topic/unit. When the whole group is identified as having prerequisite deficiencies, it may point to a lack of adequate development or coverage in previous grades. This may imply that an adjustment is required to the starting point for instruction, as well as a meeting with other grade level teachers to address these concerns as necessary.

Many topics in mathematics are also addressed in other disciplines, even though the nature and focus of the desired outcome is different. Whenever possible, it is valuable to connect the related outcomes of various disciplines. This can result in an overall savings in time for both disciplines. The most obvious of these connections relate to the use of measurement in science and the use of a variety of data displays in social studies.

V. Curriculum Outcomes

The pages that follow provide details regarding both specific curriculum outcomes and the four topics/units that comprise *Trigonometry and 3-Space 121/122*. The specific curriculum outcomes are presented initially, then the details of the units follow in a series of two-page spreads. (See Figure 4 on next page.)

This guide presents the curriculum for *Trigonometry and 3-Space 121/122* so that a teacher may readily view the scope of the outcomes which students are expected to meet during the year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings in this course are part of a bigger picture of concept and skill development. (See Appendix B for a complete listing of the SCOs for grades 9 and 10.)

Within each unit, the specific curriculum outcomes are presented on two-page spreads. At the top of each page, the overarching topic is presented, with the appropriate SCO(s) displayed in the left-hand column. The second column of the layout is entitled "Elaboration-Instructional Strategies/Suggestions" and provides a clarification of the specific curriculum outcome(s), as well as some suggestions of possible strategies and/or activities which might be used to achieve the outcome(s). While the strategies and/or suggestions presented are not intended to be rigidly applied, they will help to further clarify the specific curriculum outcome(s) and to illustrate ways to work toward the outcome(s) while maintaining an emphasis on problem solving, communications, reasoning, and connections. To readily distinguish between activities and instructional strategies, activities are introduced in this column of the layout by the symbol ❐. As well, curriculum extensions intended for students in the Level 1 course are indicated with the ⁂ ⁂ symbol. This symbol not only brackets text discussing differentiation for students in the Level 1 course, but also appears at the top of each page on which such text is located.

The third column of the two-page spread, "Worthwhile Tasks for Instruction and/or Assessment," might be used for assessment purposes or serve to further clarify the specific curriculum outcome(s). As well, those tasks regularly incorporate one or more of the four unifying ideas of the curriculum. These sample tasks are intended as examples only, and teachers will want to tailor them to meet the needs and interests of the students in their classrooms. The final column of each display is entitled "Suggested Resources" and will, over time, become a collection of useful references to resources which are particularly valuable with respect to achieving the outcome(s).

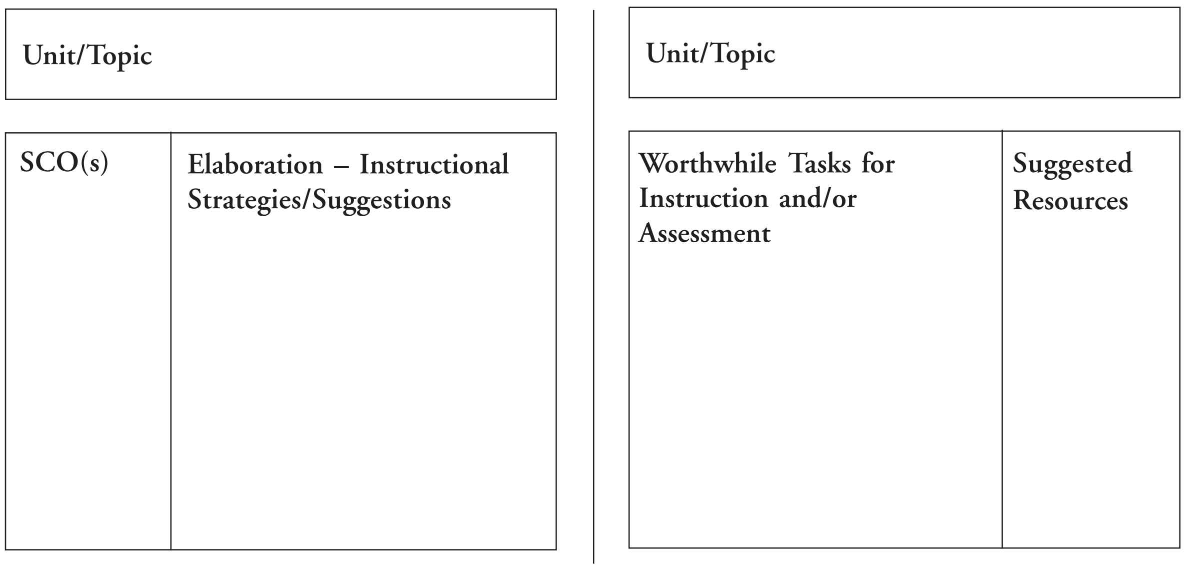


Figure 4: Layout of a 2-Page Spread

Specific Curriculum Outcomes
(by GCO)

GCO A: Students will demonstrate number sense and apply number theory concepts.

KSCO / SCO	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *ii) order real numbers, represent them in multiple ways and apply appropriate representations to solve problems* SCO: By the end of *Trigonometry and 3-Space 121/122* students will be expected to	
A1 demonstrate an understanding of irrational numbers in applications	**A1** When determining side lengths in right triangles and the values of trigonometric ratios, students will need to make use of irrational numbers (e.g., $\sqrt{2}$). Students will distinguish between exact and approximate values, and will be encouraged to make use of exact values. They will need to perform operations on expressions involving radicals. These would include rationalizing the denominator and evaluating expressions such as $2\cos 45^0 + 3\sin 135^0$. This SCO will be addressed in connection with SCOs such as C15 and D2. Unit 3, pp. 84, 86, 88, 106
A5 demonstrate an understanding of properties of matrices and apply them	**A5** Students will need to identify that only a square matrix can have an inverse and that the product of any square matrix and its inverse yields an identity matrix. Students will understand that, with the exception of commutativity under multiplication, properties for real numbers apply to matrices. Unit 1, pp. 52, 60, 62
iii) demonstrate an understanding of the real number system and its subsystems by applying a variety of number theory concepts in relevant situations	
A4 demonstrate an understanding of the conditions under which matrices have identities and inverses	**A4** When solving systems of equations using matrices, students will need to be able to identify whether a matrix is square or not, and whether a non-zero value can be found for its determinant. This outcome will be used in conjunction with B13 and B11. Unit 1, pp. 50, 52, 54, 56, 58

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *i)* *explain how algebraic and arithmetic operations are related, use them in problem-solving situations, and explain and demonstrate the power of mathematical symbolism* SCO: By the end of *Trigonometry and 3-Space 121/122*, students will be expected to	
B1 demonstrate an understand ing of the relationship between operationson fractions and rational algebraic expressions	**B1** Students will simplify algebraic expressions which will require recall and understanding of earlier work with fractions, such as finding common denominators for combining terms and removing common factors for division. These techniques will be used in proof of trigonometric identities (C25). Unit 3, p. 98
B2 demonstrate an understanding of the relationship between operations on algebraic and matrix equations	**B2** Students will need to apply algebraic manipulation skills, previously used in solving equations, to the solving of systems of equations with matrices. The isolating of a variable matrix develops from the process of isolating a variable. Unit 1, p. 50
ii) *derive, analyze and apply computational procedures in situations involving all representations of real numbers*	
B4 use the calculator correctly and efficiently	**B4** When solving trigonometric equations, students will need to become competent in using their calculators (e.g., to solve for θ given y or for y given θ). They will need to be conscious of using the correct MODE setting when working in degrees or in radians. This outcome will be used in conjunction with C27, C30 and C28. Unit 3, pp. 90, 92, 106, 108

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration

iii) derive, analyze and apply algebraic procedures (including those involving algebraic expressions and matrices) in problem situations

B5 analyze and apply the graphs of the sine and cosine functions

B5 Students will examine the graphs of sine and cosine functions to explore and identify the various characteristics of these functions. These characteristics, such as, period, amplitude, phase shift and vertical translation, will be connected to the writing of equations of sinusoidal functions, given the graph. (See SCO C3) Unit 2, p. 74

B8(121) derive and apply the general rotational matrix

B8(121) Students will drive a matrix for rotation of a vector through an angle of θ about centre of rotation (0,0). This outcome will be used in conjunction with C21 (121). Unit 4, pp. 124, 126, 128

B11 develop and apply the procedure to obtain the inverse of a matrix

$$_4\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

B11 Students will explore and develop a procedure for finding the inverse of a 2x2 matrix with a determinant of 1. They will then consider how to find the inverse if the determinant is other than one. The inverse matrix method will be used to solve systems of equations as stated in SCO B13. Unit 1, pp. 50, 52, 54, 56

B12(121) derive and apply the procedure to obtain the inverse of a matrix

B12(121) This outcome is clearly related to B11 but serves to differentiate for those students in the level one course. They will need to derive and analyse the procedures required to solve systems of linear equations using the inverse matrix method as mentioned in B13 and B14(121). Unit 1, pp. 50, 52, 54

B13 solve systems of equations using inverse matrices

B13 When solving systems of equations, students will experience the need for use of the inverse matrix method (B11, B12(121)). They will encounter situations involving systems of two, three or more equations with two, three or more variables. Unit 1, pp. 50, 52, 58

B14(121) determine the equation of a plane given three points on the plane

B14(121) Students in the level one course will be required to create the equation of a plane. Using the model equation of a plane $Ax+By+Cz+D=0$, and three given points on the plane, students will need to form three equation with three unknowns. This system will then be solved (using the inverse matrix (B12(121)), thus producing values for A, B, C, and D for the equation of the plane containing the given points. Unit 1, p. 66

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

B15 solve systems of "m" equations in "n" variables with and without technology.

Elaboration

B15 Students will solve systems of equations of various dimensions. They will find the elimination method (eliminating one variable at a time) to be useful in many cases. (See also SCOs $C14_2$ and $C19_2$.)
Unit 1, p. 46

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

i) *model real-world problems using functions, equations, inequalities and discrete structures*

SCO: By the end of *Trigonometry and 3-Space 121/122*, students will be expected to

Elaboration

C1 **model situation with sinusoidal functions**

C1 Students will need to explore real-world situations which can be modelled as sinusoidal functions (for example, tides, breathing, Ferris wheel). Interpretation of such sinusoidal functions is connected to SCOs C28, C18, and C30. Unit 3, pp. 90, 92, 110

$C2_4$ **model problem situations with combinations of functions**

$C2_4$ Students will work with situations in which modeling requires combining (e.g., adding or subtracting) functions. (Modeling musical tones and overtones would be an example.) This outcome will be addressed in connection with SCO $C19_4$. Unit 4, p. 120

C4 **model situations with periodic curves**

C4 This outcome extends SCO C1 to include modeling involing the tangent and reciprocal trigonometric functions. (See also $C17_4$ and C6.) Unit 4, p.116

ii) *represent functions in multiple ways and describe connections among these representations*

$C2_2$ **create and analyze scatter plots of periodic data**

$C2_2$ For this outcome, students will plot given data values and interpret whether the data is sinusoidal in nature or not. This is preliminary to SCOs $C17_2$ (121) and F6(121). Unit 3, p. 110

C3 **determine the equations of sinusoidal functions**

C3 Students will use previous knowledge about transformations (e.g., stretch, horizontal translation, vertical translation) to write the equations of sinusoidal functions in transformational form, e.g.,

$\frac{1}{a}(y = d) = b(\theta - c)$ Unit 2, pp. 74, 76; Unit 3, pp. 106, 110

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
C5 **determine quadratic functions using systems of equations**	**C5** Students will use systems of 3 equations in 3 unknowns to determine the equation of a quadratic relationship, given 3 points satisfying the relation. Unit 1, p. 64
C6 **demonstrate an understanding for asymptotic behavior**	**C6** Students will discover that the reciprocal functions for sine and cosine (namely, cosecant and secant) exhibit asymptotes. This property will also occur in the tangent function, since $\tan\theta = \frac{\sin\theta}{\cos\theta}$. (See also SCOs C4 and $C17_4$.) Unit 4, pp. 114, 116
C8 **demonstrate an understand ing of real-world relationships by translating between graphs, tables and written descriptions**	**C8** Students will need to model situations in 3-space to develop some ease for translating tables into graphs, graphs into written descriptions, and so on. This ability to translate will also apply to other relations (such as those that are periodic in nature). Unit 1, pp. 34, 38; Unit 2, p. 70
$C9_2$ **analyze tables and graphs of various sine and cosine functions to find patterns, identify characteristics and determine equations**	**$C9_2$** Students will need to identify the relationship between $\sin\theta$ and $\cos\theta$ (i.e., that one is a horizontal translation of 90 or $\frac{\pi}{2}$ radians of the other), identify such characteristics as amplitude (vertical stretch) and phase shift (horizontal translation), and associate values of the functions with the coordinates of a rotating point on the unit circle. This outcome will be developed in connection with SCOs such as C3, C21 , C15 and A1. Unit 2, pp. 76, 78; Unit 3, pp. 84, 86, 88, 102, 104, 106
iii) *interpret algebraic equations and inequalities geometrically and geometric relationships algebraically*	
$C9_4$ **examine, interpret, and apply the relationship between trigonometric functions and their inverses**	**$C9_4$** Students will examine and name (e.g., arc sin *x*) inverse trigonometric relationships, consider under what conditions they might be functions (see $C14_4$), and use them to solve trigonometric equations (see SCO C10). Unit 4, pp. 138, 140

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
C12 interpret geometrically the relationships between equations in systems	**C12** Students will distinguish among 2x2 systems that are represented by parallel, coincident, or intersecting lines, and will recongnize that 3x3 systems may correspond to planes that are parallel, coincident, intersect at a point, or intersect at one or more lines. They will use this knowledge to help solve problems involving systems ($C19_2$). Unit 1, pp. 34, 44
C13 demonstrate an understanding that an equation in three variables describes a plane	**C13** By identifying and plotting ordered triples that satisfy given equations in 3 variables, students will understand that these equations describe planes in 3-space. Students will use several methods (e.g., trace method) to locate and sketch planes. This outcome will be addressed in connection with SCO E2. Unit 1, pp. 38, 40, 42
$C14_2$ demonstrate an understanding of the relationships between equivalent systems of equations	**$C14_2$** To solve systems of equations using the elimination method (see $C19_2$ and B15), students need to understand the relationships between equivalent systems of equations and why equivalent systems generate the same solution. Unit 1, p. 48

iv) solve problems involving relationships, using graphing technology as well as paper-and-pencil techniques

C10 analyze and solve trigonometric equations	**C10** Students will solve trigonometric equations involving either degrees or radians, involving sine, cosine, or tangent functions (see $C14_4$), and using inverse trigonometric relationships ($C9_4$). Unit 4, pp. 114, 138
$C17_2$ (121) solve problems by determining the equation of the curve of best fit using sinusoidal regression	**$C17_2$ (121)** This outcome addresses the need for students in the Level 1 course to explore the sinusoidal or periodic nature of data with the use of technology. Students will use sinusoidal regression to determine the curve of best fit and thus find an equation which models the given data. The equation will then provide students with a means to solve further extensions of the problem. (Also see F6(121).) Unit 3, p. 110

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
C18 interpolate and extrapolate to solve problems	**C18** Students will model situations with functions and interpolate and/or extrapolate to answer questions relating to the situations. This outcome will be addressed in connection with SCOs such as C1 and C28. Unit 3, pp. 90, 92
$C19_2$ solve problems involving systems of equations	**$C19_2$** Students will initially recall previously-learned techniques for solving two linear equations in two variables, then develop the elimination method for solving 2x2 systems, solve higher order systems (e.g., 3x3), and apply matrix methods to solving systems. These techniques will be developed and applied in relation to contextual situations, and in connection with SCOs such as C12 and C28. Unit 1, pp. 34, 48, 58
v) analyze and explain the behaviors, transformations, and general properties of types of equations and relations	
$C14_4$ analyze relations, functions and their graphs	**$C14_4$** Students will analyze the tangent and reciprocal trigonometric functions (particularly with respect to asymptotic beahavior (C6)), as well as the inverses of trigonometric functions ($C9_4$). Unit 4, pp. 114, 138, 140
C15 demonstrate an understand ing of sine and cosine ratios and functions for non-acute angles	**C15** Students will relate the trigonometric values of first quadrant (acute) angles to those of 2^{nd}, 3^{rd}, and 4^{th} quadrant angles, working both in degrees and radians. This outcome connects with SCOs $C9_2$ and A1. Unit 3, pp. 88, 106
C16 analyze the effect of parameter changes on the graphs of functions	**C16** Students will connect parameter changes with variations in the graphs of the corresponding functions. In particular, this work will be extended to reflections, stretches and translations of the tangent function. Unit 4, p. 118

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

Elaboration

$C17_4$ explore and analyze the graphs of the reciprocal trigonometric functions

$C17_4$ Students will become familiar with the reciprocal trigonometric functions (i.e., $\sec\theta$, $\csc\theta$ and $\cot\theta$). They will apply these functions and analyze them in relation to such characteristics as asymptotic behavior (SCO C6). Unit 4, p.116

$C21_4$ (121) perform various transformations using muliplication of matrices

$C21_4$ (121) Students will perform transformations (e.g., reflections, rotations) by performing multiplications involving special matrices. In particular, SCO B8 addresses using matrix multiplication to perform rotations. Unit 4, pp. 122, 124

$C21_2$ describe how various changes in the parameters of sinusoidal equations affect their graphs

$C21_2$ Students will identify the parameters A, B, C, and D in equations of the form

$$y = A\ \sin B\left(\theta - C\right) + D$$

$\left(\text{or } \frac{1}{A}\left(y-D\right) = \sin B\left(\theta - C\right) \text{ in transformational form}\right)$

with amplitude (vertical stretch), period (horizontal stretch), phase shift (horizontal translation), and verical translation, respectively. They will be able to translate among algebraic, graphical and mapping representations of sinusoidal relationships. While angles will initially be expressed in degrees, this will be extended to radians. Unit 2, p. 80; Unit 3, p.106

C22 explore and verify trigonometric identities

C22 Students will be able to derive compound angle identities by at least one method. Students in the Level 2 course will explore both the use of right triangle trigonometry and the distance formula to derive these identities. Students in the Level 1 course will also derive the identities using the general rotational matrix. (See SCOs $C21_4$ (121) and B8(121).) Once derived, all students will use the compound angle identities to prove other identities and solve problems. Unit 4, pp. 130, 132, 134, 136

C23 identify periodic relations and describe their characterstics

C23 Students will distinguish between periodic relations and others (e.g., linear, quadratic, exponential) which they have examined previously. They will identify periodic relationships as ones which repeat in a regular fashion (one portion of the corresponding graph can be exactly translated onto other portions of the graph), and recognize sinusoidal relationships (see C3 and B5) as periodic relationships with special defining characteristics. (See also SCO $C9_2$). Unit 2, p. 72

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

C24 derive and apply the reciprocal and Pythagorean identities

C25 prove trigonometric identities

C27 apply function notation to trigonometric equations

C28 analyze and solve trigonometric equations with and without technology

C30 demonstrate an understanding of the relationship between solving algebraic and trigonometric equations

Elaboration

C24 While reciprocal trigonometric identities will generally be given or conjectured from examples, students will derive or prove quotient and Pythagorean identities. All of these basic identities will then be available to prove other, more complex identities (SCO C25). Unit 3, pp. 94, 96

C25 Students will use basic trigonometric identities (See SCO C24) to prove more complex identities. Students will develop and use a variety of strategies, including techniques associated with manipulating rational algebraic expressions (B1). Unit 3, p.96

C27 Students should recognize that a trigonometric relationship such as $y = 2\sin\theta - 1$ is a function and may be expressed using function notation (e.g.,). Students should then understand what it means to evaluate, for example, $f\left(45^0\right)$, or to determine θ when, say, Unit 3, p. 90

C28 Students will solve trigonometric equations, both in context and not. Solutions will be determined graphically (with and without the use of technology) and algebraically. Students will find that solving for an angle typically generates infinitely many solutions, although these may be limited by the context. While angles will initially be determined in degrees, this will be extended to radians. This outcome will be addressed in connection with SCOs such as C1 and C18. Unit 3, pp. 90, 92, 108

C30 Students will solve trigonometric equations for both angle size and other variables. By and large they should understand that standard algebraic techniques apply and should be used. They will find, however, that solving for an angle typically generates infinitely many solutions, although these may be limited by the context. This outcome will be addressed in connection with other SCOs such as C28, C18, and B4. Unit 3, pp. 90, 92

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vi) perform operations on and between functions

$C19_4$ investigate and interpret combinations of functions

$C19_4$ In connection with sinusoidal functions, students will, in particular, investigate the addition of two or more functions. This finds application in the area of sound and music (see SCO $C2_4$). (Note: Other combinations and/or compositions of functions are addressed in *Advanced Mathematics with an Introduction to Calculus 120*.) Unit 4, p. 120

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

i) *measure quantities indirectly, using techniques of algebra, geometry and trigonometry*

SCO: By the end of *Trigonometry and 3-Space 121/122*, students will be expected to

D1 derive, analyze, and apply angle and arc length relationships

D2 demonstrate an understand ing of the connection between degree and radian measure and apply them

Elaboration

D1 Within the context of the unit circle students will determine arc length in connection with rotating points. Students will also examine how the arc length (and the coordinates of the rotating point) are affected by changes in circle radius. This outcome will be addressed in close connection with SCO D2. Unit 3, pp. 100, 102

D2 Students will define angle measure in radians as the ratio of arc length to radius. They will identify and use the relationship between radians and degrees, and use radians (interchangeably with degrees) when working with the unit circle (see SCO D1), special triangles, and trigonometric ratios. Ultimately, students will apply radian measure when dealing with trigonometric functions (C4, C10, C16), equations (C22) and identities. (See also SCO D1.) Unit 3, pp. 102, 104, 106

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

i) *extend spatial sense in a variety of mathematical contexts*

SCO: By the end of *Trigonometry and 3-Space 121/122*, students will be expected to

E1 **demonstrate an understanding of the position of axes in 3-space**

Elaboration

E1 Students will recognize and use the convention for the positioning of axes in 3-space that places the x-axis running toward $(+ve)$ and away from $(-ve)$ the observer in a horizontal plane, the y-axis running left $(-ve)$ to right $(+ve)$ in the horizontal plane, and the z-axis positioned vertically (with *+ve* up and *-ve* down). The three axes intersect at right angles at a single point, the origin, which can be expressed as the ordered triple (0, 0, 0). This outcome will be addressed in conjunction with SCO E2. Unit 1, p. 36

ii) *interpret and classify geometric figures, translate between synthetic (Euclidean) and coordinate representations, and apply geometric properties and relationships*

E2 **locate and identify points and planes in 3-space**

E2 Students will locate and identify points and planes both by concrete modeling and sketching on isometric dot paper. Physical models will assist in building 3-dimensional visualization skills and will continue to be valuable as a reference for many students when working with sketches. Students will learn that 3 points define a plane, and will practice three techniques (plotting points, the intercept method and the trace method) for sketching planes. This outcome will be developed in conjunction with SCOs C8 and C13. Unit 1, pp. 36, 38, 40

GCO F: Students will solve problems involving the collection, display and analysis of data.

Elaboration

KSO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

iii) *use curve fitting to determine the relationship between, and make predictions from, sets of data and be aware of bias in the interpretation of results*

SCO: By the end of *Trigonometry and 3-Space 121/122*, students will be expected to

F6(121) explore periodic data to determine the equations of sinusoidal curves using regression analysis

F6(121) Students in the Level 1 course will create scatterplots in situations involving periodic data and use graphing technology (sinusoidal regression) to determine curves of best fit. This outcome will be addressed in conjunction with SCO $C17_2$ (121). Unit 3, p.110

UNIT 1: THE ALGEBRA OF 3-SPACE

Unit 1
The Algebra of 3-Space

(25 Hours)

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

C8 demonstrate an understanding of real-world relationships by translating between graphs, tables, and written descriptions

$C19_2$ solve problems involving systems of equations

C12 interpret geometrically the relationships between equations in systems

Elaboration – Instructional Strategies/Suggestions

C8 When given a situation that involves one or more relationships, students should be expected to translate among the representations of the relationships, e.g., between and among graphs, tables, written descriptions, and equations. For example, given three different taxi companies:

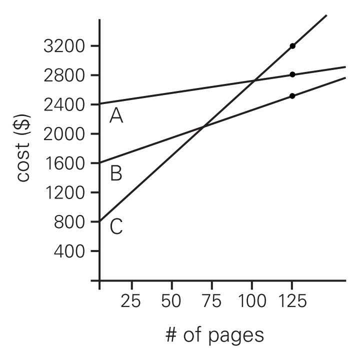
taxi A: charges $2.50 for pick up, then 50¢ per km
taxi B: charges $2.75 for pick up, then 45¢ per km
taxi C: charges $3.00 for pick up, then 40¢ per km

Students should be able to create a table of values relating the total taxi charge and the number of kilometres travelled. The table might help students construct a graph of the situation.

