Atlantic Canada Mathematics Curriculum

New Brunswick
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New Nouveau Brunswick

CURRICULUM

Functions and Relations 111/112

(Implementation Edition)

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- The teachers and other educators and stakeholders across Atlantic Canada who contributed to the development of the *Functions and Relations 111/112* curriculum guide.

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I. Background and Rationale

A. Background

Mathematics curriculum reform in Atlantic Canada is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in an increasingly technological society. Curriculum reform has been motivated by a desire to ensure that students in Atlantic Canada benefit from world-class curriculum and instruction in mathematics as a significant part of their learning experiences.

The *Foundation for the Atlantic Canada Mathematics Curriculum* (1996) firmly establishes the *Curriculum and Evaluation Standards for School Mathematics* (1989) of the National Council of Teachers of Mathematics (NCTM) as a guiding beacon for pursuing this vision, which embraces the principles of students learning to value and become active "doers" of mathematics and advocates a curriculum which focusses on the unifying ideas of mathematical problem solving, communication, reasoning, and connections. These principles and unifying ideas are reaffirmed with the publication of NCTM's *Principles and Standards for School Mathematics* (2000). The *Foundation for the Atlantic Canada Mathematics Curriculum* establishes a framework for the development of detailed grade-level documents describing mathematics curriculum and guiding instruction.

Mathematics curriculum development has taken place under the auspices of the Atlantic Provinces Education Foundation (APEF), an organization sponsored and managed by the governments of the four Atlantic Provinces. APEF has brought together teachers with department of education officials to co-operatively plan and execute the development of curricula in mathematics, science, language arts, and other curricular areas. Each of these curriculum efforts has been aimed at producing a program that would ultimately support the Essential Graduation Learnings (EGLs), also developed regionally. These EGLs and the contribution of the mathematics curriculum to their achievement are presented in the "Outcomes" section of the mathematics foundation document.

B. Rationale

The *Foundation for the Atlantic Canada Mathematics Curriculum* provides an overview of the philosophy and goals of the mathematics curriculum, presenting broad curriculum outcomes and addressing a variety of issues with respect to the learning and teaching of mathematics. This curriculum guide is one of several which provide greater specificity and clarity for the classroom teacher. The *Foundation for the Atlantic Canada Mathematics Curriculum* describes the mathematics curriculum in terms of a series of outcomes—General Curriculum Outcomes (GCOs), which relate to subject strands, and Key-Stage Curriculum Outcomes (KSCOs), which articulate the GCOs further for the end of grades 3, 6, 9, and 12. This guide builds on the structure introduced in the foundation document, by relating Specific Curriculum Outcomes (SCOs) to KSCOs for *Functions and Relations 111/112*. Figure 1 further clarifies the outcome structure.

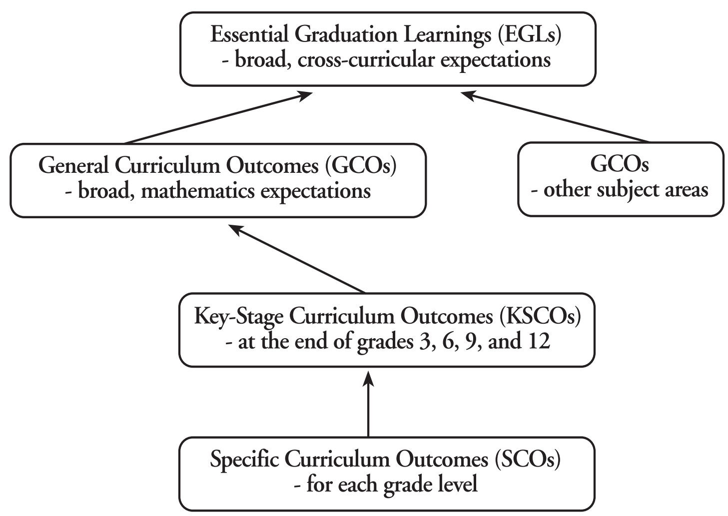


Figure 1: Outcome Framework

This mathematics guide is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice, including the following: (i) mathematics learning is an active and constructive process; (ii) learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates; (iii) learning is most likely when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and nurtures positive attitudes and sustained effort; (iv) learning is most effective when standards of expectation are made clear and assessment and feedback are ongoing; and (v) learners benefit, both socially and intellectually, from a variety of learning experiences, both independent and in collaboration with others.

II. Program Design and Components

A. Program Organization

As indicated previously, the mathematics curriculum is designed to support the Atlantic Canada Essential Graduation Learnings (EGLs). The curriculum is designed to significantly contribute to students meeting each of the six EGLs, with the communication and problem-solving EGLs relating particularly well with the curriculum's unifying ideas. (See the "Outcomes" section of the *Foundation for the Atlantic Canada Mathematics Curriculum*.) The foundation document then goes on to present student outcomes at key stages of the student's school experience.

This curriculum guide presents specific curriculum outcomes for *Functions and Relations 111/112*. As illustrated in Figure 2, these outcomes represent the step-by-step means by which students work toward accomplishing the key-stage curriculum outcomes, the general curriculum outcomes, and, ultimately, the essential graduation learnings.

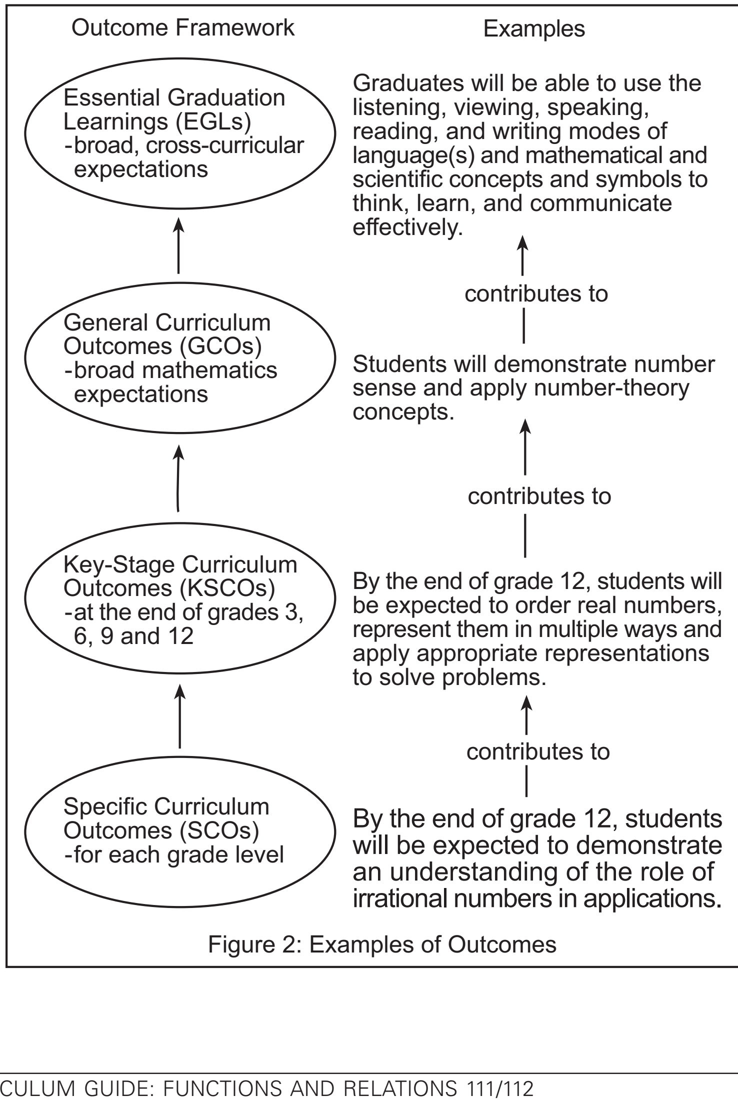


Figure 2: Examples of Outcomes

It is important to emphasize that the initial presentation of the specific curriculum outcomes for this course (pp. 17-30) follows the outcome structure established in the *Foundation for the Atlantic Canada Mathematics Curriculum* and does not represent a natural teaching sequence. In *Functions and Relations 111/112*, however, a suggested teaching order for specific curriculum outcomes has been given within a sequence of four topics or units (i.e., Applications of Trigonometry; Quadratics; Rate of Change; and Exponential Growth). While the units are presented with a specific teaching sequence in mind, some flexibility exists as to the ordering of units within the course. It is expected that teachers will make individual decisions as to what sequence of topics will best suit their classes. In most instances, this will occur in consultation with fellow staff members, department heads, and/or district level personnel.

Decisions on sequencing will depend on a number of factors, including the nature and interests of the students themselves. For instance, what might serve well as a "kickoff" topic for one group of students might be less effective in that role with a second group. Another consideration with respect to sequencing will be co-ordinating the mathematics program with other aspects of the students' school experience. An example of such co-ordination would be studying aspects of measurement in connection with appropriate topics in science. As well, sequencing could be influenced by events outside of the school, such as elections, special community celebrations, or natural occurrences.

B. Unifying Ideas

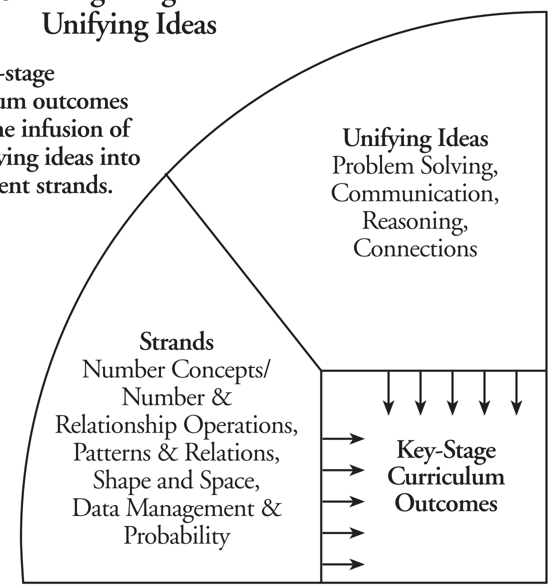
The NCTM *Curriculum and Evaluation Standards* (1989) and *Principles and Standards* (2000) establishes mathematical problem solving, communication, reasoning, and connections as central elements of the mathematics curriculum. The *Foundation for the Atlantic Canada Mathematics Curriculum* (pp. 7-11) further emphasizes these unifying ideas and presents them as being integral to all aspects of the curriculum. Indeed, while the general curriculum outcomes are organized around content strands, every opportunity has been taken to infuse the key-stage curriculum outcomes with one or more of the unifying ideas. This is illustrated in Figure 3.

These unifying concepts serve to link the content to methodology. They make it clear that mathematics is to be taught in a problem-solving mode; classroom activities and student assignments must be structured so as to provide opportunities for students to communicate mathematically; via teacher encouragement and questioning, students must explain and clarify their mathematical reasoning; and mathematics with which students are involved on a day-to-day basis must be connected to other mathematics, other disciplines, and/or the world around them.

Students will be expected to address routine and/or non-routine

Figure 3 - Integrating Strands and Unifying Ideas

The key-stage curriculum outcomes reflect the infusion of the unifying ideas into the content strands.



mathematical problems on a daily basis. Over time, numerous problem-solving strategies should be modelled for students, and students should be encouraged to employ various strategies in many problem-solving situations. While choices with respect to the timing of the introduction of any given strategy will vary, strategies such as try-and-adjust, look for a pattern, draw a picture, act it out, use models, make a table or chart, and make an organized list should all become familiar to students during their early years of schooling, whereas working backward, logical reasoning, trying a simpler problem, changing point of view, and writing an open sentence or equation would be part of a student's repertoire in the later elementary years. During middle school and the 9/10 years, this repertoire will be extended to include such strategies as interpreting formulas, checking for hidden assumptions, examining systematic or critical cases, and solving algebraically. *Functions and Relations 111/112* will continue to develop students' problem-solving reportoires.

Opportunities should be created frequently to link mathematics and career opportunities. Students need to be aware of the importance of mathematics and the need for mathematics in so many career paths. This realization will help maximize the number of students who strive to develop and maintain the mathematical abilities required for success in higher-level mathematics programming in senior high mathematics and beyond.

C. Learning and Teaching Mathematics

The unifying ideas of the mathematics curriculum suggest quite clearly that the mathematics classroom needs to be one in which students are actively engaged each day in the doing of mathematics. No longer is it sufficient or proper to view mathematics as a set of concepts and algorithms for the teacher to transmit to students. Instead, students must come to see mathematics as a vibrant and useful tool for helping them understand their world, and as a discipline which lends itself to multiple strategies, student innovation, and, quite often, multiple solutions. (See the "Contexts for Learning and Teaching" section of the foundation document.)

The learning environment will be one in which students and teachers make regular use of manipulative materials and technology, and actively participate in discourse and conjecture, verify reasoning, and share solutions. This environment will be one in which respect is given to all ideas in which reasoning and sense making are valued above "getting the right answer." Students will have access to a variety of learning resources, will balance the acquisition of procedural skills with attaining conceptual understanding, will estimate routinely to verify the reasonableness of their work, will compute in a variety of ways while continuing to place emphasis on mental computation skills, and will engage in homework as a useful extension of their classroom experiences.

D. Meeting the Needs of All Learners

The *Foundation for the Atlantic Canada Mathematics Curriculum* stresses the need to deal successfully with a wide variety of equity and diversity issues. Not only must teachers be aware of, and adapt instruction to account for, differences in student readiness, but they must also remain aware of avoiding gender and cultural biases in their teaching. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom.

NCTM's *Principles and Standards* (2000) cites equity as a core element of its vision for mathematics education. "All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study – and support to learn – mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students"(p. 12).

At grade 11 in New Brunswick, variations in student readiness, aptitude, and post-secondary intentions are addressed in significant part by the provision of courses at levels 1, 2 and 3. Students at all levels will work toward achievement of the same key-stage and general curriculum outcomes, and many of the course-specific curriculum outcomes will also be the same or similar. As well, the instructional environment and philosophy should be the same at all levels, with high expectations maintained for all students. The significant difference between levels will be the depth, breadth, and degree of sophistication and formalism expected with respect to each general outcome. Similarities between courses should allow some students to move from one course level to another.

By and large, Level 3 courses will be characterized by a greater focus on concrete activities, models, and applications, with less emphasis given to formalism, symbolism, computational or symbol-manipulating facility, and mathematical structure. Level 1 and 2 courses will involve greater attention to abstraction and more sophisticated generalizations, while Level 3 courses would see less time spent on complex exercises and connections with advanced mathematical ideas. Level 1 courses, which are designed for particularly talented students of mathematics, will be characterized by both more sophisticated engagement with mathematical concepts and techniques, and the extension of some topics beyond the scope provided at Level 2. These extensions will be included in Level 2 curriculum guides and identified with a *** ** *   * ** *** symbol.