$C19_2$ Students might then analyse and interpret the graph to determine common values between the taxi companies. These common values are in fact the intersection points of the graphs of the equations relating total charge to kilometers travelled. Students should determine these by reading the graph. Graphing technology may also be used. Students might discuss why each taxi company has different pick up charges and how this is manifested on the graph. They might talk about the kinds of numbers and the values that should be in the domain and range.

C12 Students might discuss what conditions would have to change so that all three lines are parallel. In contrast to that, have students create a new situation so that the third line contains the intersection point for the graphs of taxi A and taxi B.

C8/C12 Students might be asked to describe the graphs, tables, or written descriptions as equations, and discuss the intersection points of the graphs of the equations. Even though a given situation may involve discrete data, lines are often drawn for convenience of analysis. Students should consider how they might change an equation so that it contains a different intersection point. For example, given three cost equations and their graphs, students might be asked to interpret each of the intersection points. They may also be asked to change equation *B* so that it shares the same intersection point as that for *A* and *C.*



Students might be given equations and graphs that represent a situation and be asked to describe the situation by looking at the values given in the equation. For example, given $c = 800 + 19.2p$, $c = 1600 + 20.8p$, and $c = 2400 + 22.4p$, where c is the cost of producing a book and p is the number of pages, students should describe the three cost plans represented by the equations and their graphs. Ask students what questions can be answered from the graph that would be difficult to answer from the equation only.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Performance (C8/C12/$C19_2$)

1) Mackin Industries manufactures toy wooden furniture. They have two plants—one in New Brunswick and one in Newfoundland and Labrador. At the moment they are focussed on manufacturing dining room suites for doll houses. Each plant has certain fixed costs that must be paid regardless of how many dining room suites are sold. The New Brunswick plant has fixed costs of \$3 000, and the Newfoundland and Labrador plant has fixed costs of \$7 000. The Newfoundland and Labrador plant is newer and can manufactur goods more economically. It produces a profit of \$250 on each dining suite manufactured. The New Brunswick plant produces a profit of only \$150. Which plant is more profitable in the long run?
   a) Use a graph to represent the profit for both plants.
   b) Is the data discrete or continuous? Explain.
   c) Determine and interpret the intersection point of the two lines that pass through the points.
   d) For what levels of production is the New Brunswick plant more profitable? The Newfoundland and Labrador plant?
   e) Interpret the vertical intercepts in terms of the problem.
   f) When you graphed the profit for both plants, what values did you use on the horizontal and vertical axes?
   g) When is it that each plant begins to have more income than expenses ? Explain.
   h) Describe from your graphs how it is clear that each plant has a different profit rate.
   i) Make up a situation for a new plant in Nova Scotia whose aim is to cover it's fixed expenses by selling only 25 dining sets.
   j) Change the Nova Scotia fixed expenses so that the graph for the Nova Scotia plant shares the same intersection point as the other two graphs.
   k) State the equation that represents each situation at each plant.
   l) Show how to solve algebraically to verify your answer in (b) above, and then (i) above.

Suggested Resources

The Algebra of 3-Space

Outcomes

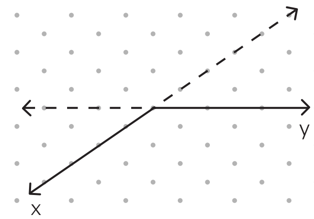
SCO: In this course, students will be expected to

E1 demonstrate an understanding of the position of axes in 3-space

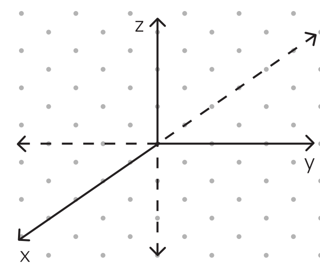
E2 locate and identify points and planes in 3-space

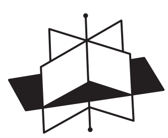
Elaboration – Instructional Strategies/Suggestions

E1/E2 The situations described on the previous 2-page spread involve relationships between two variables (e.g., kilometres travelled and total cost). It is common, however, for situations to involve more than two variables. When three variables are involved, representing the situation graphically requires moving from a 2-dimensional coordinate system (involving ordered pairs) to a 3-dimensional system (involving ordered triples). This 3-dimensional space is often referred to simply as 3-space.



When representing points in 3-space students need to agree on the positioning and use of the axes, and how they corresponds to the numbers in an ordered triple. The x-axis has positive values that get larger as they "come toward" the reader; the y-axis goes left and right, with numbers getting larger from left to right. Students can model this by simply laying a piece of graph paper flat on their desks and drawing perpendicular lines, with the x-axis being the one running towards them. The intersection is the origin. Ask them to place their pencils vertically at the origin. This would represent the vertical axis, or z-axis. Students can now locate points in four of the eight octants by building "cube towers" with height "*z*" and placing them on the graph paper, using the x- and y-values given in the table, or by the ordered triple (*x*, *y*, *z*). For example, given the point (1, 2, 3), students might represent this by placing a tower 3 units high at the x, y-coordinate (1, 2). The top of the tower would represent the point (1, 2, 3). The other four octants would be under the students' desks as the vertical axis passes through the table top. The axes and octants can be modeled very nicely using pipe cleaners, or cube-a-links and rods. It might be a good idea for each group of students to have a model of the 3-space coordinate system to help with visualization.





E2 It is understood that if *A*, *B*, and *C* are non-collinear points, then *A*, *B*, and *C* determine a plane. Students should learn notation that would save a lot of unnecessary writing. Instead of writing "the set of all points (x, y) such that *x* and *y* satisfy some condition," students can write $\{(x, y) | (\text{some condition})\}$. For example, the x-axis in 3-space can be written as $\{(x,0,0) | \text{x} \in \text{R}\}$. This reads "the set of all triples of real numbers with 2^{nd}, 3^{rd} coordinates 0."

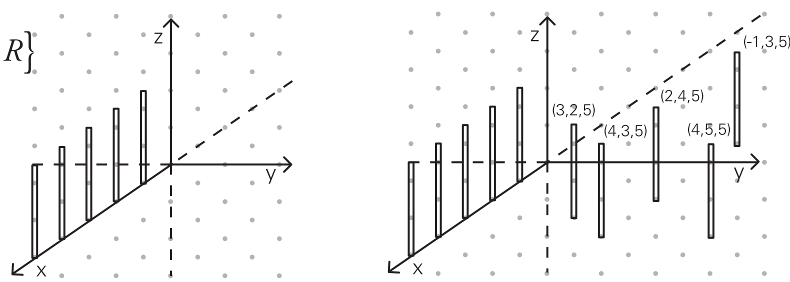
The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Activity **(E1/E2)**

1) This activity will help students become more comfortable with the visualization of points in 3-space and with graphing those points/lines/planes on two-dimensional paper. The diagrams given are there for the teacher's benefit.
   a) Position a piece of centimetre graph paper on a flat surface and draw an x - and y-axis anywhere on the grid. Position the paper so that the positive x-axis is "coming toward" you. The positive y-axis is to the right of the origin.
   b) Place your pencil vertically at the origin. This would represent the z-axis. The *x–y* plane is called the horizontal plane.
   c) Using cubes from a cube-a-link set, or straws and a geoboard, build towers or place straws to represent the following points:
      i) (1, 0, 5) ii) (2, 0, 5) iii) (3, 0, 5) iv) (5, 0, 5)
   d) Describe how your towers or straws represent these points. [Ans: The points are represented by the tops of the towers or straws.]
   e) Describe two planes to which all of these points belong. [Ans: The vertical x-z plane, and the horizontal *z* = 5 plane.]
   f) Determine five other points that would also be on the *z* = 5 plane. State their coordinates and build and position towers to represent them. Describe how you could represent the plane.
   g) Using a pencil and isometric dot paper, represent all that you have done above.

$\{(3,4,z) \mid -2 \le z \le 5 \mid z \in R\}$



Performance **(E1/E2)**

2) a) Graph the line that passes through (3, 4, 0) and is parallel to the z-axis. Represent this line, using set notation. [Ans: $\{(3,4,z) \mid z \in R\}$]
   b) Graph the same line, only make it a segment going from z = –2 to z = 5.

      Represent this segment, using set notation. [Ans:]
   c) State the coordinate for the "top" of this segment. [Ans: (3, 4, 5)] State 3 more coordinates on the plane z = 5.

3) Graph the points (3, 0, 0), (0, 2, 0) and (0, 0, 5). Join these three points and describe what the triangle represents. [Ans: a portion of a plane] What would you call the 3 points? [Ans: x-, y, and z-intercepts]

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

C8 demonstrate an understanding of real-world relationships by translating between graphs, tables, and written descriptions

C13 demonstrate an understanding that an equation in three variables describes a plane

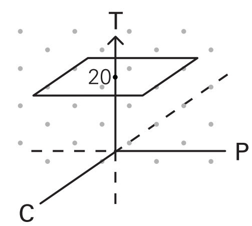
E2 locate and identify points and planes in 3-space

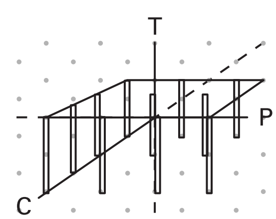
Elaboration – Instructional Strategies/Suggestions

C8 Students should model real-world situations to help them understand, represent, and visualize the planes in 3-space that represent the situations.

> The math club at Alma Mater High School produces a newsletter each month of the school year. Their total costs, "*T*," are dependent upon the number of pages using colour, "*c*," and the number of pages with pictures, "*p*." The first newsletter was produced with no colour and no pictures. The club will be charged \$20 regardless of the number of pages. The club was then told that the total cost of the next edition would increase (if colour was used) at the rate of \$5.00 per page. They were also told they could use pictures at no additional cost. Ask students to determine ten ordered triplets (c, p, T) based on different numbers of pages, and use cubes to build towers to model the situation. Some teachers may want to use straws and geoboards as the model. Students can cut straws to appropriate lengths and stick them onto the pegs of the geoboard to represent the coordinates.

C13/E2 Some students should model the 1st edition of the newsletter, others the 2nd edition. Students should be asked to write the equations for the planes they have represented. Students working with the 1st edition should determine the equation $T = 20$ to represent their horizontal plane. With the 2nd edition, students should lay a piece of paper across the tops of the towers to "see" the plane. They should realize that this plane is sloped away from them, and represents all the total costs, depending on the number of pages and unlimited use of pictures. The equation must be $T = 20 + 5c$.





Students could then be told that there is an additional cost factor of \$10 for every page on which a picture is placed. They should use towers to model this plane and represent it with an equation. Ask students to consider whether this plane represents the solution. They should realize that the solution is represented by a subset of points on that plane.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Performance (C8/C13/E2)

1) The total cost *"T"* of producing a newsletter each month is dependent upon the number of pages with colour "*c*" and whether there are pages with pictures *"p"* or not. There is an up-front charge of $50 for set-up costs (unlimited copies), an extra fee of $7.00 per page if colour is used on the page, and $3.00 if a picture appears on a page.
   a) Create a model for this situation, using cubes or straws.
   b) Place a piece of paper on the tops of the towers to represent the plane. State the equation for the plane.
   c) Describe in your own words where the plane will intersect the $c-p$ plane, if at all.

2) Suppose that 250 copies are produced of the first edition of the newsletter described in (1) above.
   a) Describe how this situation can be modeled and represented with a plane.
   b) State the equation for the plane.
   c) What would be the cost of 250 copies?
   d) Would there be an extra charge for more copies? Explain.

3) The second edition of the newsletter described in (1) above will include 3 pages with pictures. It is expected to produce 125 copies.
   a) Describe how this situation can be modeled and represented with a plane.
   b) State the equation for the plane.
   c) Describe in your own words where the plane will intersect the $c-p$ plane, if at all. If the intersection is a line, what is the equation for that line?

Suggested Resources

The Algebra of 3-Space

Outcomes

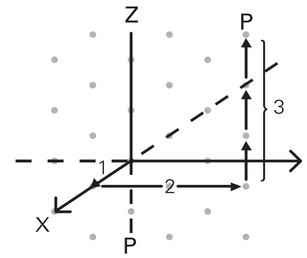
SCO: In this course, students will be expected to

E2 locate and identify points and planes in 3-space

C13 demonstrate an understanding that an equation in three variables describes a plane

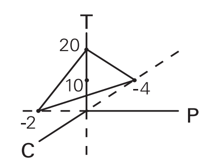
Elaboration – Instructional Strategies/Suggestions

E2/C13 Students should explore sketching in 3-space, using isometric dot paper. Three techniques for plotting points and sketching planes follow. It should be noted, however, that such sketches have significant visual limitations and should generally be accompanied by corresponding concrete modeling. Also note that SCO E2 does not require that students become proficient at both modeling and sketching.



A first technique for sketching a plane in 3-space is to identify and plot a random selection of ordered triples that satisfy a particular equation. For example, if the ordered triplet (1, 2, 3)satisfies the given equation, students would begin with their pencils at the origin and move along the positive x-axis one unit (move the pencils toward them), then to the right two units, then up three units. Once several points had been plotted that should lie on a plane, students might still find it very difficult to 'see' the plane that contains the point. Consequently, modeling is important for visualization.

A second way to sketch planes on graph paper is to use the three intercepts. First, have students find x, when $y = 0$ and $z = 0$. The point (x, 0, 0) is the x-intercept. Similarly, (0, y, 0) is the y-intercept and (0, 0, z) is the z-intercept. Students can visualize the plane that passes through these intercepts by joining the three intercepts and shading the interior of the triangle formed to represent a portion of the plane. For example, in the 3rd edition of the newsletter for the math club, the total cost is determined using the equation $T = 20 + 5c + 10p$.



Using pipe cleaners for the axes and wires for the lines joining intercepts, this plane could be projected into the (+,+,+) octant for practical purposes.

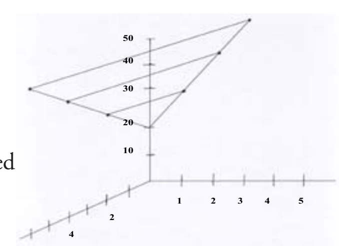
A third method is the "trace" method. It is again exemplified here to sketch T=20+5c+10p.

Choose 3 different values for total cost, T.

If T=30, then 30=20+5c+10p, so 10=5c+10p
If T=40, then 40= 20+5c+10p, so 20=5c+10p
If T=50, then 50=20+5c+10p, so 30=5c+10p

Each equation represents a line which is contained in the plane to be sketched.(They are, infact, parallel lines.) Sketch the lines by indentifying two ordered triples for each. For example,

for 10=5c+10p, two points could be (2, 0, 30) and (0, 1, 30)
for 20=5c+10p, two points could be (4, 0, 40) and (0, 2, 40)
for 30=5c+10p, two points could be (6, 0, 50) and (0, 3, 50)



The triangular region shown in the sketch is part of the plane desired Again, modeling will help with the visualization process.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

Performance (E2/C13)

1) The total cost of producing a newsletter is given by the equation $T = 5 + .3c + .5p$ where "T" is the total cost, "c" is the number of colour pages, and "p" is the number of pages with pictures.
   a) Determine at least 5 ordered triples that belong to the set of points described by the given equation.
   b) Plot these points on a 3-space grid, using isometric dot paper.
   c) Using the coordinates determined in (a) and plotted in (b), describe the plane that represents the given equation. Draw the plane on the dot paper.

2) a) Revisit #1 above. This time find the three intercepts, graph them and use these to draw the plane that represents the given equation.
   b) Find equations to represent the 3 traces for the equation given in #1 above.
   c) What is the connection between the traces and the intercepts?
   d) Find a way to model the traces, using pipe cleaners for the 3-space axes and fine wire for the traces.
   e) Use the model to produce the plane that represents the given equation in the (positive, positive, positive) octant.

3) In 2-space, y = 3x describes a line with slope 3 that passes through the origin. Using notation the line is described as $\{(x, y) | y = 3x\}$. In 3-space, this same line is described as $\{(x, y, 0) | y = 3x\}$. Its slope is 3, and it passes through the origin (0, 0, 0).
   a) Where in 3-space is $\{(x, y, 3) | y = 3x\}$? What is its slope and y-intercept?
   b) Where in 3-space is $\{(x, y, -2) | y = 3x\}$? What is its slope and y-intercept?
   c) Where in 3-space is $\{(x, y, z) | y = 3x + 1\}$? Describe its appearance.

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

C13 demonstrate an understanding that an equation in three variables describes a plane

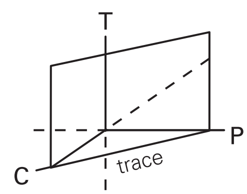
Elaboration – Instructional Strategies/Suggestions

C13 Students should be able to give the equation for a plane by

1) describing the relationship among the variables in words, then translating that description into symbols
2) being given intercepts

For example, in the first case, given the newsletter situation, students could plot many points (model with cubes or straws) and describe the plane as the total cost of the newsletter, \$20, plus additional amounts that depend on the number of pages using colour (at \$5.00/page), and the number of pages with pictures (at \$10.00/page). They would translate this into symbols: $T = 20 + 5c + 10p$.

In the second case, have the students join two given intercepts, producing a trace through which an infinite number of planes can pass. If the intercepts are (2, 0, 0) and (0, 1.5, 0), the trace is a line in the c-p plane. It's equation can be found by first determining the slope , then the p-intercept (1.5).



The resulting equation is $p = \frac{-3c}{4} + 1.5$, or $3c + 4p = 6$.

On the trace, the T-coordinates are 0. The vertical plane that contains the trace $3c + 4p = 6$ is made up of all the lines parallel to the trace, but with T-coordinates that are non-zero. Thus, the vertical plane with intercepts (2, 0, 0) and (0, 1.5, 0) can be represented with the equation $3c + 4p = 6$. When any coefficient for T other than zero is added to the equation, such as $3c + 4p - 5T = 6$, the resulting plane contains the trace $3c + 4p = 6$, but the plane is oblique and will also have a T-intercept).

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C13**)

1) State the equation for the plane that represents each of the following situations.
   a) A music store sells CDs and cassettes. It makes a profit of $5.50 on each CD sold and $3.50 on each cassette sold. It's daily expenses are $250.00. What is the total profit per day?
   b) One litre of a 15% acid solution contains 150 ml of acid and 850 ml of water.
   c) There is a sale on socks and ties at George's favourite store. Socks sell for $5.00 a pair, and ties for $10 each. How many of each will George buy?
   d) The cost of buying computer chips varies, depending on the chip purchased. One type "*x*" sells for $7.50, the other "*y*" $12.50. Determine an expression for the cost of buying some "*x*" chips and some "*y*" chips.

2) Given the following information, determine the equation for the plane:
   a) (3, 0, 0) and (0, 5, 0)
   b) (0, 2.5, 0) and (0, 0, –7.5)
   c) (–5, 0, 0) and (0, 0, –3.5)

Suggested Resources

The Algebra of 3-Space

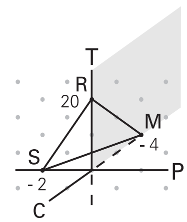
Outcomes

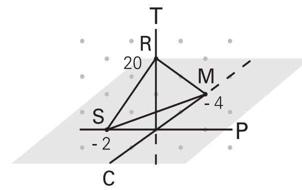
SCO: In this course, students will be expected to

C12 interpret geometrically the relationships between equations in systems

Elaboration – Instructional Strategies/Suggestions

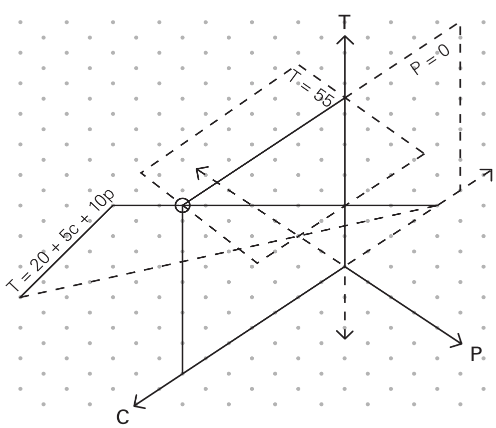
C12 A trace is a line that results from the intersection of a plane that represents a situation, with the plane determined by two axes. For example, students should use shading to see that the plane *RSM* intersects the $T - c$ plane in the line *RM*. Also, they should visualize the line *SM* as the intersection of the *RSM* plane and the horizontal $c - p$ plane. A plane determines a set of points. Another plane determines another set of points. The intersection of the two planes determines a common set of points that determine a straight line. For example, the costs for the 3rd edition of the newsletter can be represented as a plane and as the equation



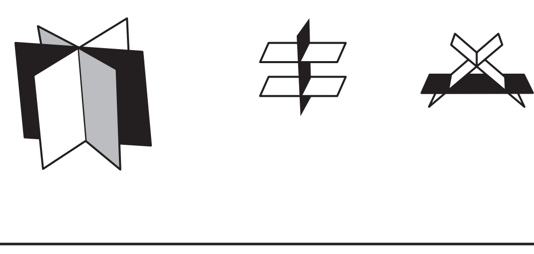


$T = 20 + 5c + 10p$. Suppose that the math club used no pictures, then $p = 0$ would represent the T–c plane. Where that plane intersects the plane $T = 20 + 5c + 10p$ would describe all the values for T that might result, depending on the number of pages using colour. Since the total number of pages in the newsletter might range from 1 to some large number, there would be many solutions to the problem. Students should examine a situation where the two planes do not intersect and examine the equations to see how the corresponding coefficients in the two equations are related.

If students know that the total cost for the newsletter was $55.00, they can represent this as the constant plane $T = 55$. When this is added to the graph just studied, there would be three planes intersecting at one point, the point (7, 0, 55). Students should interpret this point as a newsletter with 7 pages with colour, no pictures and with a total cost of $55.00. Students should also realize that three planes may intersect not only at a single point, but also to form a single line (like the pages in a book), two different lines, or three different lines.



They should examine the corresponding coefficients in the three equations to see how these situations differ.



The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Activity (C12)

1) The total cost in dollars of producing one high school yearbook is given by the equation $T = 1250 + 2c + 1.5p$, where "T" is the total cost, "c" the number of pages with colour, and "p" the number of pages with pictures.
   a) Draw the graph of the plane that represents the total cost in dollars.
   b) Draw the graph of the plane $T = 1250$. What does this plane represent? [Ans: Various numbers of pages where there is no extra charge for colour or pictures.]
   c) Look at the intersection of these two planes. How many points are on the intersection? Describe what they represent. (They represent points that form a line. These points are various values for "c" and "p" that result in yearbooks that will result in a total cost of $1250. The only meaningful point in this set of points for this context would be (0, 0, 1250); the rest must include either a negative number of pages with colour or pictures).
   d) Draw the graph of the T–c plane. Describe what this plane represents. [Ans: The total cost for a book with any number of pages that have colour and no pages with pictures.]
   e) Describe the intersection of the $T - c$ plane and the plane of the original equation $T = 1250 + 2c + 1.5p$. [Ans: The $T - c$ plane represents $p = 0$, a book with colour pages, but no pictures. The intersection then would be the various total costs resulting as the number of colour pages vary.]
   f) Is it possible that all three planes discussed above would all intersect at a point? What is the meaning of that point? [Ans: Should intersect at (0, 0, 1250)—the number of pages of colour and the number of pages with pictures that result in the cost of the book being $1250.]
   g) If the initial set up costs were reduced to $1000, but the other costs don't change, describe the relationship between this plane and the plane representing $T = 1250 + 2c + 1.5p$. [Ans: parallel]
   h) The company offers a sale: the cost per picture-page is reduced by 50 . How does this affect the plane $T = 1250 + 2c + 1.5p$?

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

B15 solve systems of "*m*" equations in "*n*" variables with and without technology

Elaboration – Instructional Strategies/Suggestions

B15 In grade 10, students solved systems of 2 equations with 2 variables, using graphs, technology, and the substitution method. In this course, students will solve 2 equations with 2 variables, using graphs, substitution, elimination, and the inverse matrix method (B13, p. 50). They will solve three or more equations with three or more variables, using technology (B13, p. 50). In the process of moving from solving 2×2 systems to m x n systems, students should experience solving 3 equations with only 2 variables and 2 equations with 3 variables. The need for solving systems like these has arisen from examining the graphs of intersecting planes. For example, the elimination method could be introduced (see C14, p. 48) to solve the following system of 3 equations in 2 variables:

1) $x + y = 5$
2) $x - y = 1$
3) $3x - 5y = -1$

Students will learn that they can add equation 1) and 2) to get

$$2x=6$$
$$x=3$$

or subtract equation 2) from 1) to get

$$2y=4$$
$$y=2$$

They should realize that substituting both of these values into equation 3),

$$\text{LHS}=3(3) - 5(2) = 9 - 10 = -1=\text{RHS}$$

satisfies equation 3), that (3, 2) is the solution to the given system, and that the equations are consistent (i.e., they intersect at one point). They should know that if the 3rd equation is not satisfied, then the system has no solution and the equations are inconsistent (i.e., have no common intersection) (see C12). The elimination method may also be the quickest way to solve a system of 2 equations with 3 variables; for example,

1) $x - y + z = 8$
2) $3x - 2y + 5z = 4$

Multiplying equation 1) by –2, then adding equation 2) gives

Similarly, multiplying equation 1) by –3 and adding equation 2) leads to $y = -20 - 2z$. Thus, the solution to the system is $(x, y, z) = (-12 - 3z, -20 - 2z, z)$. The "$z$" is called a parameter and leads to the understanding that there are an infinite number of solutions (i.e., the intersection is a line) (see C12, p. 44).