By way of a brief illustration, students at all levels should develop an understanding of exponential relationships. Students taking Level 3 courses have as much need as others to understand the nature of exponential relationships, given the central place of these relationships in universal, everyday issues such as investment, personal and government debt, and world population dynamics. The nature of exponential relationships can be developed through concrete, hands-on experiments and data analysis that does not require a lot of formalism or symbol manipulation. The more formal and symbolic operations on exponential relationships will be much more prevalent in Level 1 and 2 courses.

Finally, within any given course at any level, teachers must understand, and design instruction to accommodate, differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Further, the practice of designing classroom activities to support a variety of learning styles must be extended to the assessment realm; such an extension implies the use of a wide variety of assessment techniques, including journal writing, portfolios, projects, presentations, and structured interviews.

E. Support Resources

This curriculum guide represents the central resource for the teacher of *Functions and Relations 111/112*. Other resources are ancillary to it. This guide should serve as the focal point for all daily, unit, and course-long planning, as well as a reference point to determine the extent to which the instructional outcomes should be met.

Nevertheless, other resources will be significant in the mathematics classroom. Textual and other print resources will be significant to the extent that they support the curriculum goals. Teachers will need professional resources as they seek to broaden their instructional and mathematical skills. Key among these are the NCTM publications, including the *Principles and Standards for School Mathematics, Assessment Standards for School Mathematics, Curriculum and Evaluation Standards for School Mathematics*, the *Addenda Series*, *Professional Standards for Teaching Mathematics,* and the various NCTM journals and yearbooks. As well, manipulative materials and appropriate access to technological resources (e.g., software, videos) should be available. Calculators will be an integral part of many learning activities.

F. Role of Parents

Societal change dictates that students' mathematical needs today are in many ways different than were those of their parents. These differences are manifested not only with respect to mathematical content, but also with respect to instructional approach. As a consequence, it is important that educators take every opportunity to discuss with parents changes in mathematical pedagogy and why these changes are significant. Parents who understand the reasons for changes in instruction and assessment will be better able to support their children in mathematical endeavours by fostering positive attitudes towards mathematics, stressing the importance of mathematics in their children's lives, assisting children with mathematical activities at home, and, ultimately, helping to ensure that their children become confident, independent learners of mathematics.

G. Connections Across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating learning experiences—through teacher-directed activities, group or independent exploration, and other opportune learning situations. However, it should be remembered that certain aspects of mathematics are sequential, and need to be developed in the context of structured learning experiences.

The concepts and skills developed in mathematics are applied in many other disciplines. These include science, social studies, music, technology education, art, physical education, and home economics. Efforts should be made to make connections and use examples which apply across a variety of discipline areas.

In science, for example, the concepts and skills of measurement are applied in the context of scientific investigations. Statistical concepts and skills are applied as students collect, present, and analyse data. Examples and applications of many mathematical relations and functions abound.

In social studies, knowledge of confidence intervals is valuable in intrepreting polling data, and an understanding of exponential growth is necessary to appreciate the significance of government debt and population growth. As well, students read, interpret, and construct tables, charts, and graphs in a variety of contexts such as demography.

Opportunities for mathematical connections are also plentiful in physical educaiton, many technological courses and the fine arts.

III. Assessment and Evaluation

A. Assessing Student Learning

Assessment and evaluation are integral to the process of teaching and learning. Ongoing assessment and evaluation are critical, not only with respect to clarifying student achievement and thereby motivating student performance, but also for providing a basis upon which teachers make meaningful instructional decisions. (See "Assessing and Evaluating Student Learning" in the *Foundation for the Atlantic Canada Mathematics Curriculum*.)

Characteristics of good student assessment should include the following: i) using a wide variety of assessment strategies and tools; ii) aligning assessment strategies and tools with the curriculum and instructional techniques; and iii) ensuring fairness both in application and scoring. The *Principles for Fair Student Assessment Practices for Education in Canada* elaborate good assessment practice and serve as a guide with respect to student assessment for the mathematics foundation document. (See also, Appendix A, "Assessing and Evaluating Student Learning.")

B. Program Assessment

Program assessment will serve to provide information to educators as to the relative success of the mathematics curriculum and its implementation. It will address such questions as the following: Are students meeting the curriculum outcomes? Is the curriculum being equitably applied across the region? Does the curriculum reflect a proper balance between procedural knowledge and conceptual understanding? Is technology fulfilling a proper role?

IV. Designing an Instructional Plan

It is important to develop an instructional plan for the duration of the course. Without such a plan, it is easy to run out of time before all aspects of the curriculum have been addressed. A plan for instruction that is comprehensive enough to cover all outcomes and topics will help to highlight the need for time management.

It is often advisable to use pre-testing to determine what students have retained from previous grades relative to a given topic or set of outcomes. In some cases, pre-testing may also identify students who have already acquired skills relevant to the current course. Pre-testing is often most useful when it occurs one to two weeks prior to the start of a a topic or set of outcomes. When the pre-test is done early enough and exposes deficiencies in prerequisite knowledge/ skills for individual students, sufficient time is available to address these deficiencies prior to the start of the topic/unit. When the whole group is identified as having prerequisite deficiencies, it may point to a lack of adequate development or coverage in previous grades. This may imply that an adjustment is required to the starting point for instruction, as well as a meeting with other grade level teachers to address these concerns as necessary.

Many topics in mathematics are also addressed in other disciplines, even though the nature and focus of the desired outcome is different. Whenever possible, it is valuable to connect the related outcomes of various disciplines. This can result in an overall savings in time for both disciplines. The most obvious of these connections relate to the use of measurement in science and the use of a variety of data displays in social studies.

V. Curriculum Outcomes

The pages that follow provide details regarding both specific curriculum outcomes and the four topics/units that comprise *Functions and Relations 111/112*. The specific curriculum outcomes are presented initially, then the details of the units follow in a series of two-page spreads. (See Figure 4 on next page.)

This guide presents the curriculum for *Functions and Relations 111/112* so that a teacher may readily view the scope of the outcomes which students are expected to meet during the year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings in this course are part of a bigger picture of concept and skill development. (See Appendix B for a complete listing of the SCOs for grades 9 and 10.)

Within each unit, the specific curriculum outcomes are presented on two-page spreads. At the top of each page, the overarching topic is presented, with the appropriate SCO(s) displayed in the left-hand column. The second column of the layout is entitled "Elaboration-Instructional Strategies/Suggestions" and provides a clarification of the specific curriculum outcome(s), as well as some suggestions of possible strategies and/or activities which might be used to achieve the outcome(s). While the strategies and/or suggestions presented are not intended to be rigidly applied, they will help to further clarify the specific curriculum outcome(s) and to illustrate ways to work toward the outcome(s) while maintaining an emphasis on problem solving, communications, reasoning, and connections. To readily distinguish between activities and instructional strategies, activities are introduced in this column of the layout by the symbol ❐. As well, curriculum extensions intended for students in the Level 1 course are indicated with the *** ** *   * ** *** symbol. This symbol not only brackets text discussing differentiation for students in the Level 1 course, but also appears at the top of each page on which such text is located.