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**B15**)

1) Solve the following systems, and determine whether they are consistent or inconsistent.

$$\begin{cases} x = 2y = 3z = 0 \\ 2x + y = 0 \\ x + z = 0 \end{cases} \qquad \begin{cases} x - y - 3z = 3 \\ 3x - y - z = 5 \end{cases}$$

2) Show that the line of intersection of the planes
$x + 2y - 4z + 8 = 0$
$3x - y - z + 6 = 0$
lies in the plane $5x - 11y + 13z - 14 = 0$.

3) Determine which pairs of planes are parallel. For each pair that is not parallel, find the solution:

   a) $x + y - 3z = 4$ and $x + 2y - z = 1$
   b) $5x - 2y + 2z + 1 = 0$ and $5x - 2y + 2z - 3 = 0$
   c) $x + 3y - z - 4 = 0$ and $2x + 6y - 2z - 8 = 0$
   d) $x - 3y - z + 3 = 0$ and $2x + 4y - z - 5 = 0$

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

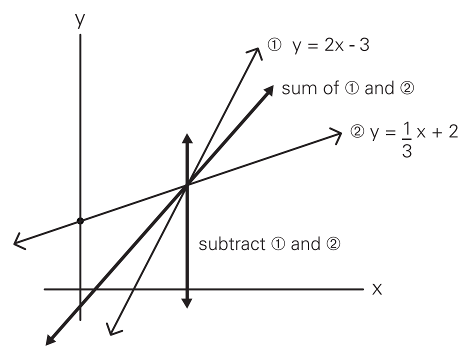
C14$_2$ demonstrate an understanding of the relationships between equivalent systems of equations

C19$_2$ solve problems involving systems of equations

Elaboration – Instructional Strategies/Suggestions

C14$_2$ Before students use the elimination method to solve systems of equations, the method must be developed with the students. The development that follows is in two dimensions. Students must understand the relationships between equivalent systems of equations before they can use the method with understanding. To develop this understanding, have students graph two equations that represent related situations so that the intersection of the graphs can be interpreted with meaning. Have students add the equations together and graph the resulting equation. Have them subtract the 2 equations and graph the resulting equation. Have students conjecture about the effect on the intersection point of the system when a new system is developed from the old by adding and/or subtracting (new graphs are produced, but the intersection point remains the same – equivalent systems of equations).

❒ Have students graph $y = 2x - 3$ and $y = \frac{1}{3}x + 2$ and find the intersection point [Ans: 3, 3]. Then have students add the two equations and graph, then subtract the two equations and graph. Compare the two systems (one dotted line, the other not).



Students should then investigate systems with equations that have been altered through multiplication by a constant and/or addition and subtraction. For example,

beginning with the system have students multiply the first equation by

3 and add it to the second equation multiplied by $\frac{1}{2}$ to form a new 3rd equation.

Then have them subtract the 2nd equation from three times the first to form a 4th equation. Graphing equations 3 and 4 will produce other algebraic system with the same intersection point (equivalent system).

C19$_2$ Students will need to practice the elimination method. Remember practise for a purpose, and in context as much as possible. They should also practise other algebraic methods and with systems that are a mixture of "*m*" equations in "*n*" variables.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Students should discuss and defend their choices for the methods they choose. It is not intended that an assessment item be focussed on any one method.

Performance (**C14$_2$**)

1) Given $3y - 2x + 5 = 0$ and $4y + 3x - 7 = 0$,
   a) explain, without graphing, how you could state another linear equation that shares the same intersection point. What is the equation?
   b) state a different equation that also passes through the same intersection point. Explain how this was determined.

2) Given $3x - 5y = 2$,
   a) sketch the graph.
   b) multiply each term in the equation by 3.
   c) explain in words why the graph of the equation (b) will be the same as the graph in (a).

3) Given this system and $3x - 4y - 1 = 0$, describe
   a) your first steps in solving this system, using elimination, and explain why.
   b) your first steps in solving the system $3x - y = -2$ and $5y + x - 7 = 0$, using substitution and explain why.
   c) what advantage you think there is in using the elimination method versus the substitution method.

$$\left\{ 2x - \frac{1}{2}y + 5 = 0 \right.$$

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

4) Ron told Dick that, when given a system of equations in the form
   if the values of *a*, *b*, *c* and *d*, *e*, *f* are each an arithmetic sequence, the solution will be (–1, 2). Do you think Ron is correct? Find five examples that work.

5) Three cab companies charge different rates. After how much time will the cost be the same no matter which cab you take?

   Yellow Cab: cost = \$1.50 + 5¢/second
   White Cab: cost = \$2.75 + 4.5¢/second
   Blue Cab: cost = \$2.13 + 5.25¢/second

6) System A has a solution. Find it. System B does not. Why not?

$$A = \begin{cases} x - 3y = 0 \\ 2x + 4y = 10 \\ x + 5y = 8 \end{cases} \qquad B = \begin{cases} x - 3y = 0 \\ 2x + 4y = 10 \\ x + 5y = 4 \end{cases}$$

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

B11 develop and apply the procedure to obtain the inverse of a matrix

B12(121) derive and apply the procedure to obtain the inverse of a matrix

B13 solve systems of equations using inverse matrices

B2 demonstrate an understanding of the relationship between operations on algebraic and matrix equations

A4 demonstrate an understanding of the conditions under which matrices have identities and inverses

Elaboration – Instructional Strategies/Suggestions

B11/B12(121)/B13 In previous grades students have used matrices to represent network graphs and inventory lists. In this course, students will use matrices to represent systems of equations. This will enable students to solve large systems with less manipulation, and with technology. It will also enable students to explore the properties of matrices.

Students will develop (Students in the level 1 course will derive and analyse) and apply procedures for solving systems of linear equations (2×2 and 3×3), using matrices (inverse matrix method). Students should learn to use these procedures, using appropriate technology.

One of the more common uses of matrices is to solve a system of equations (see B13, p. 58 for the solving). In general, every linear system can be represented by a matrix equation of the form $AX = B$, where A is the matrix made from the numerical coefficients of the linear equations, B is the column matrix made from the constant terms, and X is the column matrix whose elements are the variables.

For example, given the system, , $A=\begin{bmatrix}2 & 5\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}8\\ 5\end{bmatrix}$.

B2/A4 As is the case with regular algebraic equations, students strive to isolate the term with the variable, then isolate the variable. For example, when solving $2x + 5 = 9$, students first isolate the variable term by subtracting 5 from both sides, giving $2x = 4$, then they divide both sides by two to isolate the variable itself.

Division is not an operation with matrices, so another look at the procedure above indicates that the division by 2 could be described as multiplication by the inverse of the coefficient of x. With matrices then, to isolate the variable matrix, multiplication by the inverse [A^{-1}] will take place. The matrix equation $AX = B$ has the solution $X = A^{-1}B$ if A^{-1} exists (i.e., *if* A is a square matrix, and its determinant is not zero). See the bottom of p. 54 for the definition of determinant. This equation is found by multiplying both sides of $AX = B$ by A^{-1}. For example, given the system

continued ...

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**B13**)

1) Represent the following systems using matrices.

$$A=\begin{cases}3x+5y=7\\2x+6y=10\end{cases}\qquad B=\begin{cases}-3x-7y+11=0\\-2x+y=0\end{cases}$$

$$C=\begin{cases}2x-3y+5z=11\\\frac{1}{2}x-\frac{2}{5}y-z=\frac{12}{13}\\5x-7y-3z=\frac{2}{3}\end{cases}\qquad D=\begin{cases}5x-y=2z\\2y-3z=1\\2z-3x=5y\end{cases}$$

2) Given the following matrices, write the system they represent, using algebraic terms.

a) $\begin{bmatrix}-5 & 2 & 0\\1 & -1 & 1\\2 & -3 & 5\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}8\\11\\2\end{bmatrix}$ b) $\begin{bmatrix}3 & \text{[illegible]}\\-2 & \text{[illegible]}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$

3) Barb and Helen are solving equations.

Barb:

$3x-7y=5$

$-7y=-3x+5$

$y=\frac{3}{7}x-\frac{5}{7}$

Helen:

$\begin{cases}2x+7y=5\\3z+2x=-3\end{cases}$ 1, 2

$7y-3z=8$ 3

$7y=3z+8$ 4

$\frac{1}{7}(7y)=\frac{1}{7}(3z)+\frac{1}{7}(8)$ 5

$y=\frac{3}{7}z+\frac{8}{7}$

a) What did Barb do to get from line 2 to line 3 ?

b) How did Helen get 3 ?

c) Describe what Helen is doing to get from 4 to 5 ? How is this similar to what Barb did?

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

B13 solve systems of equations using inverse matrices

A4 demonstrate an understanding of the conditions under which matrices have identities and inverses

A5 demonstrate an understanding of properties of matrices and apply them

B11 develop and apply the procedure to obtain the inverse of a matrix

B12(121) derive and apply the procedure to obtain the inverse of a matrix

Elaboration – Instructional Strategies/Suggestions

B13/A4/A5 Continuing the elaboration from the previous 2-page spead necessitates finding the multiplicative inverse of the matrix *A* (which has the notation A^{-1}). The inverse of a square matrix *A* is a matrix *B* when the product of *A* and *B* is the identity

matrix $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or, in the case of 3 by 3 matrices, .

B11 In order to generalize the inverse of a matrix, students should find particular inverses and look for a pattern. They might begin by looking at the situation

and trying to determine the values for *a*, *b*, *c*, and *d* that would satisfy the equation. Or, they might try several examples with a determinant of 1, observe the pattern, then examine matrices with a determinant of 2, 3, etc. to redefine the pattern. (Note: The inverses should be quickly found using technology.) For example, using technology,

students should be able to see that the inverse of is . When the determinant of A is one, then students should see the pattern—reverse the order of the numbers on the major diagonal: becomes ; record the opposite numbers for the numbers on the minor diagonal .

B12(121) Students in the level 1 course should derive the inverse of a matrix. When a matrix is multiplied by the inverse, the result should be the identity matrix *I* since, with real numbers, one is the result when a number is multiplied by its inverse .

To find the inverse of a matrix must exist such that

$AB = BA = I$. yields the following

system of equations:

Solving each pair produces $a = 3$, $b = -1$, $c = -5$, and $d = 2$; thus is the inverse of *A* and is written A^{-1}.

continued...

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Activity (**A4/A5**)

1) Purpose: to find the identity of a matrix. Finding the inverse of a matrix necessitates knowing the identity matrix. As in the real number system, a matrix multiplied by the identity gives the matrix. That is, if A is a matrix, and I is the identity matrix, then $AI = A$. Students need to find I:

   a) Starting with any matrix A, try and , then

   ; using matrix multiplication, set up two systems to

   solve for a, b, c, d. solve for a and c; solve for

   b and d.

   b) Replace $I = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the values for a, b, c, and d.

   c) Does $AI = A$? Does $IA = A$?

   d) Create several other matrices like A and redo step C above.

   e) Explain why a matrix must be a square matrix in order to have an identity matrix.

[illegible] $\begin{cases} b + 2d = 2 \\ 3b + 4d = 4 \end{cases}$

   f) Given and , find AB. Find BA. What conclusion do you think you can make? Try some different matrices for A and B, then make a conjecture.

Activity (**B11**)

2) Purpose: Develop mental math procedures to obtain the inverse of a matrix.

   a) Using a graphing calculator,

      i) enter matrix

      ii) look at matrix A and matrix A^{-1} on the screen. Are the same digits in A^{-1} as in A? Describe what is different about them. Record these matrices on paper.

      iii) enter matrix

      iv) look at matrix B and matrix B^{-1} on the screen. Answer ii) above for matrix B^{-1} and B.

      v) enter matrix and repeat ii) and iii) above for matrix C and C^{-1}.

   b) Describe the pattern that seems to exist that would help you obtain the inverse of a matrix. Use the terms major diagonal (upper left to lower right) and minor diagonal.

continued ...

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

B11 develop and apply the procedure to obtain the inverse of a matrix

A4 demonstrate an understanding of the conditions under which matrices have identities and inverses

B12(121) derive and apply the procedure to obtain the inverse of a matrix

Elaboration – Instructional Strategies/Suggestions

...continued

B11/A4 Students should check this result, using the MATRX functions in their graphing calculators. Teachers should provide students with several examples to try whose determinant is 1 to help them observe the pattern.

B12(121) Students should be asked to generalize this derivation of the inverse of a matrix by solving, in the same way as above, this:

Let the inverse of *M* be

(Teachers might have different groups each obtain the value for one of *a*, *b*, *c*, or *d*.)

$$\begin{array}{lll} B\times(1) & aA+bC=1 & (1) \\ A\times(2) & aB+bD=0 & (2) \\ (4)-(3) & aAB+bBC=B & (3) \\ & \underline{aAB+bAD=0} & (4) \\ & bAD-bBC=-B & \\ & b(AD-BC)=-B & \\ & b=\dfrac{-B}{AD-BC} & (5) \end{array} \qquad \begin{array}{lll} B\times(1) & cA+dC=0 & (1) \\ A\times(2) & cB+dD=1 & (2) \\ (4)-(3) & cAB+dBC=0 & (3) \\ & \underline{cAB+dAD=A} & (4) \\ & dAD-dBC=A & \\ & d(AD-BC)=A & \\ & d=\dfrac{A}{AD-BC} & (5) \end{array}$$

Similarly, by substituting the values for *b* and *d* into the given equations (1), students will be able to determine values for *c* and *d*.

From this students would conclude the inverse of the matrix is

They should also notice and discuss that the difference in the product of the diagonals (the determinant) plays an important part, and that, if it were zero, the inverse could not be found (singular matrix).

... continued

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Activity (**B11**)

... continued

2)c) If A and A^{-1} are inverses, what will happen if you multiply the two matrices? Does the order in which you multiply them matter? Try this for the 3 sets of matrices and their inverses.

d) Repeat steps a), b), and c) above for the following matrices:

$$D = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} \qquad E = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \qquad F = \begin{pmatrix} 1 & -1 \\ 4 & -2 \end{pmatrix}$$

(If you get decimals in your answers, you may want to express the matrix with fractions to help you see the pattern.)

e) Re-express D^{-1}, E^{-1}, and F^{-1} with the fraction factored out.

f) For A, B, C, multiply the two numbers on the major diagonal, then subtract the product of the minor diagonal. This number is called the determinant. Evaluate the determinant for D, E, and F.

g) How is the determinant important when determining the inverse of a matrix?

h) Find the inverse of this matrix mentally. Then check, using your calculator.

Performance (**B12(121)/A4**)

3) Given a matrix and its inverse , work with A and A^{-1}

to obtain values for a, b, c, d so that .

Pencil and Paper (**B12(121)/A4**)

4) Given the following systems, show how you could use matrices and inverse matrices to solve the system. You don't need to solve at this point.

a)

b) $\begin{cases} 3x + 2y - z = 1 \\ 5y = -3x + 4z - 5 \\ x = 3y - 5z + 3 \end{cases}$

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

B11 develop and apply the procedure to obtain the inverse of a matrix

A4 demonstrate an understanding of the conditions under which matrices have identities and inverses

Elaboration – Instructional Strategies/Suggestions

B11/A4 As students explore patterns with determinants greater than one, they should be able to include this determinant value in their pattern to obtain an inverse. For example, they should next explore a matrix and its inverse, using technology. Give students matrix $A = \begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix}$. Ask them to find the inverse, using technology, and express it with fractions if possible . The pattern they had developed (p. 50) should have given .

They should notice that technology gave them numbers that were each of those obtained from the pattern. They should try some others that will lead them to the same discovery. To answer why , ask them to subtract the product on the minor diagonal of A from the product on the major diagonal (10 – 8 = 2) in each case. Tell them this is called the determinant and that it affects the pattern. Have them describe how.

Students should now try other matrices with different determinants to see how consistent the pattern is. When using technology, they will get an error statement if they try to find the inverse of a matrix whose determinant is zero. Discuss why.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Mental Math (**B11/A4**)

1) Find the determinant, if possible, given the following 2x2 matrices.

a) b) c) d)

2) State the inverse matrix, if possible, for each of the above matrices.

Journal (**A4**)

3) Explain the conditions under which a matrix would have no inverse.
4) Explain the conditions under which a matrix would have no identity.
5) Explain why the determinant is important when finding the inverse of a matrix.

Performance (**A4**)

6) Prove that and are inverses of one another.

det A $N = \begin{pmatrix} & \\ -\frac{3}{10} & -\frac{2}{5} \end{pmatrix}$ $x - 1$

7) Given and , find

a) b) A^{-1} c) B^{-1} d) $(A + B)^{-1}$

8)

a) Find det M and state why M must have an inverse.
b) Evaluate M^2 and compare it to M^{-1}.

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

A4 demonstrate an understanding of the conditions under which matrices have identities and inverses

B13 solve systems of equations using inverse matrices

$C19_2$ solve problems involving systems of equations

Elaboration – Instructional Strategies/Suggestions

A4/B13/$C19_2$ Students should understand that arranging numbers into a matrix is a useful procedure and that subsequent operations can be performed. Investigations can show that there is a parallel between the identity (I) matrix and the number 1 in the real number system, although it isn't a true parallel because (I) only exists for square matrices (see A5.) Students will identify matrices, identify and apply inverse matrices, and use these and appropriate technology to solve problems involving systems of equations.

The guide on p. 48 was demonstrating the solution to a 2x2 system using

matrices, but never finished. Students had $AX = B$, or .

Multiplying both sides by the inverse: $A^{-1}AX = A^{-1}B$
On the LHS, $A^{-1}A = I$ and $IX = X$.

Then all there is to do is $A^{-1}B$

They can conclude that the solution to the system is the point (4, –2).

Students will model real-world situations with two, three, or more equations, using two, three, or more variables. For example, bolts, nuts, and washers are produced using three different machines. The chart below shows the number of hours each machine spends on each of the three parts. Find how many of each can be produced in a day if machine 1 is used 6 hours per day, machine 2, 7 hours per day, and machine 3, 8 hours per day.

	Bolts	Nuts	Washers
machine 1	2	1	2
machine 2	1	1	4
machine 3	1	3	1

To solve this problem, students may first establish a system of three equations using three variables.

$$2b + n + 2w = 6$$
$$b + n + 4w = 7$$
$$b + 3n + w = 8$$

They should then re-express the system as matrices and use technology and the inverse matrix method to find values for b, n, and w.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Performance **(B13/A4)**

1) Solve each of the systems, using the inverse matrix method.

a) $\begin{cases} x-2y-0 \\ -2x+4y=3 \end{cases}$ b) $\begin{cases} 4x=11+3y \\ 2x=9+5y \end{cases}$ c) $\begin{cases} 3x-4y=-2 \\ 2x+y=5 \end{cases}$

2) Solve these systems

a) $\begin{cases} x+2y+3z=3 \\ x+y+2z=1 \\ y-4z=2 \end{cases}$ b) $\begin{cases} \frac{1}{2}x-\frac{2}{3}y+5z=2 \\ \frac{2}{3}x+\frac{2}{5}y-3z=1 \\ \frac{1}{5}x-\frac{1}{2}y-\frac{4}{7}z=0 \end{cases}$

Performance **(C19$_2$/B13)**

3) Determine the intersection of the planes P_1, P_2 and P_3, if possible. If not possible, explain why not.

a) $\begin{cases} P_1: x-2y+3z=9 \\ P_2: x+y-z=4 \\ P_3: 2x-4y+6z=5 \end{cases}$ b) $\begin{cases} P_1: x+2y+x=12 \\ P_2: 2x-y+z=5 \\ P_3: 3x+y-2z=1 \end{cases}$

4) Three robots agreed to have a race. The sum of their speeds was 30 km per hour (kph). The robots had names: C3P–oh, R2D–too, and PI3–dotcom. PI3–dotcom's speed plus one third of C3P–oh's speed was 22 kph more than the R2D–too's speed. Four times the C3P–oh's speed plus three times the R2D–too's speed minus twice the PI–dotcom's speed was 12 kph. Find how fast (to the nearest kph) each robot ran.

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

A5 demonstrate an understanding of properties of matrices and apply them

Elaboration – Instructional Strategies/Suggestions

A5 Matrices provide convenient, logical ways to store information. Operations that apply to real numbers apply to matrices. Students will discover that the properties for real numbers apply to matrices, except for commutativity under multiplication. Students should discover that the commutative property does not hold true for matrices under multiplication. They should also be aware that square matrices are the only matrices that can have an identity, and of the conditions under which square matrices have inverses. Students will be expected to discover answers to the following questions:

a) If you add matrices, is the order in which the matrices appear important? Why or why not?
b) If you multiply matrices, is the order in which the matrices appear important? Why or why not?
c) What conditions or characteristics must be considered for adding or multiplying matrices?

Commutative Property for Addition

$$A + B = B + A$$

Information can be organized into a series of columns and rows. This information can be manipulated according to specific criteria. Example: Medal standings for countries during the Atlanta Olympics were reported to the public in a variety of fashions, including total number of medals; number of gold, silver, and bronze medals; and the point standings. The information for the end-of-day results were placed in a matrix on a computer and manipulated to convey these results. The following results are the medal accumulations for four countries:

	Day 1			Day 2			Day 3		
	Gold	Silver	Bronze	Gold	Silver	Bronze	Gold	Silver	Bronze
CAN	0	0	1	0	1	0	0	2	1
USA	2	0	0	2	1	2	3	1	1
ENG	0	1	1	1	0	0	0	2	1
RUS	1	2	1	2	0	2	2	3	1

How many gold, silver, and bronze medals did each of the four countries receive at the end of the second day? Would it make any difference if the medals on Day 1 and Day 2 were added in the opposite order?

Conclusion: $$\frac{A + B = B + A}{(A+B)+C = A+(B+C)}$$

continued ...

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (A5)

1) The following table shows the numbers of different kinds of shirts sold by a store over 3 months. The store sells t-shirts, short-sleeve, crew-neck, and long-sleeve shirts.

	Jan S M L	Feb S M L	Mar S M L
T	10 15 25	5 10 20	15 5 20
SS	3 2 5	1 2 5	10 12 22
C	15 20 25	20 22 17	17 10 5
LS	12 15 20	15 15 25	1 3 2

a) The owner of the store needs to total all the small t-shirts sold in the first 3 months of the year. He wonders if he can add January and February then add March, or if he should add February and March, then add January. Do you think it will make a difference? Show that it does or doesn't.

b) Test your conclusion from (a) by adding all the medium long sleeve shirts in the same way.

c) Will the total of crew neck shirts (large) be different for Jan + Feb than for Feb + Jan? Show your work.

d) In March the price for shirts forms a matrix like this: $\begin{matrix} T \\ SS \\ C \\ LS \end{matrix}\begin{pmatrix} 12 \\ 10 \\ 15 \\ 15 \end{pmatrix}$.

The prices are the same for all sizes. How much money did the owner take in during the month of March? Keep in mind that you still want a total for each size. Determine if it makes a difference how he organizes the two matrices to multiply them. Explain.

Journal (A5)

2) Use the results from #1 above to talk about the commutative and associative properties for addition, and the commutative property for multiplication. Give examples for each

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

A5 demonstrate an understanding of properties of matrices and apply them

Elaboration – Instructional Strategies/Suggestions

... continued

A5 On their calculators, have students enter the Olympic results (expressed as matrices). Perform the operations in the order appropriate to the statement

$$(A + B) + C = R_1$$
$$A + (B + C) = R_2$$

Are the results identical? (Is $R_1 = R_2$?) If yes, then one might conjecture that associative property is true for addition of matrices. Restriction: *A*, *B*, *C* must have the same dimensions.

Associativity for multiplication can be demonstrated using the same test as above for associativity under addition. The assumption must be that each of the matrices can be multiplied according to the definition of matrix multiplication.

For example, if *A* is *3x2* , *B* is *2x4* , and *C* is *4x3*, then *A(BC)* becomes

multiplication with respect to the dimensions resulting in a 3×3 matrix as would

(as long as the dimensions allow for multiplication to occur as in the above example).

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Performance (A5)

1) Make up three, *3x3* matrices and call them *A*, *B*, and *C*. Show that . Does this mean that associativity under multiplication holds? Explain.