The third column of the two-page spread, "Worthwhile Tasks for Instruction and/or Assessment," might be used for assessment purposes or serve to further clarify the specific curriculum outcome(s). As well, those tasks regularly incorporate one or more of the four unifying ideas of the curriculum. These sample tasks are intended as examples only, and teachers will want to tailor them to meet the needs and interests of the students in their classrooms. The final column of each display is entitled "Suggested Resources" and will, over time, become a collection of useful references to resources which are particularly valuable with respect to achieving the outcome(s).

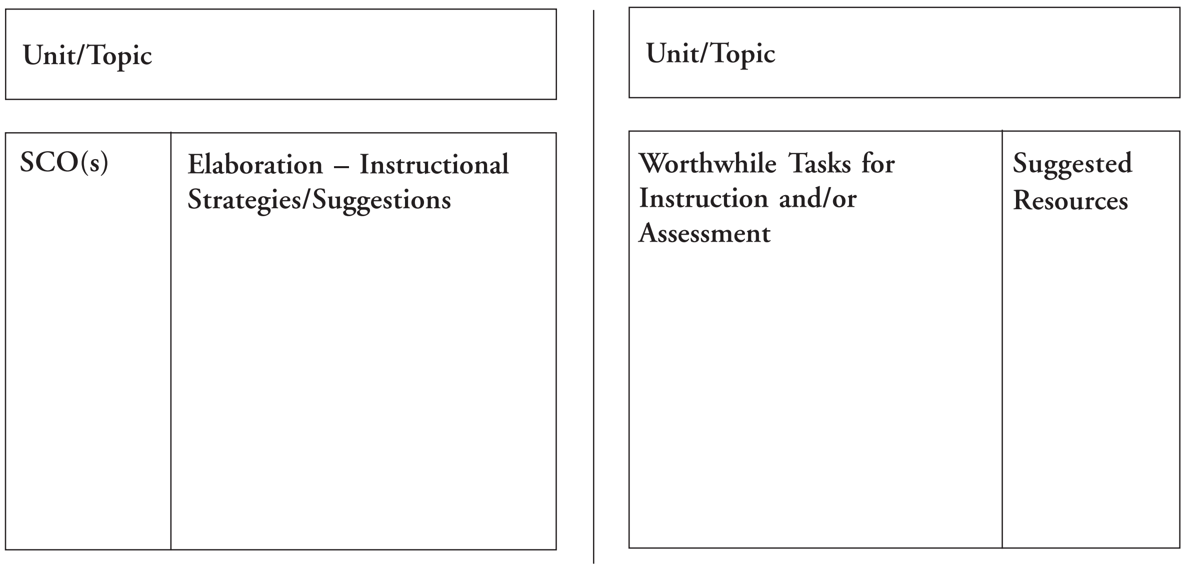


Figure 4: Layout of a 2-Page Spread

Specific Curriculum Outcomes
(by GCO)

GCO A: Students will demonstrate number sense and apply number theory concepts.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *i) demonstrate an understanding of number meanings with respect to real numbers* SCO: By the end of *Functions and Relations 111/112*, students will be expected to	
A3 demonstrate an understanding of the role of irrational numbers in applications	**A3** Students will encounter irrational numbers in the context of determining the roots of quadratic equations (see SCOs C22 and C23). They will need to make appropriate decisions with respect to expressing roots exactly (e.g., $-2 + \sqrt{7}$) or as rational approximations (e.g., 0.65). Unit 2, p. 72
A4 demonstrate an understanding of the nature of the roots of quadratic equations	**A4** Students will understand that a quadratic equation may have i) two real, distinct roots, ii) two real, equal roots (sometimes called a double root), or iii) no real roots (i.e., two complex roots – see SCO A9). They will relate this knowledge to the value of the discriminant and to the nature of the graph of the corresponding quadratic function (see C15). This outcome will also be addressed in connection with C22 and C23. Unit 2, p. 72
A5 demonstrate an understanding of the role of real numbers in exponential and logarithmic expressions and equations	**A5** Students will extend their knowledge of exponent laws involving integer exponents to include exponents involving real numbers (both rational and irrational). This understanding will be applied as well when dealing with the properties of logarithms (see SCO B13). Unit 4, p. 102
ii) order real numbers, represent them in multiple ways and apply appropriate representations to solve problems	
A7 describe and interpret domains and ranges using set notation	**A7** Students will describe and interpret domains and ranges with respect to quadratic, exponential and logarithmic functions. Domains and ranges will be addressed in connection with characteristics of the graphs of quadratic (see SCOs C31 and C32) and exponential (C33) functions, in the context of window sizes when using graphing technology, and with respect to the inverse relationship between exponential and logarithmic functions (C19). As well, students in the Level 1 course will consider domain and range in connection with transformations of exponential functions (see C35(111)). Unit 2, pp. 56, 64; Unit 4, pp. 94, 108, 116

GCO A: Students will demonstrate number sense and apply number theory concepts.

Elaboration

iv) explain and apply relationships among real and complex numbers

A9 represent non-real roots of quadratic equations as complex numbers

A9 Students will be expected to express non-real roots as complex numbers (using the representation $i^2 = -1$). This outcome will be addressed in connection with SCOs A4 and C22. Unit 2, p. 72

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

i) *explain how algebraic and arithmetic operations are related, use them in problem-solving situations, and explain and demonstrate the power of mathematical symbolism*

SCO: By the end of *Functions and Relations 111/112*, students will be expected to

B1 demonstrate an understanding of the relationships that exist between arithmetic operations and the operations used when solving equations

B1 In general, students will need to understand that many arithmetic properties (e.g., distributive property) and principles (e.g., adding equals to equals produces equals) apply in algebraic situations. In particular, students will connect algebraic perfect squares with arithmetic ones, and arithmetic applications of exponent laws with algebraic ones. This outcome will not be addressed in isolation but rather in connection with SCOs such as C9, C22, C23, C24, C25 and B12. Unit 2, pp. 66, 68; Unit 4, p. 112, 114

B2 demonstrate an intuitive understanding of the recursive nature of exponential growth

B2 Students will understand that successive terms of an exponential sequence may be generated by multiplying each preceding term by a constant factor, and that a sequence generated in this fashion is known as recursive. An in-depth study of recursive sequences is delayed until grade 12, however. This outcome will be addressed in connection with SCOs such as C2 and C4. Unit 4, pp. 90, 92

$B4_2$ use the calculator correctly and efficiently

$B4_2$ Students will need to use numerous calculator functions correctly, and make good choices with respect to their efficient use, in a variety of situations. These include trigonometric applications (see SCOs $C28_2$, D3 and D5), graphing relations and performing regression analysis (see SCOs such as C1, C2, C3, C8, C32 and F1), and considering tangents to curves (see C18, C27 and $C28_3$). Unit 1, pp. 32, 40; Unit 2, p. 58; Unit 3, pp. 84, 86; Unit 4, p. 104