Activity (A5)

2) a) Plot the points *A*(2, 3), *B*(5, –2), and *C*(7, 6).
   b) Make a 2 3 matrix called M of the coordinates for *A*, *B*, and *C*.
   c) Multiply the matrix *M* by the matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   d) Where must you place matrix *T* so that you can multiply? Explain.
   e) Record the answer matrix as coordinates, label them , plot them, and join them. Describe what you see on the graph.
   f) Describe in your own words what you have done from a) – e).
   g) Now multiply the answer matrix by . Show how you do this.
   h) Record this 2nd answer matrix as coordinates, label them , and join them. Describe the transformation observed.
   i) Using matrix *M*, *T*, and *P*, see if you can move to position with one set of multiplications. Determine all the possible orders of the matrices for this to work. For example, can they be in the order *T M P*? Describe the order in which you could multiply the three matrices once they are positioned.

$(A \cdot B)C$

$P = \begin{pmatrix} & \\ 0 & -1 \end{pmatrix}$

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

C5 determine quadratic functions using systems of equations

Elaboration – Instructional Strategies/Suggestions

C5 Students will use 3 equations with 3 variables to find the equation of a quadratic equation. For example, Wally, the human cannon ball, is fired from his cannon at a target 200m distant on the horizon and 8m above the ground. The mouth of the cannon is 3m above the ground. Assuming that his path is parabolic and that he just barely misses a tower 9.9m high, 20 metres from the target, what is his maximum height during flight?

To find the equation of the path, students might begin by overlaying the Cartesian coordinate system and determining 3 key points: (1, 3) the mouth of the cannon; (180, 9.9) the top of the tower; and (200, 8) the location of the target. To get the quadratic equation that passes through the three points (1, 3), (180, 9.9), and (200, 8), students will use the model $ax^2 + bx + c = y$ and set up a system of equations to determine values for a, b, and c.

- from (1, 3): $a(1)^2 + b(1) + c = 3$
- from (180, 9.9): $a(180)^2 + b(180) + c = 9.9$
- from (200, 8): $a(200)^2 + b(200) + c = 8$

Students will find . With these coefficients students can write the equation $-0.000671x^2 + .16x + 2.84 = y$.

To solve the problem, students need to determine the maximum height obtained by Wally. They can do this by graphing the equation, and tracing the graph, or using other calculation features with technology.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Performance (C5)

1) A man had his car repaired four times at the same service centre. From his bills, he recorded the following information.
   a) The man used a quadratic model to represent the cost of service in terms of the number of hours of work required. Is the model reasonable? Give reasons for your answer.
   b) Find an equation for the situation.
   c) Based on the equation, what might a 5-hour job cost?
   d) If the bill came to $50, how much time might you expect it would have taken?

Hours of work	Total
1	$44
3	$68
6	$104
$4\frac{1}{2}$	$86

2) A rocket attains a height of 250 m when it is fired at a target that is 200 m distant on the horizon. 180 m from this target and on the direct line of fire is an 85 m tall building. Will the rocket reach the target, and if so, by how much will it clear the building?
3) A ping-pong ball is thrown horizontally a distance of 80 m but only goes 10m off the ground. If the flight of the ball is parabolic, what equation will describe the path of the ball?
4) Water is discharged through a tap near the bottom of a water tank. The depths of water remaining in the tank with respect to the time of flow in the first 100s have been recorded in the table shown.
   a) Would you use a quadratic model to represent the situation? Give your reasons.
   b) Find an equation for the depth of water remaining in terms of the time of flow.
   c) When will all the water in the tank be emptied?

Time (s)	Depth (cm)
0	100
20	81
40	64
60	49
80	36
100	25

Suggested Resources

The Algebra of 3-Space

Outcomes

SCO: In this course, students will be expected to

B14(121) determine the equation of a plane given three points on the plane

Elaboration – Instructional Strategies/Suggestions

B14(121) In three-dimensional space, the equation of a plane is given by $Ax + By + Cz + D = 0$, as long as A, B, and C are not simultaneously zero, and

.

To find the equation of the plane that contains the three points

E = (10, 1, 1)
F = (2, –2, 2) and
G = (12, 7, 3),

students should first divide the model equation $Ax + By + Cz + D = 0$ by A,

so that the coefficient of x is 1. .

This may be written as $x + Py + Qz + R = 0$.

Students can now plug in the given values for x, y, and z and solve the resulting system for P, Q, and R:

Rearrange $P + Q + R = -10$

$-2P + 2Q + R = -2$

$7P + 3Q + R = -12$

Students would now enter the coefficients into matrix M and N.

matrix matrix

Then, using the inverse matrix method FRAC ENTER, they would get

and substitute this back into the equation, getting , the equation of the plane which contains the given points E, F, and G.

The Algebra of 3-Space

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**B14(121)**)

1) Find the values for A, B, C, and D that would give the equation for each of the following planes:
   a) the $y-z$ plane
   b) the $x-z$ plane
   c) the plane parallel to the $x-y$ plane, 4 units above it
   d) a plane parallel to the $x-z$ plane, 2 units to the left of it

2) Find the plane that contains each of the following sets of points:
   a) (2, 4, 7), (1, 0, 5), and (–2, –3, 8)
   b) (1, –2, 3), (4, 12, 0) and (1, 3, –2)

Performance (**B14(121)**)

3) Find the equation of the plane that passes through the origin and the line of intersection of the planes defined by $3x + 4y - 7z - 2 = 0$ and $2x + 3y - 4 = 0$.

4) Ted is building a tree house for his son, Junior. There are 3 branches in the tree on which Ted thinks he could build the main platform. However, he doesn't think that it will be level and he needs to know the measurement of supports he could use to make the platform level. He superimposes a 3-dimension grid over a sketch of the tree branches and determines the coordinates for the 3 points of contact. He needs the equation of the plane that contains these coordinates: (0, 0, 0), (4, 3, 7), and (–2, –5, 8).

Suggested Resources

Unit 2
Trigonometric Functions
(15-20 Hours)

Trigonometric Functions

Outcomes

SCO: In this course, students will be expected to

C8 demonstrate an understanding of real-world relationships by translating between graphs, tables, and written descriptions

Elaboration – Instructional Strategies/Suggestions

C8 Students should be given lots of situations from the real-world represented in tables, charts, or with words and graphs to explore how two variables are related, and determine whether the relationships are periodic or not. For example, from words:

- depth of water at an ocean beach, and the time of the day
- temperature of a cup of coffee vs. time since the coffee was poured
- volume of air in lungs during breathing
- trampoline jumping -distance from floor over time
- distance of a waterwheel paddle from the water surface over time

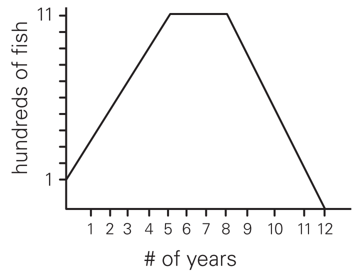
Some of these could be presented as collections of data, or displayed as a graph. Students would be expected to translate between the representations. For example, they might be asked to describe in words the motion that a graph represents. They might be asked to take measurements of how high above the ground a pebble is when stuck in a tread of a wheel, as the wheel rolls along. They might graph the relationship and try to use the graph to predict various heights as the wheel keeps rolling. They should always be careful to describe behaviour observed in one representation and explain why it behaves that way, using a different representation.

Trigonometric Functions

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**C8**)

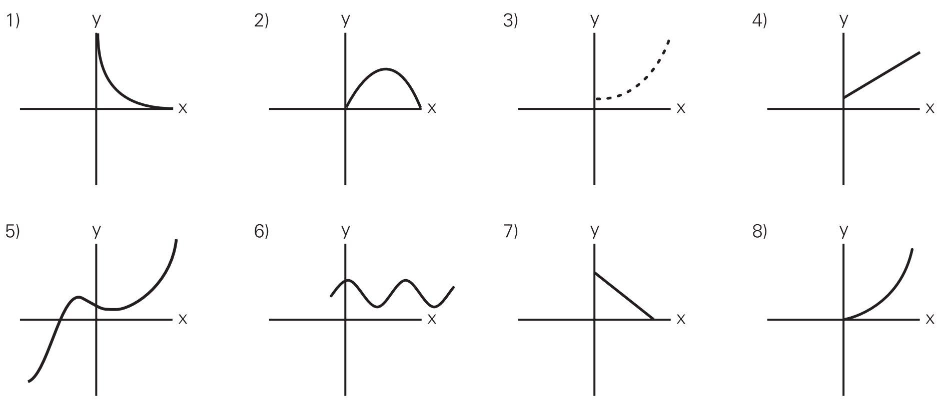
1) a) Write a story that would fit the graph.
   b) Complete the table.



x	0	2	4	6	8	10	12
f(x)							

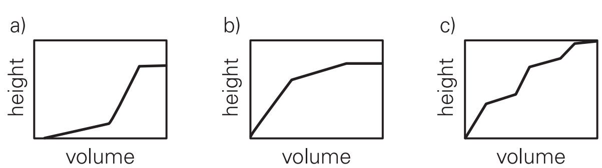
   c) About how many years would it take the declining fish stock to be reduced to about 850 fish?

2) For each of the following graphs, indicate what kind of function it could be and suggest a real-world relationship that the graph might represent.



3) a) Each student is to find an empty bottle (the more unusual the shape, the better) at home and construct a table of values relating the volume and the height as the bottle is filled with water. Students are to graph the data and bring the data, graph, and bottle to class. The class will attempt to match the various shapes of the bottles with the corresponding graphs.

   b) Sketch a possible shape of each bottle, the filling of which is shown below.



Suggested Resources

Trigonometric Functions

Outcomes

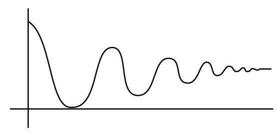
SCO: In this course, students will be expected to

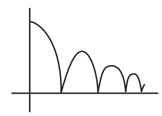
C23 identify periodic relations and describe their characteristics

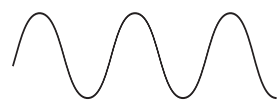
Elaboration – Instructional Strategies/Suggestions

C23 As students explore relationships between two variables in different contexts, they should note how the behaviour depicted by the variables differs. For example, when students explored the relationship between time and rental cost of snowboards, they found the longer that people rent snowboards, the more the cost of the rental. This was depicted by a straight line when graphed. The line made it easier to predict costs when given rental time. If a teacher made a bungee jump off a bridge, and the relationship between time and height above the ground or water was explored, the graph would not be a straight line. Students might think, though, that it would have similar characteristics to a height-time graph of a bouncing ball graph.

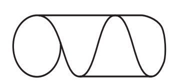
In the analyses, however, the difference should become more obvious. The bungee relationship would show rebounding heights above ground getting closer and closer to the ground, and rebounding lows getting higher and higher above the ground, both moving toward a final resting position equal to the length of the bungee cord. Different kinds of forces are involved with the bouncing ball, and a major component is the fact that the ball rebounds from a stopped position (the ground). Students might be asked to describe how these behaviours are similar to each other, but different from a linear relationship or the relationship between the distance a wheel rolls and the height above the ground of a pebble stuck in the tread. When they explore the pebble in the wheel relationship, they get a graph similar to the bungee graph, but more regular. Any part of the graph can be translated onto any other part of it.







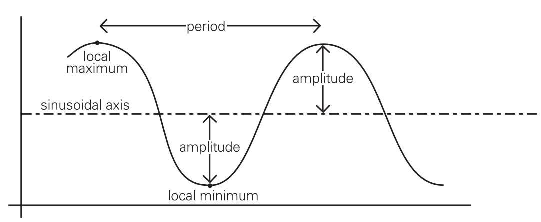
Students should learn to distinguish between periodic and sinusoidal functions. A function is periodic if the values for the dependent variables repeat over and over again as the independent variable changes like the pebble in the wheel relationship. Sinusoidal relations are special periodic relationships. The graph has a horizontal axis along which the graph can be translated onto itself, forming equal amplitudes above and below. There is a point on the sinusoidal axis on which the sinusoidal graph will have rotational symmetry.



Sinusoidal. 

Periodic, but not sinusoidal. 

Students should summarize all the characteristics of sinusoidal relationships and their meaning with respect to the context.

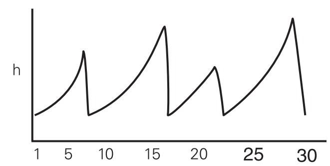


Trigonometric Functions

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (C23)

1) The graph to the right shows the height of grass for the month of July.
   a) Is this graph periodic? Explain.
   b) Write a paragraph to describe the condition of the lawn during the month of July.



2) Describe or draw a picture of a relationship that is periodic, but not sinusoidal.

3) A group of students was asked to gather data that represented a periodic function. The group decided that they would use a wheel in a vertical position. They would place a mark on the wheel and at regular time intervals measure the height of the mark. The wheel must rotate at a constant speed. Collect the data and display them graphically. Answer the following questions:
   a) Does the graph represent a periodic relationship? Explain.
   b) How would the graph change if the wheel was larger or smaller?
   c) How does the speed at which the wheel travels affect the graph?
   d) How does the speed affect the periodicity?
   e) Does the original graph represent a sinusoidal relationship? Explain.

4) Suppose that you and a friend go for a Ferris wheel ride. You are seated in your bucket and in 3 seconds you reach the top (13m above the ground) as the ride begins. The wheel rotates every 8 seconds, and has a diameter of 12m. Assume the Ferris wheel turns at a constant speed.
   a) Sketch a graph.
   b) What is the lowest you go as the wheel turns? Explain why this must be a positive number.
   c) Is the graph periodic? Sinusoidal? Explain.
   d) How high above the ground will you be after 2.5 minutes? Explain how you get your answer and why it must be correct.

5) Identify which of the following situations or relationships are i) periodic, ii) sinusoidal, iii) periodic, but not sinusoidal. Explain. (Take care to identify the independent and dependent variables).
   a) a person jogging around the block
   b) the distance a police officer is from the station over a 2-hour period
   c) a plucked tight elastic band
   d) the tide in the Bay of Fundy
   e) the apparent rising and setting of the sun over a year
   f) the distance a pendulum of a clock moves from its vertical resting position
   g) Molly swinging on her swing
   h) the bouncing of a lacrosse ball when dropped onto pavement from a height of 3m.
   i) the diving board after a dive.
   j) the teeth on a saw blade

Suggested Resources

"The Language of Functions and Graphs", Shell Material, Nottingham, England.

Trigonometric Functions

Outcomes

SCO: In this course, students will be expected to

B5 analyse and apply the graphs of the sine and cosine functions

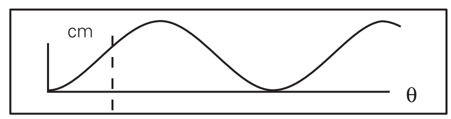
C3 determine the equations of sinusoidal functions

Elaboration – Instructional Strategies/Suggestions

B5 In trigonometry, students will develop, analyse, and apply the graphs of the sine and cosine functions. They will do this by studying real-world situations, plotting the information, recording it in tables and noticing its periodicity. Through various calculations, they will determine and apply the maximum and minimum values, the length and meaning of the period, the amplitude and the phase shift (horizontal translation). They will relate these values to the parameters in the equation stated in the functional form with in degrees, and explore what happens to the graph as each of these parameters changes.

Students might model a sinusoidal situation, using a bicycle wheel rotating on its axle. They would measure the height of the tire valve as the wheel rotates and the angle of rotation. Collecting the data in a table, then plotting points, students could then discuss how the points should be joined (smooth curve). They may need more values and want to look at other angle rotations like 30° and 60°. Students should talk about the meaning of the values near the top and bottom of the curve, and call these local maximum and local minimum. They should talk about the value of the points where the graph touches the horizontal axis (if it does) and what they represent on the wheel, and call these zeros. They should talk about why it doesn't touch the horizontal axis if it doesn't. They should draw the sinusoidal axis and determine the values of the curve that intersect this axis and what these values represent. Students should talk about the values that are in the domain and range and how these values are restricted and why. While it is understood that the domain of trigonometric functions is the set of real numbers, the first look at these graphs will often be using degree measure as the independent variable. They should graph and compare the shape of with the graph of the rotating wheel. They should answer questions like "Why is the amplitude larger for the wheel graph than for the graph?" Also, "Why doesn't the vertical axis intersect the wheel curve at the sinusoidal axis as it does with ?"

Rotation (θ^{o})	0	45	90	135	...
Height of valve (cm)	0		25		



C3 Using their knowledge about transformations, they might be able to determine the equation for the wheel graph. For example, the sinusoidal axis represents the axis of the wheel, so the amplitude represents the radius (25 cm) of the wheel. The radius for is 1 unit, so the equation should have

coefficient on the "*y*".

... continued

Trigonometric Functions

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**B5/C3**)

1) Revisit question 3, p. 73 and re-answer (b), (c), and (d) by referring to the values *A*, *B*, *C*, and *D* in the equation .
2) a) Revisit question 4, p. 73 and find the equation for the graph.
   b) Approximate how high above the ground you will be after 6 seconds. Three minutes.
   c) How high was the bucket when the ride began?
   d) At what time were you 5m above the ground? Explain.

3) If the equation that describes a situation is given by ,
describe the appearance of the graph. Describe a situation that this equation might model.

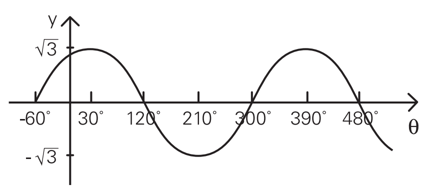
4) Graph the function described in #3 above, then answer the following:
   a) State the equation for the sinusoidal axis.
   b) State the domain and range.
   c) State the maximum values and where they occur.
   d) State the values for where the graph decreases.

Performance (**C3/B5**)

5) a) Use a graph to describe the transformations of y = sin that are evident in this image equation.
   b) Write a mapping rule.

θ $= 1$ $y = A \sin B(\theta - C) + D$ $y = \frac{4}{3}\sin 2(\theta - 45^\circ) + 8$

   c) State the equation.
   d) State all the zeros.
   e) State the minimum values and where they occur.
   f) Describe a situation this graph might represent.



Performance (**B5**)

6) A weight attached to the end of a long spring is bouncing up and down. As it bounces, its distance from the floor varies sinusoidally with time. You start a stopwatch. When the stopwatch reads 0.4 seconds, the weight first reaches a high point 50 cm above the floor. The next low point, 30 cm above the floor, occurs at 1.8 seconds.
   a) Sketch a graph of this sinusoidal function.
   b) Describe the length of a period and what it represents in this context.
   c) Predict the distance from the floor when the stopwatch reads 17.2 seconds.
   d) What was the distance from the floor when you started the stopwatch?

Suggested Resources

Trigonometric Functions

Outcomes

SCO: In this course, students will be expected to

C3 determine the equations of sinusoidal functions

C9$_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics and determine equations

Elaboration – Instructional Strategies/Suggestions

... continued

C3 The graph for passes through the origin. Translating the origin from (0, 0) on to this curve for this wheel means that it would have to move 90° to the right and 25 cm up (the axle is 25 cm above the ground), giving the two translation values—vertical translation (vt) 25, horizontal translation (ht) 90°.

Applying these transformations to the equation would give

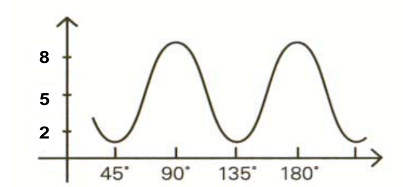
$$\frac{1}{25}(y-25)=\sin(\theta-90°)\cdot$$

There would be no horizontal stretch since the wheel rotates once every 360° and the period of is 360°. If in a situation the wheel rotated twice in 360°,

then there would be a horizontal stretch of , and the coefficient for in the

equation would be .

C9$_2$ Students should analyse the function to determine the relationship between it and . By examining graphs, they should be able to say that a graph of when translated 90° to the right matches the graph of and state that the sine curve is a +90° horizontal translation of the cosine curve.



Students might find that situations lend themselves to be described using a cosine function better than a sine function because the starting and ending points of the cosine curve are visually easier to determine. That is, the two high points stand out, and the distance between is the period. It might be easier to express the equation for the above curve as

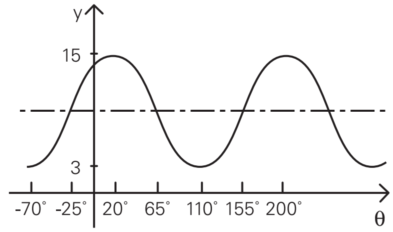
rather than $\frac{1}{4}(y-5)=\sin 4(\theta+22.5°)$.

Students should be able to sketch graphs of images of and using transformations. They could use the mapping rule to transform the table, or they could use the transformations and the 5 key points. The 5 key points are the first and last point of the period, and the three points which divide the period into quarters.

Trigonometric Functions

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (C3/C9$_2$)



1) a) Write the equation that represents this graph using as the model.

b) Write the equation again using as the model.

2) a) Create a table of values using for the equation

.

b) Create a table of values, using $0° \le \theta \le 360°$ for the equation

.

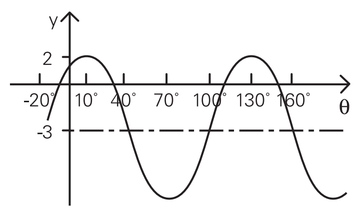
c) Describe the relationship between the *y*-values in two tables above.

d) Create a table of values using for the equation

, and describe the relationship between the functions

and .

3) From the graph:



a) State the equation for the sinusoidal axis.

b) State the amplitude.

c) State the period, domain, and range.

d) Write the equation for this graph showing a reflection for in the x-axis.

e) Approximate the zeros from the graph.

f) State the values for where the graph increases.

$\frac{1}{2}(y-1)=\cos\theta$

Suggested Resources

Trigonometric Functions

Outcomes

SCO: In this course, students will be expected to

C9$_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

Elaboration – Instructional Strategies/Suggestions

... continued

C9$_2$

❐ When asked to graph , students might use method 1 or 2.

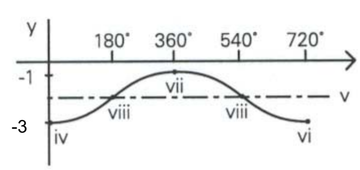
Method 1: $y = \cos\theta$

θ	y
0˚	1
90˚	0
180˚	-1
270˚	0
360˚	1

θ	y
0˚	-3
180˚	-2
360˚	-1
540˚	-2
720˚	-3

Method 2:

i) starting point (0, 1)
ii) reflect over x-axis (0, -1)
iii) multiply by horizontal stretch (0, -1) (-value times 2)
iv) apply vertical translation (-2) (0, -3) plot (P_1)
v) sinusoidal axis (vt) y = –2 sketch (dotted line)
vi) ending point 2(360°) (720°, –3) plot (P_{last})
vii) midway between high point (1unit above sinusoidal) (360°, –1) plot (P_{mid})
viii) midway between first two points and on sinusoidal axis (180°, –2) plot (P_4)
ix) midway between last two points and on sinusoidal axis (540°, –2) plot (P_5)



Trigonometric Functions

Worthwhile Tasks for Instruction and/or Assessment

Performance (C9$_2$)

1) Sketch the graphs for the following functions:

a) b) $-\frac{1}{3}(y+5)=\cos 3(\theta-45°)$

2) a) Given the function ,

i) state the period
ii) state the amplitude
iii) state the sinusoidal axis

b) assuming this equation is the image of
i) locate the image of (0°, 1)
ii) locate the image of (360°, 1)

c) State the domain and range of the function.
d) Locate all zeros.
e) Find
i) $f(120°)$
ii) $f(187.5°)$
iii) , if

Journal

[illegible]

3) Imagine that your friend Maynard phones you for help with homework. He wants to know how to graph using transformations.

Write an explanation of the procedure you should tell him over the phone. Check your explanation to make sure it works.

Suggested Resources

Trigonometric Functions

Outcomes

SCO: In this course, students will be expected to

C21$_2$ describe how various changes in the parameters of sinusoidal equations affect their graphs

Elaboration – Instructional Strategies/Suggestions

C21$_2$ Students will explore periodic behaviour and attempt to model this with graphs and equations. An important part of the modeling procedure would be to explore the graphs of sine and cosine functions to see how the shapes and locations of these graphs are affected by the values given to *A*, *B*, *C*, and *D* in the equation . There are two ways that students could go about exploring these relationships. One would be to use the function screen on the graphing calculator to enter three equations, say

Explore the three graphs and answer questions like "What is the same about all the graphs?" "What is different about all the graphs?" Students would answer with comments like "They all have the same x-intercepts." "They all pass through the origin." but "They all have different amplitudes (A)." Students would then focus on the "*Y* =" screen of the graphing calculator where they see the equations, and respond to "What is the same about the equations?" "What is different about the equations?" Using their responses, they may guess that the differences in the equations were affecting the amplitudes on the graphs—they make a conjecture. They should then make up an example and use it to verify their conjectures.

Students could continue like this to explore each of the 4 parameters, *B*, the period, *C*, the horizontal shift or translation, and *D*, the vertical translation. They should make and test conjectures for each, and describe the transformations of these graphs in words, algebraic symbols, or as mapping rules.

- ❐ Students should describe the graph of $-2(y-3)=\cos\frac{1}{3}(\theta+30°)$ as the graph of $y=\cos\theta$ that has been reflected in the x-axis, stretched vertically by a factor of and horizontally by a factor of 3, then translated up 3 and to the left 30°. Students should also describe the transformations, using a mapping rule

.

Trigonometric Functions

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**C21**$_2$)

1) Describe how the following functions are representations of

.

i) ii) $y=-\frac{1}{4}\cos(2\theta+180°)+1$

iii) iv)

2) In each of the functions in #1,
   a) state the amplitude
   b) the period
   c) the sinusoidal axis and how each of these will affect the graph or

3) In each of the functions in #1, describe the transformations of or
   a) in words
   b) as mapping rules and how these transformations will affect the graphs of or

4) Write the equation of the image of and , given the following mapping rules:

[illegible]

   a)
   b)

5) Use transformations to show each of the following:
   a)
   b) $\sin(-\theta)=-\sin\theta$
   c) $\cos(-\theta)=\cos\theta$
   d) $\sin(\theta+120°)=-\cos\theta$

Suggested Resources

Unit 3
Trigonometric Equations and Identities
(15-20 Hours)

Trigonometric Equations and Identities

Outcomes

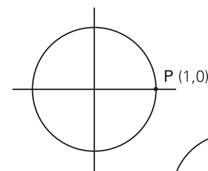
SCO: In this course, students will be expected to

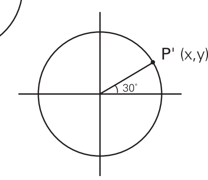
$C9_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

A1 demonstrate an understanding of irrational numbers in applications

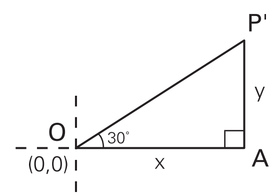
Elaboration – Instructional Strategies/Suggestions

$C9_2$/A1 Students will develop an understanding of the relationships among the angle of rotation, the radius of the circle, and the coordinates of a point that is on the circumference of the circle. Students should begin by watching the point P (1, 0) as it rotates around the unit circle centred at (0, 0). Students should rotate the point P 30° and determine its coordinates. To determine the coordinates, they would drop a perpendicular from P' to the x-axis and find the length of the two legs in the right triangle. The x-coordinate for would be the horizontal distance from the centre (0, 0) to the foot of the perpendicular (A).





Since the radius $OP' = 1$, then x can be determined as: . Similarly, . Students should conclude that when by rotating 30° centre (0, 0) in a unit circle, the coordinates for will be (cos 30°, sin 30°). These coordinates can be simplified to decimal value and , or students might determine these values using transformations and the Pythagorean Theorem: Reflect across the x-axis and join and . The is an equilateral triangle and $P'A = \frac{1}{2}P'P''$. Since each side measures 1 unit then $AP' = \frac{1}{2}$ unit and

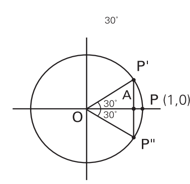


$$OA^2 = (OP')^2 - (P'A)^2 \quad \text{(Pythagorean Theorem).}$$

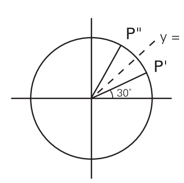
$$OA = \sqrt{1 - \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{1 - \frac{1}{4}}$$

$$= \pm\frac{\sqrt{3}}{2}$$



So, the coordinates for after a 30° rotation about (0, 0) in a unit circle can be expressed as (cos 30°, sin 30°), or approximately as (0.866, .5), or exactly as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, using the positive value . Now reflecting across $y = x$ produces the Point . This location of can be expressed as a rotation of 60°, centre (0, 0) of the point (1, 0) in the unit circle. Since now is the image of after a reflection in $y = x$, then its coordinates are the reverse of the coordinates of , or (.5, .866), or exactly as $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



continued ...

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper **($C9_2$/A1)**

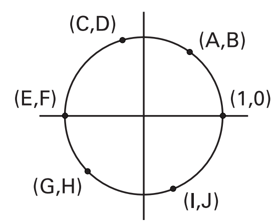
1) Complete the following table for a rotation of P(1, 0), centre (0, 0):

angle of rotation	Image of P (1,0)		
	As (x,y)	As a decimal	Exact
30˚			$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
60˚	(cos60˚, sin60˚)		
		(-0.5, 0.866˚)	
	(cos270˚, sin270˚)		
120˚			
			$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
180˚			

2) i) Locate the following points on the unit circle:
   a) (cos 330°, sin 330°) b) (cos – 120°, sin - 120°)
   c) (cos 180°, sin 180°) d) (cos 210°, sin 210°)
   ii) Write the coordinates for (i) as exact values.

$\frac{1}{2}$ $\frac{\sqrt{3}}{2}$

3) Which coordinate could represent the value of each of the following:
   a) cos(290°) b) sin(180°) c) cos(110°)
   d) sin(225°) e) cos(–70°) f) cos(420°)



Journal **(A1)**

4) Explain why can be expressed as an exact value, but is only approximately 0.866.

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

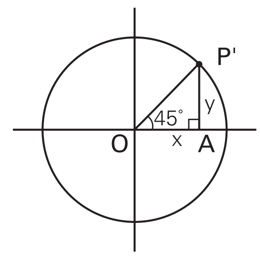
C9$_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

A1 demonstrate an understanding of irrational numbers in applications

Elaboration – Instructional Strategies/Suggestions

... continued

C9$_2$/A1 A 45° rotation of P(1, 0) centre (0, 0) in the unit circle would result in being right on the line y = x forming an isosceles when the perpendicular is dropped from .



but since P' is on $y = x$, then $y = x$ and

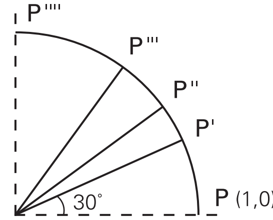
but $OP' = 1$, so

Students should be able to simplify $\sqrt{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{2}} = \frac{1}{\sqrt{2}}$, and the convention normally used suggests that the denominator be rationalized $\frac{1}{\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. In this form students are more able to think of this coordinate as 'half of root 2' or about 0.7. So, the coordinates for P' after a 45° rotation about (0, 0) in the unit circle

are , or approximately (.707, .707), or exactly , again

using the positive value for .

To quickly summarize, students should now be able to relate some of the rotations of P (1, 0), and the coordinates of the range of P through those rotations in the unit circle (radius 1). Students should develop strategies that will allow them to interplay between the relationships. Mentally they should be able to determine the values for cos 45°, sin 60°, cos 0°, sin 90°, etc. They should be able to say that, if the coordinate for R is (.707, .707), then R is located at the point that is the image of P(1, 0) after a 45° rotation, centre (0, 0) in the unit circle.



Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**$C9_2$/A1**)

1) Complete the following table for a rotation of P(1, 0), centre (0, 0):

angle of rotation	Image of P (1,0)		
	As (x,y)	As a decimal	Exact
45˚			
	(cos225˚, sin225˚)		
		(.707, -.707˚)	
	(cos270˚, sin270˚)		
			$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
330˚			
			$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

2) i) Locate the following points on the unit circle.
   a) (cos45°, sin45°) b) (cos240°, sin240°)
   c) (cos315°, sin315°) d) (–.707, .707)
   ii) Write the above coordinates as exact values.

3) Explain why $\sin(135°) = \frac{\sqrt{2}}{2}$.

4) Explain why $\cos(-60°) = \frac{1}{2}$ and not .

Journal (**A1**)

5) Explain why is called an exact value, while 0.707 is only approximate.

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

C9$_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

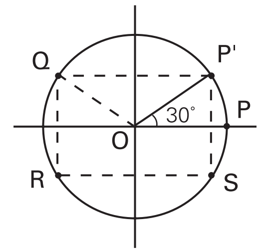
A1 demonstrate an understanding of irrational numbers in applications

C15 demonstrate an understanding of sine and cosine ratios and functions for non-acute angles

Elaboration – Instructional Strategies/Suggestions

... continued

C9$_2$/A1/C15 Students should extend this understanding of the relationships just established by visualizing the locations of these points as they are reflected across the x- and y-axes to locations in the 2nd, 3rd, and 4th quadrants. In these new locations values of the same numeric magnitude exist for the multiples of 30°, 60°, and 45° angle rotations. Note how Q is the image of P reflected in the y-axis. The or 150°. So, the coordinates for Q are the same magnitude as for , but the



x-value is negative (–.866, .5) or . Similarly R

becomes and S . Using their knowledge of these relationships,

students can evaluate trigonometric expressions like to get

and simplify to $\frac{3\sqrt{2}}{2}-\frac{1}{4}\Rightarrow\frac{6\sqrt{2}-1}{4}$.

(Notice $\sin^2 30°$ in the above notation. Students should understand that this is the common way of writing .

Students in the Level 1 course should explore these relationships even further, using the parametric mode on their graphing calculators. Have them draw the unit circle and the curve, and trace along one graph, then jump to the other. When the jump is made, the cursor moves to the corresponding point on the graph, or vice versa, from the graph to the corresponding coordinate on the unit circle. Have them analyse what is happening, then use the technology to deepen their understandings and build their mental math skills.

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper **($C9_2$/A1/C15)**

6) The point P(1, 0) on a unit circle centred at the origin is rotated through an angle of –120°.
   a) Find the coordinates of the image point of P and write a mapping rule for this rotation.
   b) Evaluate the values of cos(-120°) and sin(–120°).

7) Name the smallest positive angle of rotation which is the same as each of the following rotations:
   a) –540° b) –270°
   c) –390° d) –135°

8) For what values of will each of the following occur.
   a) sin θ is positive b) cos θ is negative c)sin is a minimum

9) The image of P(1, 0) on the unit circle is rotated about (0, 0). State the angle of rotation for each of the following image coordinates.

   a) b) c) (-1, 0) d)

10) Evaluate the following:
   a) sin30° + cos 60° b) 12 sin45° cos45°
   c) 20 sin60° cos240° d) cos180°cos45° – sin180° sin45°

   e) $\sin^2 30° + \cos^2 70°$ f)

   g) $\frac{\cos 45°}{\sin 120°} - \frac{\sin 210°}{\cos 30°}$

11) a) Plot the point , using a compass and protractor, but not a calculator.

   b) Estimate $\cos(115^0)$, perhaps using a ruler.

Journal **($C9_2$/A1)**

12) If $90° < \theta < 180°$ and , explain why must be .

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

C1 model situations with sinusoidal functions

C28 analyse and solve trigonometric equations with and without technology

C18 interpolate and extrapolate to solve problems

C30 demonstrate an understanding of the relationship between solving algebraic and trigonometric equations

B4 use the calculator correctly and efficiently

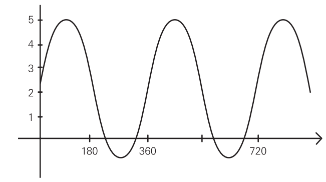
C27 apply function notation to trigonometric equations

Elaboration – Instructional Strategies/Suggestions

C1/C28/C18 All students should understand that the relationship being studied by graphing trigonometric functions is the relationship between the coordinates for the points on the unit circle and the angle rotation. If students graph y = sin and trace, they will see, for example, when , that y = 0.5. Students should now be able to use the equation of the sinusoidal relationship to predict or calculate values to help solve problems. As students model situations with sinusoidal equations, they are expected to obtain approximate answers by interpolation and extrapolation, or by tracing the graph.

C30 They should extend this to getting more exact answers by solving these equations both with and without technology.

For example, the graph of the rotation of a water wheel is given. Students could find the equation and use the equation to answer a question like "How high is a certain paddle after 2 full rotations?"



The equation: . After 2 rotations $(360° \times 2)$, they would evaluate sin 720°. Its value is the same as sin 360° or the value is zero. (Picture the unit circle - P (1, 0) does not rotate or rotates one complete or 2 complete rotations—sine is still zero.) Students would then solve and conclude that the paddle is 2 m high after 2 full rotations.

C30/B4 If students were asked how many degrees rotation would result in the paddle having a height of –0.5 m (underwater), then students would solve

This answer implies that if the wheel rotated –56° the paddle would be 0.5 m below the waterline. Students will learn on the next page that there are infinitely more correct answers, since this relationship is periodic.

C30/C27/B4 Sometimes trigonometric equations are given in the form $y = 2\sin\frac{1}{2}(\theta) + 2$. Students should recognize this equation as the standard form (different organization of the terms), and that it could be expressed as a function . Students should be able to evaluate the function, using function notation. For example, they should be able to evaluate f(45°), f(20°), and f(120°). They should also be able to determine , given , etc.

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Activity **(C28/B4)**

1) The following activity will help students understand the connection between the unit circle and the graphs of $y = \sin\theta$ and .
   a) Have students create a unit circle and make the points (1, 0), (0, 1), (-1, 0), and (0, -1). Label (1,0) as P.
   b) Have students sketch the first period of beside the unit circle so that 1 on the vertical axis is the same scale as 1 on the unit circle. Students should label the horizontal axis in degrees.
   c) Below the sketch of , have students sketch so that the axes correspond (same scale).
   d) Ask students to label the point P on their two graphs, and explain why it is labelled where it is.
   e) Ask students to rotate P to (0, 1) and label this .
   f) Ask students to locate on the two graphs. Have students explain their thinking.
   g) Ask students to rotate P to (0, -1) and label this , and locate this on their two graphs.
   h) Ask students to use their graphs to find the coordinates for , if is the image of P after a 180° rotation. Ask students to explain how they obtained the coordinates from the two graphs. (Check using the unit circle.)
   i) Ask students to use the graphs to determine the coordinates for the image of P as it rotates from (1, 0) through the following rotations.
   i) 45° ii) 270° iii) 210° iv) 300°

Performance **(C28/C27)**

2) Graph the following functions and show/explain how you would use the graph to evaluate
   a) $f\,(22.5°)$ b) $f\,(–90°)$ c)
   d) e)

   i) ii)

Performance **(C28/C30/B4)**

3) Show how to evaluate $f\,(22.5°)$ and $f\,(–90°)$ for both equations above using an algebraic approach.
4) The equation represents the relationship between the height of a frog sitting on a paddle above the water as the water wheel rotates.
   a) What is the diameter of the wheel?
   b) What do "*h*" and " " stand for?
   c) What would be a proper domain and range?
   d) How high above the water does the frog get?
   e) Describe the rotation of the wheel in as many ways as you can.
   f) When does the frog first reach 10 units above the water?
   g) How much must the wheel rotate for the frog to come out of the water?

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

C1 model situations with sinusoidal functions

C28 analyse and solve trigonometric equations with and without technology

C18 interpolate and extrapolate to solve problems

C30 demonstrate an understanding of the relationship between solving algebraic and trigonometric equations

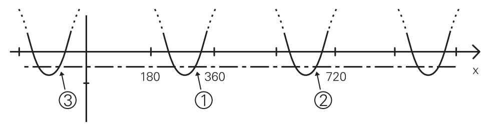
B4 use the calculator correctly and efficiently

Elaboration – Instructional Strategies/Suggestions

... continued

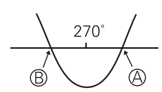
C1/C28/C18/C30/B4 On the previous two-page spread, students solved

for $y = {}^{-}0.5$, and obtained the answer . The graph doesn't show any negative degrees along the -axis. Students might have been expecting an answer



around 200°. In fact, if students drew a horizontal line through , it would intersect the graph in several places. If 56° was subtracted from 360°, the intersection point ① would be the point that answers the problem. This point repeats itself every +360° or –360°, as can be seen on the graph, marked ② and ③. Students should then show the solution to the equation mentioned above as

Also, students can see another rotation that results in a depth of –0.5 m. They need to remember the symmetry around this local minimum value. The x-value directly above the local minimum would be 270° (halfway between 180° and 360°). Since the x-value at *A* is 304°, that is 270° + 34°, so then *B* is 270° – 34° or 236°. Now students should complete the solution above as



C30/B4 While students have the option to solve trigonometric equations using graphing techniques, technology, or algebraic methods, all students need to practise using algebraic techniques to build their manipulation skills. Students in the Level 1 course will be expected to solve more complex trigonometric equations algebraically, such as the following:

1) $2\sin 2\theta = -\sqrt{2}, 0 \le x \le 720°$

$$\sin 2\theta = \frac{-\sqrt{2}}{2}$$

$$2\theta = \begin{cases} 225° + 360° \\ 315° + 360° \end{cases}$$

$$\theta = \begin{cases} 112.5° + 180°k, 0 \le k \le 4 \mid k \in I \\ 157.5° + 180°k, 0 \le k \le 4 \mid k \in I \end{cases}$$

2) $\sin^2\theta - 3\sin\theta + 2 = 0$, for all θ

$$(\sin\theta - 2)(\sin\theta - 1) = \theta$$

$$\sin\theta = 2 \text{ or } \sin\theta = 1$$

$$\theta = \varnothing \text{ or } \theta = 90^0 + 360^0 k, k \in I$$

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

Pencil and Paper **(C28/C30/B4)**

1) Show all your steps as you solve the following equations algebraically for

.

a) b)

c) d)

2) (Level 1) Solve the following for all θ :

a) b)

c) d) $\cos^2\theta = \sin\theta(1+\sin\theta)$

e) $24 - 24\cos\theta = 7\sin\theta + 5$

3) Students in the Level 1 course should revisit #4 on the previous third column and determine the answers to (d), (f) and (g) algebraically.

Pencil and Paper **(C1/C18)**

$(\sqrt{2}\sin\theta - 1)\left(\cos\theta + \frac{1}{2}\right) = 0$

4) Tommy has a tree swing near the river in his backyard. The swing is a single rope hanging from a tree branch. When Tommy swings, he goes back and forth across the shore of the river. One day his mommy (who was taking an adult math course) decided to model his motion using her stopwatch. She finds that, after 2 seconds, Tommy is at one end of his swing, 4 metres from the shore, over land. After 6 seconds, he reaches the other end of his swing, 5.2 metres from the shore over the water.

a) Sketch a graph of this sinusoidal function.

b) Write the equation expressing distance from the shore versus time.

c) Predict the distance when

i) time is 6.8 seconds

ii) time is 15 seconds

iii) time is 30 seconds

d) Where was Tommy when his mommy started the watch?

Journal **(C28/C30/B4)**

In order to solve the equations in #4(b), time must be expressed in terms of angle rotation. Assume the angle formed by the rope of the swing at its maximum is 120° and describe how to convert the time periods to accumulated angle measures. Write the equation expressing distance versus angle measure and find the angle measure from the vertical when Tommy is 1 metre from the shore, over land.

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

C24 derive and apply the reciprocal and Pythagorean identities

Elaboration – Instructional Strategies/Suggestions

C24 Students could examine tables of values to determine relationships that exist among the trigonometric functions.

Students could be asked to examine the following table and describe a relationship between and , between and , and between and .

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0^0	0	1	0	undefined	1	undefined
30^0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45^0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60^0	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Students might notice a pattern when comparing sin 0° = 0, and csc 0° is undefined, and , and csc 30° = 2. They might conjecture that the cosecant values for the same are reciprocals of the sine values. They might then test this, using . The reciprocal is $\frac{2}{\sqrt{2}}$ and rationalized, is . The trigonometric ratio for csc45° is $\sqrt{2}$. So the conjecture checks. Students should know these as the reciprocal identities:

Students might also notice or be encouraged to find that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. They should explore more and agree that $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$. Students should be asked to use the unit circle to show that $\sin^2\theta + \cos^2\theta = 1$ and call this the Pythagorean identity. They should then derive the other two Pythagorean

identities: Divide through by to get

Then divide through again by $\sin^2\theta$.

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (C24)

1) Given $\sec 32° \doteq 1.179$ and ,
   a) state the product sec 32°(csc32°) as a decimal to four places [ans: 2.2248]
   b) evaluate

      i) [ans: 1.887] ii) [ans: .624869]

   c) state the sum [ans: 2.2248] to four decimal places

   d) find a way to describe how this verifies that and $\cos\theta = \frac{1}{\sec\theta}$

2) a) Draw a unit circle centred at (0, 0). Label the origin 0.
   b) Rotate P(1, 0) through $\theta°$, where . is the image point.
   c) Drop a perpendicular from to the x-axis. Call the x-intercept A.
   d) State the length of OA and as trigonometric expressions.
   e) Calculate the slope of .
   f) Describe how this verifies that .

$\frac{\sin A}{\csc A} + \frac{\cos A}{\sec A} = 1$

3) Simplify the following expressions:

   a) $\cos\theta\tan\theta$ b)

   c) $\sin\theta + \cos\theta\tan\theta$ d)

   e) f) $\frac{1}{\sec^2\theta} + \frac{1}{\csc^2\theta} - 1$

   g)

4. Verify the following identities:

   a) $\sec\theta - \tan\theta = \frac{1-\sin^2\theta}{\cos\theta}$ b) $\frac{1-\tan^2\theta}{1+\tan^2\theta} = 1-\tan\theta$

   c) $\frac{\sec\beta}{\cos\beta} - \frac{\tan\beta}{\cot\beta} = 1$ d)

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

C24 derive and apply the reciprocal and Pythagorean Identities

C25 prove trigonometric identities

Elaboration – Instructional Strategies/Suggestions

C24/C25 Students should understand that trigonometric identities (the relationships described on p. 94) are useful in the simplification of more complex trigonometric expressions. Students will apply identities to simplify expressions. (Note: See next 2-page spread for a discussion of algebraic skills involed).

*** Students in the Level 1 course should have the opportunity to do more application of identities and in more complex situations. *** Students will also be expected to examine more complex trigonometric statements and to prove or verify that the statements are true, or that they are also identities. For example, students might be asked to prove that this statement is true: .

Typically, students would begin with the more complicated side of the statement and simplify it to look like the other side. Strategies are used to help students think their way through each verification. For example, in looking at the left side (LHS) $\cos\theta - \sin^2\theta\cos\theta$, students should decide that it is factorable: $\cos\theta\left(1-\sin^2\theta\right)$, then the Pythagorean Identity can be used to replace $1-\sin^2\theta$. Since $1-\sin^2\theta = \cos^2\theta$: they would then write $\cos\theta\left(\cos^2\theta\right)$. This combines through multiplication to give which is the same as the right side (RHS) of the original statement. The identity is verified. When written by the students, encourage them to be neat and to organize their verification. It might look something like this:

Verify if

LHS:	RHS:
$\cos\theta\left(1\text{-}\sin^2\theta\right)$ (factor)	$\cos^3\theta$
$\cos\theta\left(\cos^2\theta\right)$ (Pythagorean)	
$\cos^3\theta$ (same as RHS)	

Students should be helped to examine several strategies that will help them as they apply their identities in simplification or verification problems.

Here are some strategies:

1) Use reciprocal identities to express all trigonometric terms as sines and cosines.
2) Use Pythagorean identities.
3) Factor when possible.
4) Express two fractions as one.
5) Express one fraction as two.
6) Multiply by a clever form of one.
7) Multiply by conjugates.

Sometimes students will be helped by simplifying one side, then the other side, until both sides are equal.

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Performance (C24/C25)

1) a) Use factoring to re-express $\sin^4\theta\cos^4\theta$ in lowest terms.
   b) Use a Pythagorean identity to simplify further.

2) a) Express these two fractions as one fraction:
   b) Use a Pythagorean identity to simplify further.

3) a) Begin with $\dfrac{\sin\theta}{1+\cos\theta}$.
   b) Multiply the numerator and denominator by $1-\cos\theta$.
   c) Why could this strategy be called multiplying by a clearer form of 1?
   d) Why could this strategy also be called multiplying by conjugates?
   e) Simplify to make this fraction equal to .

4) a) Use the strategy of making one fraction two fractions with

.

$= \sec^2\theta$ $= 2\csc\alpha$ $\sin\theta\sin\alpha$

   b) Simplify and restate $\cot^2\theta$ as $\dfrac{\cos^2}{\sin^2\theta}$, then simplify again.
   c) Continue until this simplifies to -cot2x. What other strategies did you use?

5) Prove that each equation is an identity. Describe the strategies that you use.
   a)
   b) $\cos^4\beta-\sin^4\beta=1-2\sin^2\beta$
   c) $\dfrac{1}{\sec^2\theta}+\dfrac{1}{\csc^2\theta}=1$
   d)
   e) $\sec^2\theta+\csc^2\theta=\sec^2\theta\csc^2\theta$
   f) $\dfrac{1}{1+\cos\phi}-\csc^2\phi-\csc\phi\cot\phi$

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

B1 demonstrate an understanding of the relationship between operations on fractions and rational algebraic expressions

Elaboration – Instructional Strategies/Suggestions

B1 Since many identities involve working with fractional form, this may be a good time to strengthen students' skills with combining, multiplying, dividing, and simplifying rational algebraic expressions. For example, ask students to combine $\frac{3}{2}+\frac{5}{3}$. Watch as they re-express with a common denominator, .