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

	Elaboration
ii) derive, analyze and apply computational procedures in situations involving all representations of real numbers	
$B4_3$ calculate average rates of change	**$B4_3$** Students will determine average rates of change in a variety of situations, and conceptualize them as the ratio of the change in one quantity to the change in another (e.g., change in distance over a particular interval of time). Addressing this outcome will be connected closely to that of SCOs such as C17, C16 and C30. Unit 3, pp. 76, 78, 80, 82
B6 derive and analyze the Law of Sines, Law of Cosines, and the formula 'area of triangle ABC = ½ bcsinA'	**B6** Students should not only understand the Law of Sines and Law of Cosines, but should also be able to develop either of these from basic principles. This will be done in connection with SCOs D5 and D3. Unit 1, pp. 34, 36, 42
iii) derive, analyze and apply algebraic procedures (including those involving algebraic expressions and matrices) in problem situations	
B10 derive and apply the quadratic formula	**B10** One means of solving quadratic equations (see SCOs C22 and C23) is to use the general quadratic formula. All students will apply the formula, while understanding its connection to the nature of the roots (A4) and irrational (A3) and complex (A9) numbers. Deriving the quadratic formula will be an outcome for students in the Level 1 course only. Unit 2, p. 68
B11 (111) **analyze the quadratic formula to connect its components to the graphs of quadratic functions**	**B11(111)** This outcome is intended for students in the Level 1 course only. These students will associate aspects of the general quadratic formula with characteristics of the graph of the corresponding quadratic function (e.g., the equation of the axis of symmetry and the coordinates of the vertex). This outcome will connect with SCOs such as B10, C22 and C23. Unit 2, p. 70
B12 apply real number exponents in expressions and equations	**B12** Students will apply exponent laws in situations involving real number (especially rational) exponents to restate expressions and/or equations in equivalent forms. This outcome will connect to SCOs such as A5, C24 and C25. Unit 4, pp. 112, 114
B13 demonstrate an understanding of the properties of logarithms and apply them	**B13** Students will develop an understanding of the properties of logarithms based on their understanding of exponent laws (see SCO B12) and the inverse relationship between exponential and logarithmic relations (see C19). Unit 4, p. 118

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *i)* *model real-world problems using functions, equations, inequalities and discrete structures*	
SCO: By the end of *Functions and Relations 111/112*, students will be expected to **C1 model real-world phenomena using quadratic functions**	**C1** Mathematical modeling consists of providing a (simplified) mathematical description of a particular phenomenon. While mathematical models take many forms (e.g., scale diagrams, tables of values, graphs), in meeting this outcome students will describe phenomena using quadratic functions. This outcome will be addressed in connection with SCOs such as C29, C3, C8, F1 and C23. Unit 2, pp. 54, 56, 58, 60
C2 model real-world phenomena using exponential functions	**C2** Mathematical modeling consists of providing a (simplified) mathematical description of a particular phenomenon. While mathematical models take many forms (e.g., scale diagrams, tables of values, graphs), in meeting this outcome students will describe phenomena (such as compound interest) using exponential functions. This outcome will be addressed in connection with SCOs such as B2, C3, C11, F1 and C25. Unit 4, pp. 90, 92, 98, 100, 104
ii) *represent functional relationships in multiple ways (e.g., written descriptions, tables, equations and graphs) and describe connections among these representations*	
C3 sketch graphs from descriptions, tables and collected data	**C3** As part of the process of mathematical modeling (see SCOs C1 and C2), students will sketch scatter plots, with a view to distinguishing types of patterns (see C4 and C29), translating between representations (C8) and fitting curves and equations to data (F1). Unit 2, pp. 50, 56, 58; Unit 4, p. 104

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

Elaboration

C4 demonstrate an understanding of patterns that are arithmetic, power and geometric, and relate them to corresponding functions

C4 Students will focus on arithmetic and power patterns in Unit 2 and geometric ones in Unit 4. Among other ways, students should distinguish among arithmetic, power and geometric patterns in terms of levels of common differences between terms in sequences (e.g., arithmetic sequences have common differences at the first level while common differences are found at the second level in quadratic patterns). Also, students should relate arithmetic, power and geometric patterns to linear, power (especially quadratic) and exponential functions. This outcome will be addressed in connection with SCOs C29 and B2. Unit 2, pp. 50, 52; Unit 4, p. 92

C8 describe and translate between graphical, tabular, written and symbolic representations of quadratic functions

C8 Describing and translating between various representations of quadratic functions is a significant aspect of mathematical modeling (see SCO C1). While working towards achieving C8, students will also be sketching graphs (C3), analyzing tables and graphs (C29), analyzing quadratic functions (C31) and studying the affects of parameter changes on graphs (C32). Unit 2, pp. 54, 56, 60

C9 translate between different forms of quadratic equations

C9 Students will need to translate between general, transformational and standard forms of quadratic equations, so that the form of the equation suits the purpose for which it is to be employed. (Examples: transformational and standard forms reveal the coordinates of the vertex, general form does not; general and standard forms are convenient for use with graphing technology, transformational form is not.) Note: Translating from general form to either transformational or standard form requires use of the algebraic process known as “completing the square.” Students should bring knowledge of this technique from *Geometry and Applications in Mathematics 111/112*. SCO C9 will be addressed in part in connection with B1 and C23. Unit 2, pp. 64, 66

C10 (111) determine the equation of a quadratic function using finite differences

C10(111) This outcome is intended for students in the Level 1 course only. The technique capitalizes on the constant second level differences associated with quadratic relationships (see SCO C29) and students’ ability to solve linear systems (as developed in Year 10). Unit 2, p. 62

C11 describe and translate between graphical, tabular, written and symbolic representations of exponential and logarithmic relationships

C11 Describing and translating between various representations of exponential and logarithmic relationships is a significant aspect of mathematical modeling (see SCO C2). While working towards achieving C11, students will also be focusing on patterns associated with exponential functions (C4) and the nature of the inverse relationship between exponential and logarithmic functions (C19). Unit 4, pp. 92, 98, 100, 116

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

Elaboration

iii) *interpret algebraic equations and inequalities geometrically and geometric relationships algebraically*

C15 relate the nature of the roots of quadratic equations and the x-intercepts of the graphs of corresponding functions

C15 Students will connect two distinct, two equal, and no real roots of quadratic equations with, respectively, two, one, and no x-intercepts on the graphs of corresponding quadratic functions. This outcome will be addressed in conjunction with SCO A4. Unit 2, p. 72

C16 demonstrate an understanding that slope depicts rate of change

C16 Whether dealing with linear or curved graphs, students will understand that the slope of the graph at any point depicts the rate of change of one variable with respect to another at that point. This outcome will be addressed in connection with SCOs such as C17, C30 and $B4_3$. Unit 3, pp. 78, 80

C17 demonstrate an understanding of the concept of rate of change in a variety of situations

C17 Rate of change is a central concept in mathematics. Students will address rate of change in a variety of application areas, dealing with constant, variable, average (see SCO $B4_3$) and instantaneous (see C18, C27 and $C28_3$) rates of change. C17 also connects closely with SCOs C16 and C30. Unit 3, pp. 76, 78, 80, 82, 84, 86

C18 demonstrate an understanding that the slope of a line tangent to a curve at a point is the instantaneous rate of change of the curve at the point of tangency

C18 Students will understand the tangent to a curve at a point to be the straight line that best approximates the curve at that point. They will further understand that the slope of the tangent line is the slope of the curve and, hence, the instantaneous rate of change of the curve, at that point. This SCO is closely connected to C17 and C27. Unit 3, pp. 84, 86

C19 demonstrate an understanding, algebraically and graphically, that the inverse of an exponential function is a logarithmic function

C19 Since this is students' first exposure to inverse functions, they will not only understand that exponential and logarithmic functions are inverses, but will also understand, in general, the significant characteristics of inverse functions (e.g., the domain of a function is the range of its inverse). Addressing SCO C19 will allow further work on outcomes such as C11 and A7. Unit 4, p. 116

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
iv) solve problems involving relationships, using graphing technology as well as paper-and-pencil techniques	
C22 solve quadratic equations	**C22** In *Mathematics 10*, students solved quadratic equations by factoring and by identifying the x-intercepts of the corresponding graphs. In this course students will add the general quadratic formula as a tool for solving quadratic equations. This outcome will be addressed in conjunction with SCOs B10 and C23. Unit 2, pp. 68, 70
C23 solve problems involving quadratic equations	**C23** Students will solve a variety of problems involving quadratic equations. While this may sometimes be conveniently done graphically, students will often need to manipulate/solve quadratic equations to identify maximum/minimum values and/or roots. Students should also be prepared to identify inadmissible roots when dealing with problem contexts. This outcome will be addressed in conjunction with others such as C9 and C22. Unit 2, pp. 58, 66
C24 solve exponential and logarithmic equations	**C24** While students will be able to solve some exponential or logarithmic equations graphically, they will also employ exponent laws, logarithm properties and other standard techniques to solve equations algebraically. Addressing this outcome will connect with SCOs B12 and C15. Unit 4, pp. 112, 114, 118
C25 solve problems involving exponential and logarithmic equations	**C25** Students will solve problems involving exponential and logarithmic equations in a variety of real-world contexts. They will solve by considering basic patterns (see SCO C4), modeling (C2 and F1), translating between representations (C11), considering parameter changes (C34), and solving equations (C24). Unit 4, pp. 92, 94, 96, 98, 104, 112, 114
C27 approximate and interpret slopes of tangents to curves at various points on the curves, with and without technology	**C27** Students will approximate the slope of a tangent to a curve at a particular point by considering the slopes of secants to the curve in the vicinity of the point in question. They will interpret the determined slope of the tangent as the instantaneous rate of change at the point, in terms of the context of the problem being modeled. This outcome will be addressed in conjunction with SCOs $C28_3$, C17 and C18. Unit 3, pp. 84, 86
$C28_3$ solve problems involving instantaneous rate of change	**$C28_3$** Students will solve problems involving instantaneous rate of change in a variety of contexts. This outcome will be addressed in connection with other SCOs such as C17, C18, C27 and C30. Unit 3, p. 84

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

Elaboration

v) analyze and explain the behaviours, transformations and general properties of types of equations and relations

C29 analyze tables and graphs to distinguish between linear, quadratic and exponential relationships

C29 Students will differentiate between linear, quadratic and exponential relationships in a number of ways. In tables, they will identify number patterns and investigate first-, second- and third-level differences. With respect to graphs, they will not only recognize general shapes, but also identify patterns of key points (e.g., moving over 1 up 1, over 2 up 4, over 3 up 9 from the vertex of a quadratic). This SCO will be addressed in connection with others such as C4, C8, C33 and C10(111). Unit 2, pp. 50, 52, 54, 56, 62; Unit 4, pp. 94, 106

C30 describe and apply rates of change by analyzing graphs, equations and descriptions of linear and quadratic functions

C30 How fast is a given relationship changing? Is it changing at the same rate all the time? Students will analyze linear and quadratic functions to answer questions such as these. They will also apply rates of change by determining average and instantaneous rates for these functions. This outcome will be addressed in connection with others such as C17, C16, $B4_3$ and $C28_3$. Unit 3, pp. 78, 82, 84

C31 analyze and describe the characteristics of quadratic functions

C31 Students will analyze representations (e.g., tables, graphs, equations) of quadratic functions and describe such characteristics as symmetry, vertices, and intercepts in terms of equations of axes of symmetry, maximum/minimum values and range, and roots of corresponding equations, respectively. See also related SCOs A7, C8 and C32. Unit 2, pp. 56, 60, 64

C32 demonstrate an understanding of how parameter changes affect the graphs of quadratic functions

C32 For the quadratic function $\frac{1}{a}(y-k)=(x-h)^2$, a, k and h are parameters, i.e., values that are constant for a particular application or situation. When addressing this outcome, however, students will show how altering one or more of these parameters produces related families of quadratic functions. Parameter changes will be connected directly to stretches, reflections and/or translations of corresponding graphs. This outcome will be addressed in conjunction with SCOs C8 and C31. Unit 2, pp. 60, 64, 66

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
C33 analyze and describe the characteristics of exponential and logarithmic functions	**C33** Students will analyze exponential (e.g., $y = a^x$) and logarithmic (e.g., $y = \log_a x$) functions and describe characteristics such as general shape, location of asymptotes, and domain and range, for values $a > 1$ and $0 < a < 1$. SCOs such as C29 and A7 will have connections here. As well, students in the Level 1 course will connect C33 with C35(111), which examines transformations on exponential functions. Unit 4, pp. 94, 106, 110
C34 demonstrate an understanding of how parameter changes affect the graphs of exponential functions	**C34** For the exponential function $y = Ca^x$, C and a are parameters, i.e., values that are constant for a particular application or situation. When addressing this outcome, however, students will show how altering one or both of these parameters produces related families of exponential functions. Unit 4, p. 96
C35 (111) **write exponential functions in transformational form and as mapping rules to visualize and sketch graphs**	**C35(111)** This outcome is intended for students in the Level 1 course only. Students will capitalize on previous experiences with transformations (i.e., reflections, stretches and translations) and mapping rules to connect algebraic representations of exponential functions to their corresponding graphs. This SCO may be addressed in connection with C33. Unit 4, pp. 108, 110
vi) perform operations on and between functions	
$C28_2$ analyze and solve trigonometric equations, with and without technology	**$C28_2$** Students will solve simple trigonometric equations, both substituting trigonometric values for known angles to solve for unknown measurements, and using known information to determine trigonometric values for unknown angles and then solving for the angles using inverse trigonometric relationships. Unit 1, p. 32

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to
iii) apply measurement formulas and procedures in a wide variety of contexts

SCO: By the end of *Functions and Relations 111/112*, students will be expected to

D3 apply sine and cosine ratios to situations involving non-acute angles

D5 apply the Law of Sines, the Law of Cosines and the formula 'area of a triangle ABC = ½ bcsinA' to solve problems

Elaboration

D3 In dealing with obtuse triangles, students will need to determine sine and cosine values for obtuse angles. These will be applied in situations involving the Law of Sines and Law of Cosines (see SCO D5). This outcome is also connected to B6 and $B4_2$.
Unit 1, pp. 36, 38, 40, 42, 44, 46

D5 Having developed the Law of Sines and Law of Cosines, students will apply them in a variety of problem situations. This outcome is closely connected with SCOs B6 and D3.
Unit 1, pp. 34, 36, 38, 40, 42, 44, 46

GCO F: Students will solve problems involving the collection, display and analysis of data.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

iii) *use curve fitting to determine the relationship between, and make predictions from, sets of data and be aware of bias in the interpretation of results*

SCO: By the end of *Functions and Relations 111/112*, students will be expected to

F1 **analyze scatter plots, and determine and apply the equations for curves of best fit, using appropriate technology**

Elaboration

F1 Students will collect data and sketch graphs (see SCO C3) to model mathematical situations (C1, C2). As part of the modeling process, they will determine the equations of curves of best fit and use them to solve problems (see also C23 and C25).
Unit 2, p. 58; Unit 4, p. 104

Unit 1
Applications of Trigonometry

(10 Hours)

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

$B4_2$ use the calculator correctly and efficiently

$C28_2$ analyse and solve trigonometric equations with and without technology

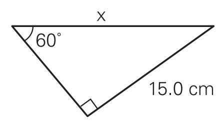
Elaboration – Instructional Strategies/Suggestions

Trigonometry is the study of the relationships of the measures of the sides and angles of triangles. All problems that deal with geometric situations where length or angle measure is required can be modelled with diagrams or constructions of the geometric figures.