Then give them a similar one with algebraic terms, . They should follow the same steps: $\frac{9+10}{6x}$, then simplify. Give them two fractions to multiply,

. Remove common factors before multiplying so that the numbers are small: . Some might multiply the remaining factors $\frac{30}{40}$, then factor and rename again, giving . Others could do all the dividing through by common factors in the first step and arrive at the same answer in one step.

Algebraically, when simplifying an expression like the following, students should factor any expressions possible, then, using division, simplify as much as possible:

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (**B1**)

1) Select some equations from below to show that you know how to combine algebraic fractions.

a) b) c)

d) $\frac{x^2}{4}+\frac{y^2}{5}$ e) $\frac{3m}{2a}+\frac{5m}{3b}$ f) $\frac{a^2+3}{2}+\frac{3a-a^2}{3}$

g) $\frac{m+2}{5}+\frac{m-2}{3}$ h) $\frac{4x-3}{15y}+\frac{12}{6y^2}$ i) $\frac{3m-5}{m^2}+\frac{4m}{m^3}$

j) $\frac{m^2-1}{2m}+\frac{m^2+1}{3m}$ k) $\frac{7x+9}{3y}+\frac{3x-2}{9y^2}$ l) $\frac{4}{8a}+\frac{5}{9a^2}$

m) $\frac{16-y}{4y}+\frac{3+y}{7y}$ n) $\frac{\sqrt{2}}{5}-\frac{3\sqrt{2}}{10}$ o) $\frac{\sqrt{18}}{2}-\frac{\sqrt{8}}{3}$

2) Select from below to show that you know how to multiply/divide algebraic fractions.

a) $\frac{3a+3b}{a^2-b^2}+\frac{6a-6b}{a^2-2ab+b^2}$ b)

c) $\frac{6a^2-a-1}{4a^2-3a-1}\times\frac{20a^2-a-1}{6a^2-5a+1}$ d)

[illegible]

e) f)

g) h)

i) j) $\frac{m}{m-7}\div\frac{m^3+6m^2+5m}{m-7}$

k) $\frac{16a-40a-24}{4a^2-4a-24}\times\frac{2a^2+3a-2}{m-7}$ l) $\frac{m^4-m^2-6}{m^4+2m^2}\times\frac{m^2-7}{m^4+5m^2}$

3) a) In question 2(g) above, substitute the value zero for "*m*" in the expression and simplify. What do you notice?

b) Try again, but use $m = -1$.

c) $m = 0$ and $m = -1$ are called restrictions on the variable. Can you explain what that means?

d) What restrictions are on "*m*" in question 2(l)? Explain.

4) Simplify. What values of the variables are not possible?

a) $\frac{1}{a^2+5a+6}+\frac{1}{a^2-9}$ b)

c) d)

e) f) $\frac{4}{3a^2+27a+60}+\frac{3}{2a^2+16a+30}$

Suggested Resources

Trigonometric Equations and Identities

Outcomes

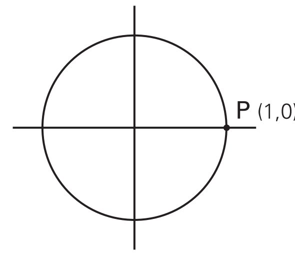
SCO: In this course, students will be expected to

D1 derive, analyse, and apply angle and arc length relationships

Elaboration – Instructional Strategies/Suggestions

D1 The final element in the development of trigonometric relationships is the arc-length a point travels as it is rotated through various angles of rotation, centre (0, 0) in the unit circle. All students should examine this element to determine the relationships.

As the point P rotates clockwise (CW) around the circumference of the unit circle, it travels certain distances. These distances are called arc lengths and can easily be calculated. For example, if P is rotated, centre (0, 0), 360°, it will travel the entire circumference. Students can determine the circumference, using $c = \pi d$. Since d = 2 in the unit circle, the entire circumference is units. A 180°



rotation would be units, a 90° rotation units (written and read “pi over two”). Students should be given the opportunity to combine arc length into their other relationship understandings and complete a detailed table like the following:

angle of rotation	coordinates for image of P (1,0)	radius	arc length
0˚	(1,0)		0
30˚	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$		$\frac{30°}{360°}$ *or* $\frac{\pi}{6}$
150˚	$\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$		$\frac{5\pi}{6}$
60˚		1	
300˚		1	
	$\left(-\frac{\sqrt{2}}{2}, -\frac{2}{2}\right)$		
		1	$\frac{3\pi}{4}$

continued ...

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper **(D1)**

1) Complete the following table for a rotation of P(1, 0) about (0, 0).

angle of rotation	coordinates for the image of P (1,0)	arc length
30˚	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	$\frac{\pi}{6}$
45˚	$\left(\frac{\sqrt{2}}{2}, \quad \right)$	
		$\frac{\pi}{3}$
90˚		
	$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$	
135˚		
210˚		
	$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$	
		$-\frac{2\pi}{3}$

Journal **(D1)**

$-\frac{3\pi}{2}$

2) Explain what these mean:

a) An arc length of is associated with an angle of rotation of –270°.

b) An arc length of 6 units on the unit circle is related to some angle rotation.

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

D1 derive, analyse, and apply angle and arc-length relationships

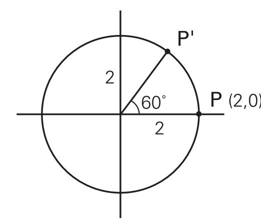
D2 demonstrate an understanding of the connection between degree and radian measure and apply them

C9$_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics and determine equations

Elaboration – Instructional Strategies/Suggestions

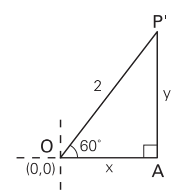
... continued

D1 Students should then examine how the relationships are affected if the radius of the circle changes. They should remember from their study of dilatation that if the radius doubles, then so will the circumference.
What happens to the coordinates of the image of P (1, 0)?



Begin with a circle radius 2 units and the point P (2, 0). Rotate it 60° about (0,0). Determine the coordinates for the image of P which is (P'). Since

in then x = 2cos 60°. Similarly,



y = 2sin 60°. So, the coordinates for P' will be (2cos 60°, 2sin 60°), thus simply a direct relationship with the radius. So if the radius is 3, the coordinates will be 3 times the value they would be on the unit circle, and the arc-length would be 3 times as long. Students should be able to explain why.

D2/C9$_2$ Up to this point when exploring the sine and cosine functions, the domain of the function has always been degree measure.

Another unit of angle measure was developed long ago that comes from the relationships being studied between arc length and radius. This measure of rotation is called radians.

Angle measure (in radians) = arc length radius

continued...

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Activity **(D1/D2)**

1) a) Using the origin as centre, draw three circles, radius 1, 2, 3 units.
   b) Make P (1, 0) on the unit circle.
   c) Rotate P (1, 0) 45°, centre (0, 0) and mark its image .
   d) State the coordinates for .
   e) Draw a line from the origin O through to hit the circumferences of the other two circles at R and S.
   f) i) What is ? [ans: 45°]
      ii) What is ? [ans: 45°]
      iii) What is ? [ans: 45°]
   g) i) What is the arc length of ? Explain why.
      ii) Show how to determine the corresponding arc lengths on the other two circles.
   h) If has coordinates (cos 45°, sin 45°), what are the coordinates for R and S in terms of cosines and sines? Show how to determine these.
      i) From the relationship between the coordinates seen in part (h), state the exact coordinates for R and S in simplified form.
   j) Find a way to justify these coordinates, using right angle calculations.
   k) The angle of rotation was 45°. What is the corresponding rotation in radians?
   l) State the relationship that seems to be true about the angle measure, arc-length, radius, and corresponding coordinates on the circumferences of circles of different radii.

[ans: $\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$]

Performance **(D1/D2)**

2) a) The point (1, 0) on the unit circle is rotated about the origin through each angle given in radian measure. For each rotation, find the arc length, to one decimal place, through which the point (1, 0) has moved.

   i) ii) iii) iv)

   v) vi) vii) viii)

   b) Redo (c), (d), and (g) above with P (2, 0) being rotated, centre (0, 0).

3) Find the length of the intercepted arc for each of the following central angles in a circle of radius 6cm. Express your answer to the nearest tenth of a centimetre.

   a) b) c) d)

4. Find the radian measure of the central angle that intercepts each of the following arcs in a circle of radius 5 cm.
   a) 12 cm b) 20 cm c) 5 cm d) 42 cm

5. The length of an arc of a circle intercepted by a central angle of 150° is 21 cm. Find the diameter of the circle correct to the nearest centimetre.

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

D2 demonstrate an understanding of the connection between degree and radian measure and apply them

$C9_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics and determine equations

Elaboration – Instructional Strategies/Suggestions

...continued

D2/$C9_2$ Radians are numbers that belong to the Real number family. In the unit circle, the radian measure is the same as the arc length. Students should spend some time visualizing, for example, a 45° rotation in the unit circle as a

rotation of radians. They might do this by completing a table like the

following , while visualizing the unit circle or its image:

angle of rotation		coordinates for the image of P (1,0)		
degrees	radians		radius	arc length
30˚			2	
		$\left(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$	1	
210˚			2	
			1	$\frac{3\pi}{4}$
120˚			3	
⋮	$\frac{5\pi}{6}$	⋮	1	⋮

All students should apply these relationships to problems involving angle measure and arc length.

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (D2/C9$_2$)

1) Complete the following table.

angle of rotation		coordinates for the image of P (1,0)	circle	
degrees	radians		radius	arc length
	$\frac{\pi}{6}$		3	
	$\frac{3\pi}{4}$		2	
210°			1	
315°				$\frac{7\pi}{4}$
	$\frac{2\pi}{3}$		3	
150°			2	
	$\frac{7\pi}{4}$	$(\sqrt{2}, -\sqrt{2})$		

[illegible] $\frac{\quad}{\sin\left(-\frac{\pi}{6}\right)}$ $+\cos\frac{5\pi}{4}$

2) Evaluate.

a)

b)

c)

d)

e)

f) $\sin^2\frac{\pi}{6}+\cos^2\frac{\pi}{6}$

g) $\sin\frac{\pi}{6}\cos\frac{\pi}{3}-\cos\frac{\pi}{6}\sin\frac{\pi}{3}$

h) $\cos\frac{2\pi}{2}-\sin\left(-\frac{\pi}{3}\right)$

i)

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

D2 demonstrate an understanding of the connections between degree and radian measure and apply them

A1 demonstrate an understanding of irrational numbers in applications

B4 use the calculator correctly and efficiently

C9 $_2$ analyse tables and graphs of various sine and cosine functions to find patterns, identify characteristics, and determine equations

C3 determine the equations of sinusoidal functions

C21 $_2$ describe how various changes in the parameters of sinusoidal equations affect their graphs

C15 demonstrate an understanding of sine and cosine ratios and functions for non-acute angles

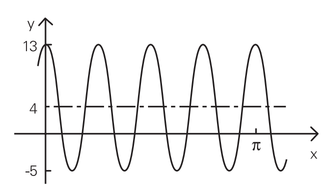
Elaboration – Instructional Strategies/Suggestions

D2/A1/B4 All students should be given the opportunity to evaluate trigonometric expressions given in degrees and radians. For example, all students can evaluate

$$2\cos\frac{\pi}{4} + 3\sin\frac{3\pi}{4}.$$

Such questions provide all students with opportunities to strengthen their application of the relationships between radian measure, arc-length and coordinates of points, as well as build their mental math and visual skills. For example, for the question above, students should visualize the point P(1, 0) rotated $\frac{\pi}{4}$ on the unit circle, read its x-coordinate and double it . Then students would visualize P(1, 0) rotated and read its y-coordinate and triple it . Then students would have to demonstrate their ability with fractions and radicals to combine and , giving . The mental math and the exact answers should be encouraged; however, some students may only be able to do this using a calculator, and approximate rationalizations. Students are expected to become as proficient with radians as they are with degrees. The radian measure will become more important to students since radian values belong to the Real number system, and calculus work with trigonometric functions is simplified when the domain is the set of real numbers.

C9 /C3/C21$_2$/C15 Students should revisit all types of questions and function work and re-explore and apply radian measure. For example, they should become very proficient with evaluating complex trigonometric expressions with radian measure. They should be able to determine equations of sinusoidal situations with a domain in real numbers. To obtain the equation of this sinusoid, students would describe the transformations of $y = \cos x$ as a vertical translation of 4, vertical stretch of 9, no shift, but a horizontal stretch of $\frac{1}{8}$.



(The period of ; the period here is $\frac{\pi}{4}$ units, and

.)

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

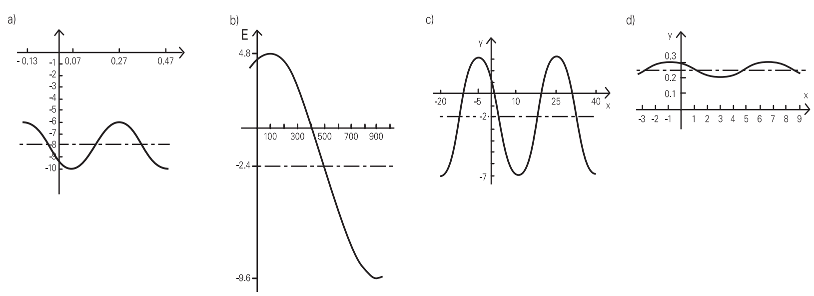
Performance (**D2/A1**)

1) Using the unit circle to visualize, determine the numerical value for each expression.

a) $2\cos\frac{2\pi}{3}+\sin\frac{2\pi}{3}$ b) $3\cos^2\frac{7\pi}{6}$ c)

d) e) $\dfrac{\sin\left(-\frac{\pi}{3}\right)}{\sin\left(\frac{3\pi}{4}+2\cos\left(\frac{7\pi}{6}\right)\right)}$

Performance (**C9 /C3**)



2) i) Determine the equations for the following graphs.
ii) State the following for each of the above graphs: domain and range, zeros, equation for sinusoidal axis, values that cause decreasing portions of the graph, minimum values and where they occur.

Performance (**C21** $_2$)

3) a) Draw the graph, given the following equations.
i) $-(y+2)=\cos\left(x+\frac{3\pi}{4}\right)$ ii)
b) State the domains and ranges, list the zeros, and give the values that cause maximum values for each of the above.
c) State the mapping rule for each of the above if they are images of y = cos x and *y* = sin *x* respectively.

Performance (**C15**)

4) Evaluate.

a) b) $\sec\frac{4\pi}{3}-\tan\frac{7\pi}{6}$

c) $5\sin^2\left(\frac{11\pi}{6}\right)-3\cos^2\left(\frac{5\pi}{3}\right)$ d)

Journal (**C3**)

5) Revisit problem 4, p. 93 and explain why and how you could solve the equation in #4b to determine the amount of time that it takes for Tommy to be 1m from the shore, over land. What other times will this occur? Explain.

Suggested Resources

Trigonometric Equations and Identities

Outcomes

SCO: In this course, students will be expected to

C28 analyse and solve trigonometric equations with and without technology

B4 use the calculator correctly and efficiently

Elaboration – Instructional Strategies/Suggestions

C28/B4

When solving equations, students should be able to solve equations in both degrees and radians. (See discussion on pp. 90, 92 for solving equations in degrees.) Students need to be careful about expressing the roots according to given domains. For example, solve

$$4\cos^2 z - 12\cos x = 16 \text{ for } x \in R$$

$$\text{Sol. } 4\cos^2 x - 12\cos x - 16 = 0$$

$$\cos^2 x - 3\cos x - 4 = 0$$

$$(\cos x - 4)(\cos x + 1) = 0$$

$$\cos x - 4 = 0 \text{ or } \cos x = -1$$

$$\cos x = 4 \text{ or } x = \cos^{-1}(-1)$$

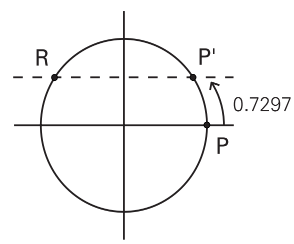
$$x = \theta \text{ or } x = \pi + 2k\pi, k \in I$$

$$\therefore \{x = \pi + 2k\pi, k \in I\}$$

Another example shows the important use of the calculator and the understanding of the role of symmetry to help determine all the roots when solving equations.

$$6\sin^2 2x - 4\sin 2x = 2 - 3\sin 2x \text{ for } -2x \le x \le 2\pi | x \in R$$

Solution:



$$2x = \sin^{-1}\left(\frac{2}{3}\right)$$

$2x \doteq .7297$

$P - P'$ represents the arc whose sine is

. But there is another arc

whose sine is $\left(\frac{2}{3}\right)$ also. To

find it, use symmetry:

.

$$\therefore 2x \doteq \begin{cases} 0.7291 + 2k\pi, k \in I \\ 2.4119 + 2k\pi, k \in I \end{cases}$$

$$\therefore x \doteq \begin{cases} 0.365 + k\pi, k \in I \\ 1.206 + k\pi, k \in I \end{cases}$$

$$2x = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$2x = \begin{cases} \frac{-\pi}{6} + 2k\pi, k \in I \\ \frac{-5\pi}{6} + 2k\pi, k \in I \end{cases} \quad (\text{dividing by } 2)$$

$$\therefore x = \begin{cases} \frac{-\pi}{12} + k\pi, k \in I \\ \frac{-5\pi}{12} + k\pi, k \in I \end{cases}$$

Since the domain is $-2\pi \le x \le (2\pi) / x \in R$, the final solution is

Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C1/C28/B4**)

1) The electricity supplied to a home is called “alternating current” (AC) because the current varies sinusoidally with time. The frequency (period =) of the sinusoid is 60 cycles per second. Suppose that at time t = 0 seconds, the current is at its maximum, i = 5 amperes, with an amplitude of 5.
   a) Draw a graph to represent several frequencies.
   b) Write an equation expressing current in terms of time.
   c) What is the current when t = 0.01?
   d) At what time intervals is the current at a maximum? Registering 1 amp?
   e) Use a CBL or a computer equivalent, the light probe, and a program that will gather data representing light intensity over time. (Use the program “Light 2” “Real World Math with the CBL System,” if available.)
      i) Locate a simple fluorescent bulb.
      ii) Start the program.
      iii) Point the probe close to the lit fluorescent tube.
      iv) From the graph on the screen, it appears that light intensity values are rising and falling in a regular pattern. What do you think the peaks represent in terms of the tube?
      v) Calculate the period, take its reciprocal, and that is the frequency (the time required for one complete on–off cycle).
      vi) How consistent is the frequency of this model compared to the given situation (60 cycles per second)?
      vii) State the equation that represents the intensity of the fluorescent tube over time and find the times when the intensity is zero. Interpret the meaning of these values.

$\cos^2 2x = 0$

60

Performance (**C28/B4**)

2) Solve these equations for $0 \le x \le 2\pi$.
   a) 3sin2x + sin x = 0 b)

3) Solve these equations for .
   a) b)

4) Beth’s and Anna’s solutions to the equation are shown. Find any errors that they have made and explain how to correct them.

Beth

$$2\cos 2x + \sqrt{2} = 0$$

$$\therefore 2\cos 2x + \sqrt{2} = 0$$

$$\cos 2x = \frac{\sqrt{2}}{2}$$

$$2x = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$2x = \frac{\pi}{4} + 2k\pi, k \in I$$

$$x = \frac{\pi}{8} + k\pi, k \in I$$

Anna

$$2\cos 2x + \sqrt{2} = 0$$

$$\cos 2x = \frac{\sqrt{2}}{2}$$

$$\cos 2x = -\frac{\sqrt{2}}{4}$$

$$x = \begin{cases} 1.932 + 2k\pi, k \in I \\ -1.932 + 2k\pi, k \in I \end{cases}$$

Suggested Resources

Brueningsen, Chris, Real-World Math with the CBLTM System, Texas Instruments, 1994

Trigonometric Equations and Identities

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Outcomes

SCO: In this course, students will be expected to

C1 model situations with sinusoidal functions

C2$_2$ create and analyse scatter plots of periodic data

C3 determine the equations of sinusoidal functions

C17$_2$(121) solve problems by determining the equation of the curve of best fit using sinusoidal regression

F6(121) explore periodic data to determine the equations of sinusoidal curves using regression analysis

Elaboration – Instructional Strategies/Suggestions

C1/C2$_2$ Students should be given situations from real-world situations, represented in charts, tables, or with words and graphs. Students should explore how the variable are related, and determine if the relationship is sinusoidal or not. For example, from words

depth of water at an ocean beaches, and the time of day
volume of air in lungs during breathing
trampoline jumping, and distance from floor over time
distance of a water wheel paddle from the water surface over time

Some of these situations could be presented as collections of data, or displayed as a graph.

❒ Ferris Wheel Problem: It is observed that when the last seat on the ferris wheel is occupied at a certain distance, *d,* metres above the ground, it takes 5 seconds to reach the top of the wheel, which is 13 metres above the ground. The 12 m wheel makes 1 revolution every 11 seconds.

***
**
* **C1/C2$_2$/C3/C17$_2$(121)/F6(121)** Students should use the given information in the Ferris Wheel Problem to create a scatterplot using technology, then use sinusoidal regression (SinReg) to determine the equation for the curve of best fit. The TI-83 fits the model equation to the data using an interactive least-squares fit. At least four datapoints are required, and at least 2 points per cycle. It is recommended to give the number of iterations in the syntax.

$$\text{SinReg } (8, L_1, L_2, Y_1)$$

Note: the output will always be in radians, regardless of the MODE setting.

Once the equation is obtained, students can use it solve problems. For example, ask students to predict the height of the seat after 35 seconds. Ask students to determine the times when the seat will be 10 m above the ground.

*
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Trigonometric Equations and Identities

Worthwhile Tasks for Instruction and/or Assessment

Activity (**C1/C2$_2$/C3/C17$_2$(121)/F6(121)**)

1) a) Using a light probe, ask students to point the probe at a single fluorescent light tube and record its intensity using the program Light 2 from the Texas Instruments book *Real-World Math with the CBL System*.
   b) Ask students to examine and interpret the graphs that result.
   c) Ask students to determine the period and frequency at which the bulb flickers. (The frequency is the reciprocal of the period.)
   From the intensity versus time graph on the calculator, ask students to explain what the peaks, or maximum values, represent in terms of the flashing bulb. Ask them what the minimum values represent.
   d) Ask students to use regression to determine the equation that represents the relationship between intensity and time.

Performance (**C1/C2$_2$/C3/C17$_2$(121)/F6(121)**)

2) While riding his bicycle, Julian noticed that a piece of glass became caught in the tred of his front wheel. Assuming the wheel to be 60 cm in diameter, sketch a graph of the relationship between the glass and distance.
   a) Determine an equation for the graph.
   b) Make up a problem in which this equation can be used.

Journal (**C2$_2$/C3**)

3) In the problem about Julian's bicycle, above, explain how Julian could create table of values for the relationship between the position of the glass with respect to the ground and the distance the bicycle travels.

Suggested Resources

Brueningsen, Chris, *Real-World Math with the CBL System*, Texas Instruments, 1994

Foerster, Paul A., *Algebra and Trigonometry*, Revised Edition, 1984, Addison Wesley.

Unit 4
Trigonometry - Further Topics
(20 hours)

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C14 analyse relations, functions, and their graphs

C6 demonstrate an understanding for asymptotic behaviour

C10 analyse and solve trigonometric equations

Elaboration—Instructional Strategies/Suggestions

C14$_4$ /C6/C10 Students should explore the tangent function in some detail discovering and describing its characteristics. They should remember from previous study that $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\theta \in R$ and $\cos\theta \neq 0$.

- Ask students to sketch the graph of for , then to explain what happens as . Have them explain why this is so. Ask students to explain why tan $\frac{\pi}{4}$ and tan have the same values. Have them explain in terms of the definition of tangent.

When a portion of the graph of a function repeats over and over again, as it does with sine function, cosine function, and tangent function, it is called **periodic**. The **period** of such a function is the length of the *x*-interval required for the graph of one complete cycle. Students should be able to graph *y* = tan *x* with and without graphing technology.