$C28_2$ In a previous course, students developed the trigonometric ratios $\sin\theta$, $\cos\theta$, and $\tan\theta$ and applied them to problems involving right triangles. Students may need to review the use of trigonometric ratios to solve problems involving right triangles.

$B4_2$/$C28_2$ Students will use their calculators correctly and efficiently for various procedures while working with trigonometric relationships. For example, when finding a missing side in a situation modeled by a right triangle, some students may set up an equation like

$$\sin 60° = \frac{\text{length of side opposite}}{\text{length of hypotenuse}} = \frac{15.0}{x},$$



then multiply both sides by x: $x \sin 60° = 15.0$, then divide by sin 60°, and they would have the expression: $x = \frac{15.0}{\sin 60°}$.

Using their calculators in degree mode, they would divide 15.0 by sin 60° to obtain 17.3 cm, the length of the hypotenuse. This would be efficient use of the calculator.

Students will also use trigonometric equations like the one to the right to find missing angles in right triangles.

$$\tan\theta = \frac{4.0}{7.0}$$

$$\frac{4.0}{7.0} = .57$$

$$\theta = \tan^{-1} .57$$

$$\theta \doteq 29.68$$

$$\theta \doteq 30°$$

Some students would first change the ratio to a decimal, $\tan\theta = .57$.

Then those students would find the angle measure θ by using "$\tan^{-1}$."

Other students may simply enter $\tan^{-1}\left(\frac{4.0}{7.0}\right)$ into their calculators and solve for theta. This would be considered efficient use of the calculator.

Since this chapter deals with measurement, students should be careful to properly use precision, accuracy, and significant digits. Teachers may wish to discuss these at this time.

Applications of Trigonometry

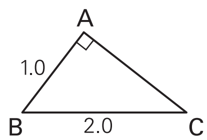
Worthwhile Tasks for Instruction and/or Assessment

$B4_2$/$C28_2$

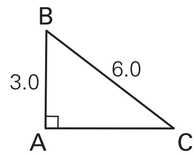
Performance

1) a) Find the measure of angle C for each of the following. From the pattern in the values, make a conjecture about a particular relationship that seems to exist in a 30º–60º–90º triangle.

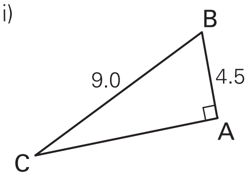
i)



ii)

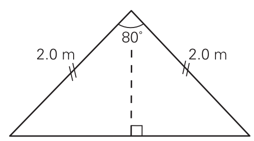


iii)



b) Test your conjecture in a situation that you make up.

2) Oliver makes a shelter in the shape of an isosceles triangle. Using the given measures find how much room above his head he will have inside the shelter. His height is 1.48 m.



3) Richard leans the top of his 7.3m ladder against the sill of a window that is 6.5m above the ground.
   a) At what angle to the ground will his ladder be?
   b) If the ladder is rotated without moving its feet, to lean in the opposite direction against a building that is 5.1 m away from the first building, at what angle will it be to the ground?

Suggested Resources

Brueningsen, Chris et. al., *Real-World Math with the CBLTM System*, Texas Instruments

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992

"Geometer's Sketchpad," Key Curriculum Press

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

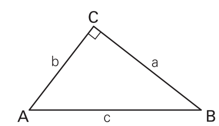
B6 derive and analyse the Law of Sines, the Law of Cosines, and the formula for "area of a triangle $ABC = \frac{1}{2}bc\sin A$"

D5 apply the Law of Sines, the Law of Cosings, and the formulas "area of a triangle $ABC = \frac{1}{2}bc\sin A$" to solve problems

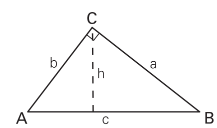
Elaboration – Instructional Strategies/Suggestions

B6 Sometimes finding the area of a right triangle can be done more efficiently by using the area formula, area of a triangle $= \frac{1}{2}bc\sin A$.. To develop this formula, students should be asked to write how they would find the area of triangle ABC.

Students would write A= $\frac{1}{2}$ base $\times$ height.



In this triangle, the base is c, so A= $\frac{1}{2}$ch



The teacher would then ask students to replace the "*h*" with an expression using sin A.

Students would write

$\sin A = \frac{h}{b}$,

or,

$h = b\sin A$,

so filling into A= $\frac{1}{2}$ base $\times$ height

$A = \frac{1}{2}c(b\sin A)$,

or without brackets

$A = \frac{1}{2}bc\sin A$.

D5 Students should apply this formula in various problem - solving situations involving area. To use this formula to find area, students should realize that they need any two sides and the included angle measure of any triangular shape. When the area of a triangular shape is given, students can use this formula to find any of the missing three measures, if the other two are given. For example, if the area of a triangular region on a stage was to be carpeted with 37 square metres of carpet, and two adjacent sides measured 12.0 m and 6.7 m, then the angle between these sides could be found:

Area = $\frac{1}{2}bc\sin A$

$37 = \frac{1}{2}(12.0)(6.7)\sin A$

$A \doteq 67°$

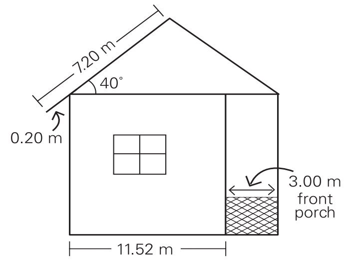
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

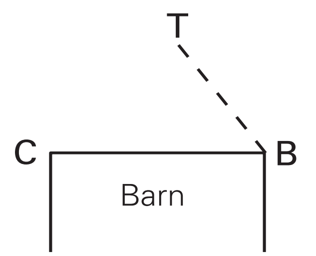
D5

Performance

1) Hilary wants to paint the triangular gable ends of her log cabin. She knows that a can of paint will cover 39 m2. She expects to have to paint two coats. If a can costs $29.95, how much money will she have to spend?



2) Cousin Barney is building a new corral on the side of his barn for his new lamb, Huey. He measures the barn length to be 15.25 m. There is already a fence from one end of the barn to a tree (T) with a length 21.62 m. Barney has just spread seed that covers 120.50 m2 inside the triangular region C–B–T. How long will the fence be that goes from C to T?



3) The area of a triangle is 100cm^2. Two sides are known, one being is 3 times the length of the other. The angle between them is 47°. What is the measure of all the sides?

B6

Journal

4) Ask students to explain why $c \sin A$ from the formula
"Area of a triangle" $= \frac{1}{2} bc \sin A$ is the same as "h" in the formula
Area of a triangle$= \frac{bh}{2}$.

Suggested Resources

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

B6 **derive and analyse the Law of Sines, the Law of Cosines, and the formula for "area of a triangle $ABC = \frac{1}{2} bc \sin A$"**

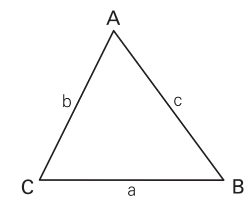
D3 **apply sine and cosine ratios to situations involving non-acute angles**

D5 **apply the Law of Sines, the Law of Cosines, and the formula "area of a triangle $ABC = \frac{1}{2} bc \sin A$" to solve problems**

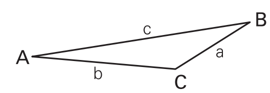
Elaboration – Instructional Strategies/Suggestions

B6 To develop the Law of Sines, students might begin like this (or they might use a geometry software package to explore in the same way):

Questions 1–3 refer to the acute $\triangle ABC$ on the right.