Students should be able to describe certain characteristics of the tangent function:

Analysis of *y* = tan *x*:

domain:

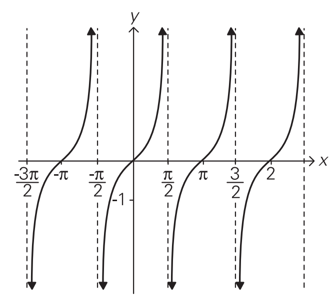
zeros:

no minimum, no maximum

increases: ; does not decrease.

odd function: tan *x* = –tan (–*x*); period: π

asymptotes:



C10 Students will solve equations that involve tangents. For example, when students solve: 5 sec^2 *x* – 2 tan *x* – 8 = 0, for $x \in [0, 2\pi]$, their solution might look like:

$$5(\tan^2 x + 1) - 2\tan x - 8 = 0 \qquad (\tan^2 x + 1 = \sec^2 x)$$
$$5\tan^2 x - 2\tan x - 3 = 0$$
$$(5\tan x + 3)(\tan x - 1) = 0 \qquad \text{(factoring)}$$
$$5\tan x + 3 = 0 \text{ or } \tan x - 1 = 0 \qquad \text{(product of 2 factors = 0)}$$
$$\tan x = -3/5 \text{ or } \tan x = 1$$
$$x \doteq -0.540 \text{ or } x \doteq -0.785$$

solution for:

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

Performance (**C14 /C6/C10**)

1) Give an analysis of the equations below (domain, range, zeros, maximum/minimum, increasing/decreasing, and asymptotes).

   a) $y = \tan\left(x + \frac{\pi}{2}\right)$ b) c) $-y = \tan x$

Paper/pencil (**C10**)

2) Solve the equation for the indicated domain.

   a) $\tan^2 x + \tan x = 0$,

   b) $\tan x - \sqrt{3} = 2\tan x, x \in R$

   c)

   d) $3\sec^2 x + \tan x - 5 = 0,\ x \in [0, 2\pi)$

3) Find the general solutions.

   a) $\tan\left(3\theta - \frac{\pi}{2}\right) = \sqrt{3}$

   b) $\tan\left(\frac{\pi}{2} - x\right) = -1$

$\tan^2 x \sec x - 5 = 0,\ x \in [-\pi, \pi)$ $y = \tan(x + k\pi)$ $\theta = \left\{ [illegible] \ -\frac{5\pi}{12} + k\pi \right\}, k \in I$

Journal (**C14** $_4$)

4) Since the model or parent sine and cosine functions have graphs with periods of 2π, Wanda wonders why the tangent function does not. Write in words how you would respond to Wanda.

5) When solving a trigonometric equation Charles presented the following solution. Sarah missed her last class. Explain in detail, using words, what Charles' solution actually represents:

6) When analysing a trigonometric function for interval(s) of increase Tony presented the interval in the following way. Explain in detail, using words, what the symbols actually represent:

$$x \in \left(2k\pi, \frac{\pi}{2} + 2k\pi\right) \cup \left(\frac{\pi}{2} + 2k\pi, \pi + 2k\pi\right), k \in I$$

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C4 model situations with periodic curves

C17 $_4$ explore and analyse the graphs of the reciprocal trigonometric functions

C6 demonstrate an understanding for asymptotic behaviour

Elaboration—Instructional Strategies/Suggestions

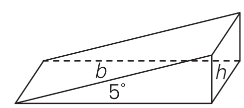
C4/C17 $_4$ **/C6** Many situations can be modeled with the sine, cosine, or tangent functions. However, in some applications the equation or model can be written more concisely with variations to these three functions (namely, their reciprocals—see the example below). These reciprocal functions share many common properties with the basic related trigonometric functions, but they also have some interesting properties of their own, namely the addition of asymptotes for the cosecant and secant functions. By doing an activity like the one in the next column students will discover these properties.

C4 Example 1: Bob Hammer has been hired to construct various ramps for ABC Movers. The ramps must be built with a 5 degree inclined angle. The movers need ramp heights of 21 cm, 36 cm, 45 cm, 72 cm, and 100 cm. Find the length of the boards that Bob will need for the inclined surface of each ramp.

Solution: In Bob's drawing of the ramp, he labels the height h and the length b.

For each ramp, the angle is 5 degrees. Bob writes this equation $\sin 5° = \frac{h}{b}$.

Because he needs to know the length of the board, he solves



for b and gets .

Now he can find the length of each board by substituting the different height measures.

Each equation involves the reciprocal of the sine function. This reciprocal is called the **cosecant** function, abbreviated csc. So another way to write Bob's final equation is $b = h \bullet \csc 5°$.

Similarly, the reciprocal of the cosine has a special name, **secant**, abbreviated, **sec.** And the reciprocal of the tangent is the **cotangent** or **cot.**

Calculators do not have special keys for these reciprocal functions. So students must convert them to sin , cos θ , or tan . Or, students can enter the reciprocal key (x^{-1}). To enter csc x, enter (sin (x)), then **press the x^{-1} key.**

Trigonometry- Further Topics

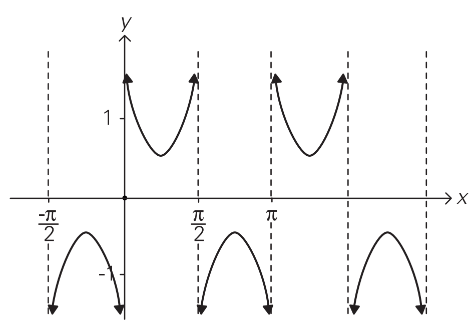
Worthwhile Tasks for Instruction and/or Assessment

Activity (**C4/C17   /C6**)

1) In this activity, students will explore the graphs of the reciprocal trigonometric functions.
   a) Graph y = sin x and y = csc x in the same graphing window. Carefully sketch the graph of each function on your paper. Compare the y-values of sin x and csc x for several different x-values. How does the range for sin x compare to the range for csc x? What is the period of the cosecant curve? (Note: The cosecant curve has vertical asymptotes that are not actually part of the graph. The function actually breaks at these values and starts up again on the other side of the asymptote. However, it may appear that the asymptotes are drawn on your screen. These “apparent asymptotes” or “drag lines” occur when the calculator connects points from the top and bottom of the screen. If the window is set so that a pixel has the exact value of the asymptote, then there will be no drag line drawn (use a friendly window). Otherwise, the calculator will connect the pieces of the graph. Where are the asymptotes located? How can you explain these with reference to the ramp-building example in the second column?
   b) Graph y = cos x and y = sec x in the same graphing window. Carefully sketch the graph of each function on graph paper. What is the period of the secant curve? Where are the vertical asymptotes? Describe similarities and differences between the secant graph and the cosecant graph. (Hint:
4 Look at the similarities and differences of the sine and cosine graphs.)
   c) Graph y = tan x and y = cot x in the same graphing window. Carefully sketch the graph of each function on graph paper. What is the period for the cotangent curve? Where are its vertical asymptotes? How does it compare to the graph of the tangent? Explain.

Paper and Pencil (**C17** $_4$)

2) State the equation of this graph



Journal (**C4/C17** $_4$)

3) Describe how you can tell the graph of y = csc x from the graph of y = sec x.
4) Explain why there are local maxima and minima in the graphs of y = csc x and y = sec x. How are these points related to the corresponding points of their reciprocal graphs y = sin x and y = cos x ?

Suggested Resources

Trigonometry - Further Topics

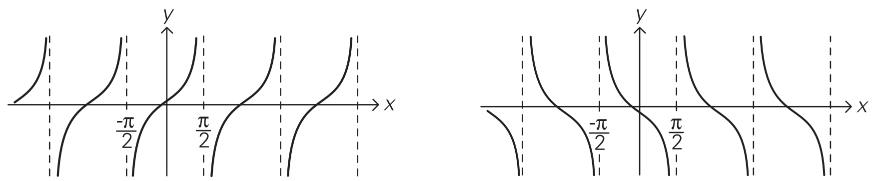
Outcomes

SCO: In this course, students will be expected to

C16 analyse the effect of parameter changes on the graphs of functions and express these changes using transformations

Elaboration—Instructional Strategies/Suggestions

C16 Students should realize that they can reflect, stretch, and translate the graph of $y = \tan x$, just as they could $y = \sin x$ and $y = \cos x$. For example, the graph of $y = \tan(-x)$ is the graph of $y = \tan x$ reflected in the y-axis.



y = tan (x)   y = tan (-x)

❑ Have students picture the graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ in their minds. [Answer:

The graph of is the graph of $y = \tan x$ translated horizontally

by]. Ask students how this graph differs from . [Answer:

the zeros and the asymptote are different.] Have them check their answer using technology. What transformation changes the period? [Answer: A horizontal stretch changes the period.]

a) Ask students to predict the period of $y = \tan 2x$ and what the graph might look like. Ask students to check using their calculators.

b) Ask students to predict the period of and what the graph might

look like. Have them check using their calculators.

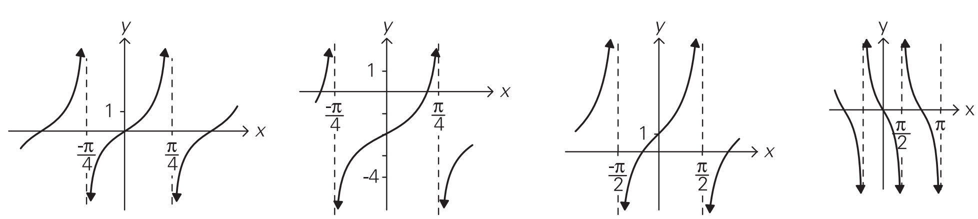
c) Ask them how the period changes compared to other transformations of functions they have studied.

d) Ask students to describe using words and mapping rules the transformations of $y = \tan x$ for the following image graphs:

i) $y = \tan\left(\frac{1}{2}x\right)$

ii) $3(y - 2) = \tan^2(x - 1)$

e) Each of the following graphs is an image of $y = \tan x$. Describe the transformations, and write the equations for each:

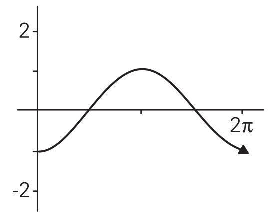


Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

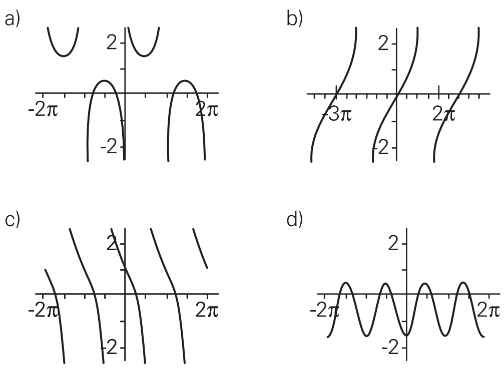
Pencil and Paper (C4/C17 /C16)

1) Draw a sketch of this function on your paper and label it *f(x)*. Then sketch a graph of $y = \frac{1}{f(x)}$ on the same axis.



Pencil and Paper (C16)

2) State the equation for each graph.



3) State the transformations and graph the following:

$\frac{2}{3}(y-3) = \cos\left(x+\frac{\pi}{2}\right)$

a) $2y = \cos\left(x - \frac{\pi}{2}\right)$ e)

b) $y + 5 = \sin\left(x + \frac{\pi}{2}\right)$ f)

c) $y = \tan\left(x - \frac{\pi}{2}\right)$ g)

d) $y = \cos(-x) + 5$ h)

Performance (C4/C17 /C16)

4) Naomi is standing 20 m from the base of a cliff. Looking through her special binoculars, she sees the cliff dwellings of swallows in the cliff face. To map the area, she must know the height of the dwellings above the canyon floor. Naomi holds her binoculars at eye level, 1.2 m high.
   a) Write an equation which relates the angle at which she holds the binoculars to the height of the object she sees.
   b) The top of the cliff is at an angle of 54°. How high is the cliff?
   c) If cliff swallows built a nest in the cliff face 11 m above the canyon floor, at what angle should Naomi focus her binoculars to observe the birds?

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C2 $_4$ model problem situations with combinations of functions

C19 $_4$ investigate and interpret combinations of functions

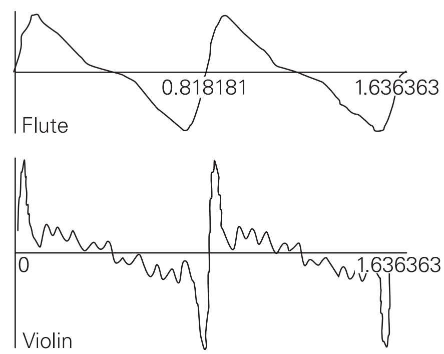
Elaboration—Instructional Strategies/Suggestions

C2 $_4$ **/C19** $_4$ Some applications and problem situations require a combination of more than one function to make a realistic model. For example, in the area of sound production, to accurately reproduce an instrumental sound on an electronic synthesizer, the individual components of sounds and pitches for various instruments must be considered. Each of these individual pitches can be represented by a sine function, and these sine functions are added together to create the desired effect.

The following graphs and equations show a flute and a violin playing the same musical note (A above middle C). * See suggested resources for where this discussion originated.

Flute: $y = 16 \sin 440x + 9 \sin 880x + 3 \sin 1320x + 2.5 \sin 1760x + 1.0 \sin 2200x$

Violin: $y = 19 \sin 440x + 9 \sin 880x + 8 \sin 1320x + 9 \sin 1760x + 12.5 \sin 2200x + 10.5 \sin 2640x + 14 \sin 3080x + 11 \sin 3520x + 8 \sin 3960x + 7 \sin 4400x + 5.5 \sin 4840x + 1.0 \sin 5280x + 4.5 \sin 5720x + 4.0 \sin 6160x + 3 \sin 6600x$.



The fundamental period for both graphs is the same. This period is determined by the "440" in the equation $y = A \sin 440x$, and the period represents the basic tone (remember the music extension in Mathematical Modeling, Book 2, pages 122-124). The "bumps" in the graphs are caused by the overtones (a series of faint tones that are generated when any one tone is sounded.) For example, when a string vibrates it does so as a whole, but also in smaller segments (halves, thirds, fourths, fifths, and so on), producing harmonics), which correspond to the remaining parameters in the equations. The coefficients of these parameters indicate the loudness of each overtone and, because the coefficients differ, the heights of these bumps also differ. Both instruments have equations that are composed of different sine functions. The numbers 440, 880, 1320, 1760, and so on, are the frequencies of the tones (measured in cycles per second). All of the frequencies are multiples of the basic frequency, 440. One cycle is completed when $\sin 440x = \sin 360^0$ and this occurs when the period is $\frac{360^0}{440}$, or when $x = 0.82$ seconds. The variable x is a measure of time and y represents the loudness or amplitude of the sound.

Each musical instrument has its own typical sound and graph. The sound of the note, and also the shape of the graph is affected slightly by the musician, but the basic shape remains the same. Students should enter these equations into their calculators, then modify the coefficients of some of the terms and observe how the graph is affected.

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

Activity (**C2 /C19** $_4$)

1) How can you predict the graph of the sum of two different sine functions? In this activity students will investigate how the period of a function depends on the periods of each part of the equation. Have students:

a	b	Period
1	2	
2	3	
3	6	
2	4	
4	12	

a) Set their graphing window to $0 \le x \le 4\pi$ and . Mark the *x*-axis in units of . Ask them to graph equations of the form *y* = sin *ax* + sin *bx* using the pairs of *a*- and *b*-values listed in the table. Then record the period for each pair.

b) Write a statement that explains how to find the period of a function *y* = sin *ax* + sin *bx* where *a* and *b* are whole numbers.

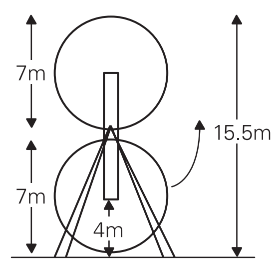
c) Set their graphing window to and . Mark the *x*-axis in units of . Graph equations of the form *y* = sin *ax* + sin *bx* where the values of *a* and *b* are fractions. Record the period for each (*a, b*) pair.

d) Write a statement that explains how to find the period of a function *y* = sin *ax* + sin *bx* when *a* and *b* are fractions. Hint: it may be helpful to rewrite each period as a multiple of . For example, if the period is , write it as .

$4\pi\ 2\pi$ $y = \sin\frac{1}{3}x + \sin 4x + \sin\frac{1}{8}x$

e) Predict the period for .
Explain your reasoning.

Performance (**C2** $_4$ **/C19** $_4$) *



2) A unusual amusement park ride is the double Ferris wheel. Islay is on the ride. Each small wheel takes 25 sec to make a single rotation. The two-wheel set takes 35 sec to revolve once. The dimensions of the ride are as shown.

a) Islay gets on at the foot of the bottom wheel. Write an equation that will model her position as this wheel spins around.

b) The entire ride (two-wheel set) starts rotating at the same time that the two smaller wheels begin to rotate. Write an equation that models the height of the centre of Islay's wheel as the entire ride revolves.

c) Because the two motions occur simultaneously, write a final equation (a combination) for Islay's position.

d) During a 5-minute ride, how many times is Islay within 2 m of the ground?

Suggested Resources

* Murdock, Jerald et al. *Advanced Algebra Through Data Exploration*. Berkley, CA: Key Curriculum Press, 1998.

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C21 $_4$ (121) perform various transformations using multiplication of matrices

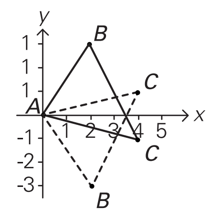
Elaboration—Instructional Strategies/Suggestions

C21 $_4$ (121) Matrices can be used to describe transformations. They provide a convenient way of representing reflections, rotations, and other transformations. The coordinates of a point, (*x, y*), are represented by a column matrix $\begin{pmatrix} x \\ y \end{pmatrix}$.

In earlier studies, students have observed the following about the product of matrices: The product of a matrix and a matrix results in a

matrix:

If transformation matrices can be represented by matrices, then the results of transformations can be related to the product of matrices. For example, plot with vertices *A* (0, 0) *B* (2, 3) and *C* (4, –1) and reflect it in the *x*-axis to obtain .



If describes , then describes

its image, , under this reflection. There is some matrix for which

.

To find *a, b, c* and *d*, perform the matrix multiplication

.

By equating the elements of each matrix:

Solving these two systems:

$4a + 6b = 4$	$4c + 6d = -6$
$\underline{4a - b = 4}$	$\underline{4c - d = 1}$
$7b = 0$	$7d = -7$
$b = 0$	$d = -1$
So, $2a + 3(0) = 2$.	So, $2c + 3(-1) = 01503$
Therefore, $a = 1$	Therefore $c = 0$

(Some students may choose to solve these systems using matrices.)

Thus, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

A reflection in the *x*-axis can be described by the matrix under multiplication.

... continued

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

Activity **C21** (121)

Plot ΔABC with vertices A (0, 0), B (2, 3) and C (4, –1) and reflect it in the y-axis.

Following the procedure described on the previous page:

a) State the matrix that describes and the matrix that describes its reflected image in the y-axis.

b) Using set up the matrix multiplication, and perform it.

c) Solve the systems.

d) What matrix under multiplication produces a reflection in the y-axis?

e) Using similar procedures as above find the matrix which under multiplication produces

- i) a reflection in $y = x$
- ii) a 90° rotation about centre (0, 0)
- iii) a 180° rotation about centre (0, 0)

2) Using matrix multiplication, find the image of quadrilateral $ABCD$ with vertices A (2, –1), B (3, 1), C (6,0) and D (5, –2) when

a) reflected in the line $y = x$

ΔABC 4 $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

b) rotated 90° about (0, 0)

c) reflected in the line $y = x$, followed by a 90° rotation about the origin

d) rotated 90° about (0, 0), followed by a reflection in $y = x$

e) Does the order of these transformations result in different images?

... continued

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C21 (**121**) perform various transformations using multiplication of matrices

B8(**121**) derive and apply the general rotational matrix

Elaboration—Instructional Strategies/Suggestions

... continued

C21 $_4$ (**121**) The significance of this matrix multiplication is that work with reflections, and transformations in general, can be more easily programmed on a computer. Teachers may be familiar with some software packages for drawing where transformations can be applied to an original figure. Many complex figures required in designs and in engineering can be handled more easily using matrices on computers. Animation can be achieved by writing a program that includes a series of multiplications of matrices. Each multiplication results in an object defined by a set of points being moved to a new location. A series of these movements simulates animation.

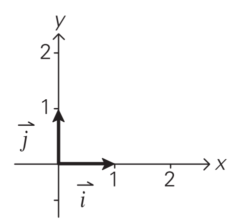
Students should be involved in several activities where they can derive a particular matrix that produces a reflection in the *y*-axis, in the line *y* = *x*, a combination of reflections, a rotation of 90° and 180° about the origin. Some students could investigate animation and produce a program that uses a series of matrix multiplications which result in an object moving, for example, around another stationary object.

B8(**121**) In activities on the previous two-page spread, students conjectured that

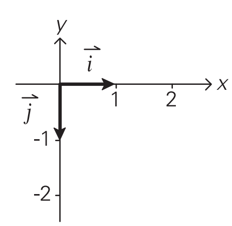
the matrix describing a rotation of 90°, centre (0, 0) is

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and a rotation of 180°, centre (0, 0) as .

Students should now derive a matrix for a rotation of , centre (0, 0). To begin, they should consider the matrix



as made up of two unit vectors.



The images of these unit vectors under a given transformation will determine the matrix that describes this transformation. For example, under a reflection in the *x*-axis, the following changes take place.

$$\vec{i}: \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{j}: \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Therefore, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix for the reflection in the *x*-axis.

This column is the image of $\vec{i}$ ↑ ↑ this colum is the image of

...continued

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

... continued

Pencil and Paper **C21 (121)**

3) A triangle represented by the matrix $\begin{pmatrix} 3 & 3 & 5 \\ 2 & 1 & 2 \end{pmatrix}$ is relocated after multiplication by the matrix .

   a) Find the coordinates of the resulting figure.
   b) Describe the transformation that is equivalent to the result obtained in (a).

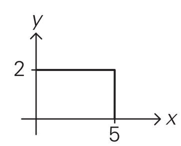
Pencil and Paper (**C21** $_4$ **(121)/B8(121)**)

4) Find the matrix that performs each transformation.
   a) a reflection in the *x*-axis
   b) a rotation of + 180° about the origin
   c) a reflection in the line $y = x$
   d) a rotation of +90°
   e) a reflection in the *y*-axis
   f) a reflection in the line $y = -x$
   g) a rotation of +270°
   h) a reflection of –90°

4 (−0 1 −1 0 / −1 0 0 1)

Performance (**C21** $_4$ **(121)/B8(121)**)

5) Use the matrices found above to find the image coordinates for the following.
   a) reflect ΔABC, A (1, 2), B (4, –5), C (–2, –3) in $y = x$, then rotate 90° centre (0, 0).
   b) rotate the line $2y + 3x - 5 = 0$, –90° about the centre (0, 0) and find the equation of the image.
   c) In the beginning of an animation project, students were asked to write a program that would transform a rectangle as stated below:



      i) Begin with rectangle *ABCD* as in the diagram.
      ii) Rotate 90° centre (0, 0).
      iii) reflect in the line $y = x$.

   What final transformation will return the rectangle to its original location? Using matrices explain why this happens.

6) Ask students to write a few paragraphs in which they are to describe in words and with matrices how they could use matrix multiplication to make it look like a triangle is rotating around the origin of the Cartesian Plane.

7) A triangle is located at A (–5, 1), B (–2, 1), and C (–5, –2). A matrix is produced to represent this triangle and is multiplied by the matrix then the resulting matrix is multiplied by the matrix . Explain what is happening to the original triangle, and describe how the triangle would then reflect across the line $y = x$, and then the *x*-axis using matrix multiplication. State its final location.

Suggested Resources

Trigonometry - Further Topics

Outcomes

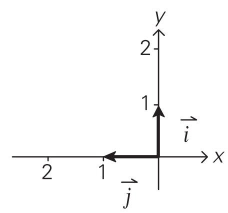
SCO: In this course, students will be expected to

B8(121) derive and apply the general rotational matrix

Elaboration—Instructional Strategies/Suggestions

B8(121)...continued

Similarly, for a 90° rotation, with centre (0, 0) the following changes take place.



Therefore, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the matrix for 90° rotation, centre (0,0).

This column is the image of $\vec{i}$ ⌐ ⌙ this column is the image of

... continued

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper **B8(121)**

1) The vertices *A* (–3, 2), *B* (–6, 2), and *C* (–6, 7) of Δ*ABC* are recorded in matrix form as , and a rotation of radians about the centre (0, 0) is applied to .
   a) Write the general rotational matrix for this rotation.
   b) Find the coordinates of the vertices of the image of
   c) Plot and its image,
   d) What other transformation of gives the same image?
2) Using unit vectors on the unit circle, find the matrix that performs each rotation about the centre (0,0).
   a) 270°
   b) 225°
   c) 300°
   d) radians
   e) –120°
   f) radians
   g) radians
   h) 210°
   i) radians

ΔA′B′C′ –6 –6
3π/4 2 2 7

... continued

Suggested Resources

Trigonometry - Further Topics

Outcomes

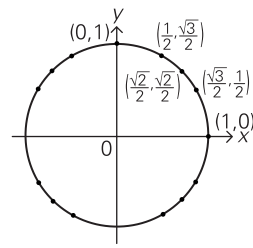
SCO: In this course, students will be expected to

B8(121) derive and apply the general rotational matrix

Elaboration—Instructional Strategies/Suggestions

... continued

B8(121) Unit vectors on the unit circle, shown on the right, can be used to find matrices of various rotations with centre (0, 0). For example, when rotating 45° the unit vectors *i* and *j* , their components become:



$$\vec{i}:\begin{pmatrix}1\\0\end{pmatrix}\rightarrow\begin{pmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix} \text{ and } \vec{j}:\begin{pmatrix}0\\1\end{pmatrix}\rightarrow\begin{pmatrix}-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}.$$

Thus, $\begin{pmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{pmatrix}$ is the matrix that through multiplication would rotate a

figure radians. Also, when rotating the unit vectors –90° the unit vector

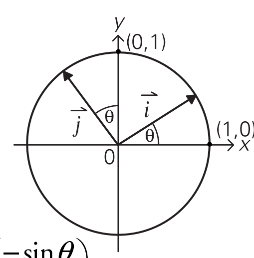
components become:

.

Thus, multiplication by the matrix $\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ performs a rotation of –90°.

In general, for a rotation of , with centre (0, 0), and, since $\vec{j}$

is perpendicular to , .



Since $\cos\left(\theta+\frac{\pi}{2}\right)=-\sin\theta$ *and* $\sin\left(\theta+\frac{\pi}{2}\right)=\cos\theta$, $\begin{pmatrix}0\\1\end{pmatrix}\rightarrow\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$, the matrix

that will perform a rotation of , with centre (0, 0) is . This

matrix is called the **general rotational matrix**.

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

... continued

Pencil and Paper **B8(121)**

3) a) Construct a graph for $y = \sin x$ over the domain .

   b) If the curve in (a) is to be rotated about the origin for 30°, what matrix would perform this rotation?
   c) Write this matrix in decimal form to the nearest tenth.
   d) Perform matrix multiplication on the ordered pairs above to plot the image of $y = \sin x$ under this rotation.
   e) Does the resulting graph represent a function? Why?

4) The vertices A (–7, 0), B (–9, 4), and C (–6, 4) of a triangle are recorded in matrix form as $\begin{pmatrix} -7 & -9 & -6 \\ 0 & 4 & 4 \end{pmatrix}$. A rotation of radians is applied to the triangle.
   a) Write the matrix that performs the rotation.
   b) Find the image of the vertices of the triangle in matrix form.

5) Using a graphing calculator, and the 1:Pt-On feature on the Draw Points menu, plot a point on the graph, rotate it 45°, centre (0, 0) by multiplying

$\frac{5\pi}{4}$ $x \in \left(-\frac{\pi}{2}, \frac{5\pi}{2}\right), x \in R$

by the general rotational matrix on the Home screen, then plot its image using the PT-On feature. Make sure your Window selection is on Zoom, 5:ZSquare, so that your rotation looks like 45°. Plot a triangle and follow the same steps to plot its image after a rotation of 60°.

Journal **B8(121)**

6) When performing matrix multiplication to rotate a figure on a plane, is it important when recording the multiplication which side (left or right) of the coordinate matrix the rotation matrix is placed? Explain.

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C22 explore and verify trigonometric identities

Elaboration—Instructional Strategies/Suggestions

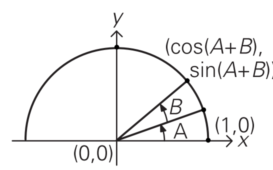
C22 Students should derive the compound angle identities. Three methods will be discussed on this and the next two two-page spreads. Students can derive the compound angle identities by using the general rotational matrix, the distance formula (p. 132), and the right triangle trigonometric method (p. 134), but **only one method is necessary**.

General Rotational Matrix Method

Have students examine the diagram to the right. Imagine that the point (1, 0) is rotated about the origin (0, 0) by an angle *A*. Then it is rotated from there through angle *B*, centre (0, 0). After these two rotations students should conclude that (1, 0) maps to the point (cos (*A* + *B*), sin (*A* + *B*)).

Thus, maps to .

Using matrix multiplication, and rotating through



Then, from there, rotating through :

But, according to the diagram this location should be .

Thus, students should conclude .

So, by the equality of matrices, students should deduce:

-
- $\sin(A+B) = \cos A \sin B + \sin A \cos B$

Since *A* and *B* are arbitrarily chosen angles, students can replace *B* by –*B*. Therefore, $\cos(A+(-B)) = \cos A \cos(-B) - \sin A \sin(-B)$.

Recall that Then,

-
- $\sin(A-B) = \sin A \cos B - \cos A \sin B$

... continued

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

Pencil and Paper (C22)

1) The identities that result from the following are called the **Double-Angle Identities.**
   a) Use the identity sin (2A) = sin (A + A) to show that sin (2A) = 2 sin A cos A.
   b) Use cos (2A) in the form cos (A + A) to show:
      i) $\cos(2A) = \cos^2 A - \sin^2 A$
      ii) $\cos(2A) = 2\cos^2 A - 1$
      iii)
   c) Use tan (2A) = tan (A + A) to show that $\tan(2A) = \dfrac{2\tan A}{1-\tan^2 A}$.

Performance (C22)

2) a) i) Graph $y_1 = \cos^2 x$ and $y_2 = \sin^2 x$ on your calculator screen.
      ii) Complete a table of values for the difference between y_1 and y_2 .

x	0	$\frac{\pi}{6}$					...
cos² *x* – sin² *x*							

$\cos(2A) = 1 - 2\sin^2 A$

   b) i) Describe what a plot of all possible pairs from the above table would look like. Explain what it means if you connect those points with a smooth curve.
      ii) Graph $y = \cos^2 x - \sin^2 x$. What other equation would give the same graph?
      iii) State what you discovered in (ii) as an identity.
3) Graph y = 2 sin x cos x on your calculator. What other equation would give the same graph? State your results as an identity.
   a) Graph $y = (\sin x - \cos x)^2$ on your calculator. What other equation, without a horizontal translation, would give the same graph?
   b) Expand the right side of the equation and use the identities you discovered in Problems 2 and 3 to write an equivalent expression.

... continued

Suggested Resources

Trigonometry - Further Topics

Outcomes

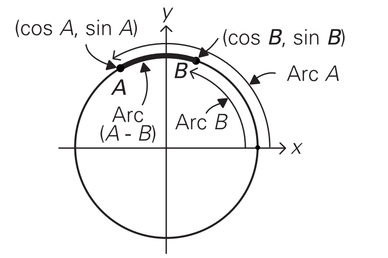
SCO: In this course, students will be expected to

C22 explore and verify trigonometric identities

Elaboration—Instructional Strategies/Suggestions

... continued

C22 Distance Formula Method:



Have students examine the diagram to the right. Imagine that the point (1, 0) moves along the arc marked arc *A* to the point *A*. Then, from there, moves back to point *B*, located at the end of arc (*A* – *B*). The arc between these two points *A* and *B* has a measure (*A* – *B*), and is called arc (*A* – *B*). The length (*d*) of the chord (see second diagram) for this arc can be found using the distance formula:

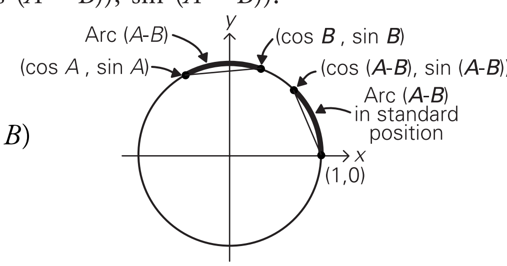
using $\sin^2 A + \cos^2 A = 1$

$d^2 = 2 - 2\cos A\cos B - 2\sin A\cos B$

Have students rotate (see second diagram below) arc (*A* – *B*) to standard position, then the terminal point has coordinates (cos (*A* – *B*)), sin (*A* – *B*)).

Now apply the distance formula:

$$\begin{aligned} d^2 &= (\cos(A-B)-1)^2 + (\sin(A-B)-0)^2 \\ &= \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B) \end{aligned}$$



From the two expressions for d^2:

2 – 2cos (*A* – *B*) = 2 – 2cos *A* cos *B* – 2sin *A* sin *B*

Subtracting 2, then dividing by –2:

cos (*A* – *B*) = cos *A* cos *B* – sin *A* sin *B*

Since *A* and *B* are arbitrarily chosen angles, students can replace *B* with (–*B*) to get cos (*A* + *B*).

$$\begin{aligned} \cos(A+B) &= \cos(A-(-B)) \\ &= \cos A\cos(-B) + \sin A\sin(-B) \\ &= \cos A\cos B - \sin A\sin B \end{aligned}$$

The expression sin (*A* – *B*) can be transformed to a cosine using the co-function property $\sin x = \cos\left(\frac{\pi}{2} - x\right)$. Thus, .

Using cos (*A* + *B*): $\sin(A-B) = \cos\left(\frac{\pi}{2} - A\right)\cos B - \sin\left(\frac{\pi}{2} - A\right)\sin B$

$$= \sin A\cos B - \cos A\sin B$$

then, $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

... continued

Performance (C22)

5) Solve: for $2\cos\theta = \sin\left(\theta + \frac{\pi}{6}\right)$ for .

6) Show that .

7) Derive a formula for $\cot(A+B)$ and $\cot(A-B)$ in terms of $\cot A$ and $\cot B$.

8) Verify

a)

b) $\sin\left(\frac{\pi}{2} - A\right) = \cos A$

$\theta \in R$

$\cos\left(\frac{\pi}{2} - A\right) = \sin A$

$\sec(A+B) = \frac{\sec A \sec B}{1 - \tan A \tan B}$

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C22 explore and verify trigonometric identities

Elaboration—Instructional Strategies/Suggestions

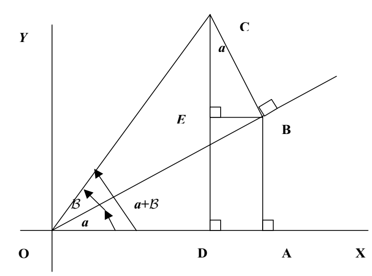
... continued

C22 Right Triangle Trigonometry Method:

Ask students to examine the graph to the right. The ray *OB* is the ray *OA* rotated through the angle α. The ray *OC* is the ray *OB* rotated through the angle . Have students express in terms of cosines and sines.

Their thinking might follow this pattern: *OD* = *OA* - *DA*

but $\cos(\alpha+\beta)=\dfrac{OD}{OC}\Rightarrow OD=OC\cos(\alpha+\beta)$



similarly

In

Since *DA* = *EB*, then $DA=BC\sin\alpha$

$$\therefore OC\cos(\alpha+\beta)=OB\cos\alpha-CB\sin\alpha \text{ (i.e., } OD=OA-DA)$$

$$\cos(\alpha+\beta)=\frac{OB}{OC}\cos\alpha-\frac{CB}{OC}\sin\alpha$$

But, $\dfrac{OB}{OC}=\cos\beta$ and $\dfrac{CB}{OC}=\sin\beta$

So, $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

By replacing β with , students will obtain:

(since $\sin(-\theta)=-\sin\theta$ and $\cos(-\theta)=\cos\theta$)

To develop formulas for $\sin(\alpha-\beta)$, and $\sin(\alpha+\beta)$, students should replace α

by and by in each of the above:

Now, using $\cos\left(\dfrac{\pi}{2}-\theta\right)=\sin\theta$, and $\sin\left(\dfrac{\pi}{2}-\theta\right)=\cos\theta$

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

and by replacing β with , students will obtain:

... continued

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

... continued

Pencil and Paper (C22)

9) Jeff said he could find sin 105° without using his calculator. He began by saying that sin 105° = sin (60° + 45°), then he continued with the right hand side = sin 60° cos 45° – sin 45° cos 60°.
   a) Has Jeff made any errors? Explain.
   b) Correct Jeff's work using exact values and expressing your answer using radicals.
   c) Explain how you would check your answer.

10) William asked the teacher for the formula for . The teacher reminded him about $\tan x = \frac{\sin x}{\cos x} \left(\cos x \neq 0\right)$ and suggested that he use this fact when determining . She also suggested: "... divide each term in the numerator and denominator by $\cos\alpha\cos\beta$"
   a) Why do you think this would be helpful? Explain.
   b) Show what William did to complete his work.
   c) Determine a formula for .

11) The teacher asked the class to determine a formula for cos (2*A*). Belinda asked if she could write this as cos (*A* + *A*).

$\tan\left(\alpha + \beta\right)$

   a) Kate said she could and proceeded to determine a formula for cos (2*A*) which had a ($\cos^2 A$) and a ($\sin^2 A$) in it. Show the work that Kate must have done.
   b) Naomi said her answer only had one of those in it. What was her answer?
   c) Is there any other way to correctly state the formula? Explain.

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C22 explore and verify trigonometric identities

Elaboration—Instructional Strategies/Suggestions

... continued

C22 Using the identity $\tan x = \dfrac{\sin x}{\cos x}$, $\cos x \neq 0, \tan(A+B)$ can be derived:

Similarly, $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$.

Students should know that they can use a form of this formula, $\tan\theta = \dfrac{m_1 - m_2}{1+m_1 m_2}$

to determine the angle, θ, between two intersecting lines with slopes m_1 *and* m_2.

Students can summarize their work by producing a list of all the compound angle identities.

-
- $\sin(A-B) = \sin A\cos B - \cos A\sin B$
- $\cos(A+B) = \cos A\cos B - \sin A\sin B$
- $\cos(A-B) = \cos A\cos B + \sin A\sin B$
- $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A\tan B}$
- $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A\tan B}$

Students can use these identities to prove other identities, and solve equations (see the problems and equations on the next page).

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

.. continued

Pencil and Paper (C22)

12) Prove these identities.

a) $(\sin A + \cos A)(\sin B + \cos B) = \sin(A+B) + \cos(A-B)$

b)

13) a) Express as a single trigonometric ratio. Do not use a calculator.

i) 2 sin 14° cos 14°

ii) $\frac{2\tan 35°}{1-\tan^2 35°}$

iii)

iv) $\cos^2 26° - \sin^2 26°$

v) $2\cos^2 34° - 1$

vi)

vii) $\frac{1+\tan x}{1-\tan x}$

b) Explain how you can use a graphing utility to check your answer to vii).

14) Given that is acute, find the value of , , and for each of the following without finding .

[illegible]

a)

b)

15) a) Prove the following identities.

i) $\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

ii) $\tan x + \cot x = 2\csc(2x)$

iii)

iv) $\sec(2A) + \tan(2A) = \frac{\cos A + \sin A}{\cos A - \sin A}$

v) $\cos(4x) = 8\cos^4 x - 8\cos^2 x + 1$

b) What should be true about the graphs of the functions that correspond to each side of these identities? Verify your conjectures with (i) and (ii).

16) Determine the angle between the two lines *AB* and *CD* that pass through *A* (–3, –2), *B* (1, 2), and *C* (–1, 7), *D* (1, –2).

17) If *A, B*, and *C* are the angles of a triangle, prove:

a)

b) $\sin C = \sin(A+B)$

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

C9 examine, interpret, and apply the relationship between trigonometric functions and their inverses

C10 analyse and solve trigonometric equations

C14 $_4$ analyse relations, functions, and their graphs

Elaboration—Instructional Strategies/Suggestions

C9 $_4$ **/C10** When given a trigonometric function value, students have previously found the approximate measure of the angle or length of the corresponding arc. Finding the angle measure from the function value involves using the inverse function keys on a calculator. These are usually marked $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$, and are read "sine inverse," "cosine inverse," and "tan inverse." Their meanings are:

- **$\sin^{-1}x$: an angle whose sine is *x*.**
- **$\cos^{-1}x$: an angle whose cosine is *x*.**
- **$\tan^{-1}x$: an angle whose tangent is *x*.**

For example, given a right triangle with an hypotenuse 21.5 cm, and the leg opposite the angle marked θ is 17.4 cm. Find . Students would set up the equation . To find θ, students must find $\sin^{-1}$ (0.8093), or "the angle whose sine is 0.8093".

Students should not confuse the expression $\sin^{-1} x$ with the reciprocal of sin *x*

, or csc *x*. The expression $\sin^{-1}x$ is read: "the arc whose sine is ...".

Students have been using $y = \cos x$ or $y = C + A\cos B\,(x - 2)$ to find values for *y* when *the angle* *x* is known. Using $y = \cos^{-1} x$ students can find values for the angle (or arc length) *y* when *the ratio of the sides* *x* is known.

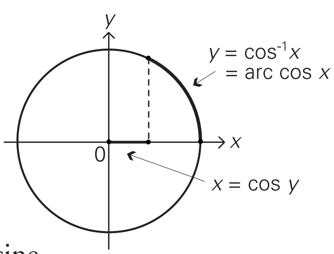
C14 The geometrical meaning of $\cos^{-1} x$ can be seen by drawing a unit circle. If $x = \cos y$, then *y* is the arc whose cosine is *x*. For this reason, $\cos^{-1} x$ is often written as arccos *x*. Since any arc coterminal with *y* has *x* as its cosine, there are many values of arccos *x*. Thus the inverse cosine is not a function.

so, $y = \arcsin x = \sin^{-1} x$ means $x = \sin y$

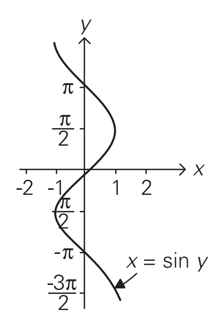
$y = \arccos x = \cos^{-1} x$ means $x = \cos y$

$y = \arctan x = \tan^{-1} x$ means $x = \tan y$

and so on.



Students should now graph the inverse of sine and cosine graphs. Ask them if they can visualize what such a graph would look like. They probably recall that (*x, y*) and (*y, x*) are inverses of each other. (See the activity on the next page.) To draw $y = \sin^{-1} x$, students should simply plot the relation $x = \sin y$. This is most easily done by picking values of *y* and finding *x*. Since *x* and *y* are interchanged for the inverse relation, everything that happened along the *x*-axis for $y = \sin x$ will happen along the *y*-axis for $y = \sin^{-1} x$.



y = arcsin x

... continued

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

Activity (C9 /C14 $_4$)

1) This activity will help students understand the relationship between the sine and cosine functions and their inverses. Have students enter the *A*-values in this table (including the values between π and) into a calculator data list and use the list functions (like a spreadsheet) to calculate the other table entries.

A	0									...	...	
sinA												
sin^{-1}(sin *A*)												

a) Plot the points (*A*, sin *A*) on your calculator screen. You should see a familiar graph. Record a sketch of this on graph paper.

b) Plot the points (sin *A*, sin $^{-1}$ (sin *A*)) on your calculator screen. (You will need to redefine your window when you do this.) Is this graph a function? Compare the number of points plotted to the number of actual data

$\pi \le y \le 2\pi$

points in your lists. Trace the graph and look through your data lists and then explain what has happened. Record a sketch of this on your paper.

c) Your first scatter plot, (*A*, sin *A*), was a graph of points from the sine function. Inverse relations are formed by switching the *x*- and *y*-coordinates of the points. Now graph the inverse sine by switching the coordinates. Use window values of $-1.5 \le x \le 1.5$ and . Create a plot of (sin *A, A*). Use the smallest mark on your calculator to mark these points. Is this graph a function? Explain. Now add a plot of the points (sin A, sin $^{-1}$ (sin *A*)) to your calculator screen using a different mark. Describe the similarities and differences in the two data plots. Why do you suppose the calculator gives only restricted values in the plot of (sin *A*, sin^{-1} (sin *A*))?

d) In previous work you discovered that $f^{-1}(f(x)) = x$. Is this true for the sine function? Study the values and graph of (*A*, sin $^{-1}$(sin *A*)). Describe and explain the relationship between *A* and sin^{-1} (sin *A*). You may need more angles, including negative angles, to be confident of your conclusions.

e) Now investigate the inverse of the cosine and tangent functions by following steps similar to those above. Again, you may need to use more angles, including negative angles, to be confident of your conclusions.

f) Draw accurate graphs of $y = \sin^{-1} x$, $y = \cos^{-1} x$, and $y = \tan^{-1} x$ relations. Write a few sentences describing each, and explaining the domain and range of each.

... continued

Suggested Resources

Trigonometry - Further Topics

Outcomes

SCO: In this course, students will be expected to

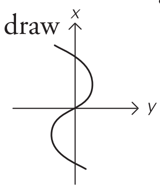
C9 examine, interpret, and apply the relationship between trig functions, and their inverses

C14 $_4$ analyse relations, functions and their graphs

Elaboration—Instructional Strategies/Suggestions

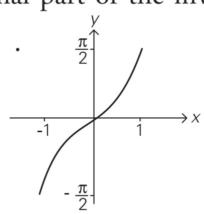
... continued

Calculators will draw an inverse that is not a function using "Draw Inverse". If $y = \sin x$ then do "draw inverse y_1". It will draw

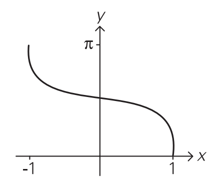


for $y = \sin^{-1}(x)$. But if we go to y_2 and enter $y_2 = \sin^{-1}(x)$ it then draws only the function on part of the curve.

C9 /C14 $_4$ Students should understand that calculators only give angle measures for the functional part of the inverse graph, i.e., $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ for sin-1 and for .



$y = \text{Sin}^{-1}\,\theta$



$y = \text{Cos}^{-1}\,\theta$

These parts of the curves are called the principal parts of the curves and are denoted in many texts as Arcsin θ or Sin-1 , and Arccos or Cos-1 (using capital letters to indicate the "principal" part of the curve, or the part that is a function). Students could use **parametric** mode to explore inverses. Enter the following keystrokes. The first two equations produce the function $y = \sin x$, while the second two equations produce the inverse $y = \sin^{-1} x$.

$x_{1T} = T$

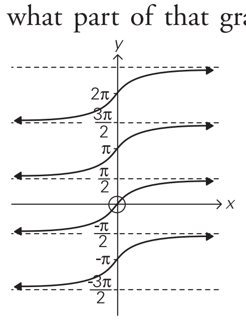
$y_{1T} = \sin T$

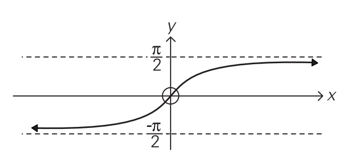
$x_{2T} = y_{1T}$

$y_{2T} = x_{1T}$

To view the graphs of $y = \cos x$ and $y = \cos^{-1} x$, change $y_{1T} = \sin T$ to $y_{1T} = \cos T$. To view, use a MODE set for radians and use a Window screen set for trig-function (Zoom, 7:Trig). Students might want to use "Zoom Fit".

As required in the activity on the previous page, students should extend their understanding of trigonometric functions and their inverses to the tangent function. Students should draw the graph of $y = \tan^{-1} x$ (sometimes written arctan x), and indicate what part of that graph would be the "principal" part.





$y = \tan^{-1} x$

Trigonometry- Further Topics

Worthwhile Tasks for Instruction and/or Assessment

... continued

Pencil and Paper **(C9 $_4$ /C14 $_4$)**

2) Express each of the following without using inverses.

a) $\arccos 0 = \frac{\pi}{2}$ c)

b) $-\frac{\pi}{2} = \sin^{-1}(-1)$ d)

3) Evaluate

a) Arctan (–1)
b) Arctan $\sqrt{3}$
c) Arctan 1
d) Arctan 0.2401
e) Arctan (-57.29)

4) Evaluate .

5) Find the value(s) of x.

a) $\arctan x = \frac{7\pi}{18}$ b) $\arctan x = -\frac{\pi}{6}$ c)

[illegible]

6) Find the first five positive x-values that make each equation true.

a) sec x = –2.5 b) (csc x – cot x)(sec x + 1) = 0.8

7) Solve for x: .

8) Evaluate

a) c)

b) d)

9) Express x in terms of y. What is the domain of the new equation?
$3y = 4 - 2\cos^{-1}x$.

10) A mass hanging from a spring is pulled 3.0 cm down from its resting position and released. It makes 12 complete bounces in 10.2 sec. At what times during the first 3.0 sec was it 0.5 cm below its resting position?

11) a) The time between high and low tide in a river harbour is approximately 7 hours. The high-tide depth of 15.7 m occurs at noon and the average river depth is 2.3 m. Write an equation modeling this (*time, depth*) relationship.
b) If a boat required at least 5 m of water, find the next two time periods when the boat will not be able to enter the harbor.

12) The height of the water (measured in decimetres) at a river's mouth varies during the tide cycle. The height of the water is $h(t) = 7.5 \sin 30t + 15$, where t is measured in hours. If you use $t = 0$ for the present time, during what time intervals over the next 48 hours is the river depth 11.5 dm or more?

Suggested Resources