1) Measure each side to the nearest tenth of a centimetre.
   a) c b) a c) b
2) Measure each angle to the nearest degree.
   a) $\angle A$ b) $\angle B$ c) $\angle C$
3) Calculate each of the following to 1 decimal place.
   a) $\dfrac{a}{\sin A}$ b) $\dfrac{b}{\sin B}$ c) $\dfrac{c}{\sin C}$



4) Repeat questions 1–3 for the obtuse $\triangle ABC$.
5) Draw an acute triangle, $\triangle ABC$, of your own and repeat questions 1–3.
6) Draw an obtuse triangle, $\triangle ABC$ of your own and repeat questions 1–3.
7) Based on the results of question 3, what can you conclude about the relationship between $\dfrac{a}{\sin A}, \dfrac{b}{\sin B}, \dfrac{c}{\sin C}$?

Students should conclude that these ratios are equal, and they should understand that they should be able to prove that they are equal.

To help students with this proof, teachers might ask some students to write a trigonometric equation, using sin B, that shows the height AD, and others to use sin C.

Students would respond with $\sin B = \frac{AD}{c}$

so AD = c sin B,

and others with $\sin C = \frac{AD}{6}$ so *AD* = *b* sin *C*.

Teachers and students would reason that the conclusion would have to be *c* sin *B* = *b* sin *C,* since they are both equal to *AD*.

Students should then verify that $\frac{c}{\sin C} = \frac{b}{\sin B}$ is the same as c sinB = b sin C.

(Multiply both sides of the equation by sin *C* sin *B*—the lowest common denominator.)

D3/D5 Students should apply the Law of Sines to solve various problems. See the next two-page spread.

Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

B6

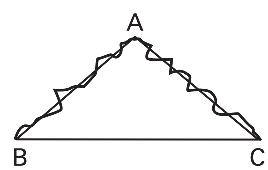
Performance

1) (Level 1 only) Draw any triangle ABC, and the altitude from A. Use this to derive the Law of Sines.
2) (Level 2 only) Draw any triangle ABC, and the altitude from A. Use this to derive the Law of Sines. Hint: Express the length of the altitude in terms of the sine of an angle, and again as the sine of a different angle.
3) If $\frac{\sin A}{a}=\frac{\sin B}{b}$, how do you know for sure that $\frac{\sin C}{c}=\frac{\sin B}{b}$? Explain.

B6/D5

Performance

4) What information must be given in a problem that would lead you to use the Law of Sines?
5) a) While visiting a ski lodge in Switzerland, Real noticed the profile of a mountain in the distance, and how triangular it appeared. He estimated the angle measure at C to be about 30°, and B to be about 40°. The sign advertises a T-bar ride up the slope AB, a distance of 2917 m. Show that the ski run down the side AC must be at least 3750 m.



   b) If someone told you that the distance from A to C was actually 3827 m, determine how close your estimate of it is.
6) a) Make up a problem where a side length can be determined using Law of Sines. Show the solution.
   b) Make up a problem where an angle measure can be determined using Law of Sines.

Suggested Resources

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

D3 **apply sine and cosine ratios to situations involving non-acute angles**

D5 **apply the Law of Sines, the Law of Cosines, and the formula "area of a triangle $ABC = \frac{1}{2} bc \sin A$" to solve problems**

Elaboration – Instructional Strategies/Suggestions

D3/D5 Students will use the Law of Sines to find a missing side length when two angle measures are given. Or, they will use the Law, whether they have one angle already and are looking for another, given the opposite sides. Or, they will use it when they've got 2 angles and one side and they need the other side. Any one proportional statement from the formula called the Law of Sines is all that is used at one time.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \text{ or } \frac{b}{\sin B} = \frac{c}{\sin C}$$

Students should realize that some students may use the formula in this form:

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

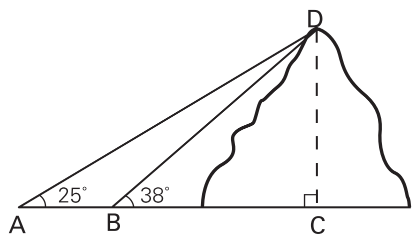
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

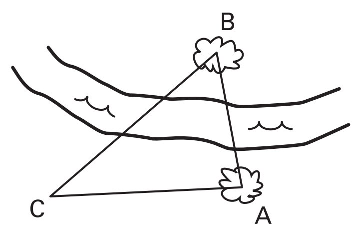
D3/D5

Performance

1) Surveyors cannot get to the inside centre of a mountain easily. Therefore, a mountain's height must be measured in a more indirect way. Find the height of the mountain.



2) Surveyor Sally had to determined the distance between two large trees situated on opposite sides of a river. She placed a stake at C, 100.0 m from point A. Sally then determines the angle measures at points C and A to 45º and 80º respectively to B.
   a) Ask students to help her find the distance between the trees.
   b) Ask students to create a different problem,using the above diagram.



Journal

3) In class, Marlene said that she could find AB in question 2(above), using the Law of Cosines. Is she correct or not? Explain.

4) Billy is using $\frac{\sin A}{a} = \frac{\sin B}{b}$ to solve a problem. Billy's dad said that in his day he would have used $\frac{a}{\sin A} = \frac{b}{\sin B}$. Is Billy or his dad correct? Explain.

Suggested Resources

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992

"Geometer's Sketchpad," Key Curriculum Press

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

D3 apply sine and cosine ratios to situations involving non-acute angles

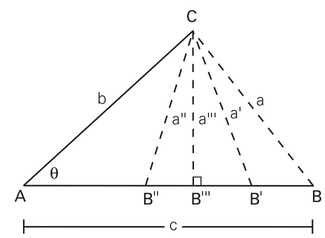
D5 apply the Law of Sines, the Law of Cosines, and the formula "area of a triangle $ABC = \frac{1}{2} bc \sin A$" to solve problems

$B4_2$ use the calculator correctly and efficiently

Elaboration – Instructional Strategies/Suggestions

$D3/D5/B4_2$ When applying the Law of Sines, students must consider the possibility that there may be more than one triangle with certain given measurements. One will lead to an angle measure greater than 90°, but the calculator will give the corresponding supplementary measurement. They should consider what happens given the following criteria:

When given a measure for $\angle A$, a length "*b*" and a length "*a*", the points A and C are fixed, but the point B is not. The side CB could be positioned in any of the locations indicated by the dotted line depending on how long CB (a) is and the measure of the $\angle C$.



The shortest "*a*" can be is the perpendicular length from C to AB''' and is determined by b sin A. If "*a*" is a little longer, then B might be in location B' or B''. This will be true until *a* exceeds *b*, then with θ given, *B* can be the only location on *AB*.

Thus:

1) if $a = b \sin A$, then there is only one triangle and it is a right triangle with one $\angle B = 90°$.
2) if $a > b \sin A$, but $a < b$, then two triangles are possible with *B* at *B*'or *B*''.
3) if $a > b \sin A$ and $a > b$, then there is only one $\triangle ABC$ with *B* at location *B*.

Situation (2) is called the "ambiguous" case, and students should be given the opportunity to develop this through an activity.

Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

D5/D3/$B4_2$

Performance

1) Sketch a diagram, including some measurements that would ask students to solve for a particular side in a given triangle. Have them show how the diagram could be drawn differently, resulting in a different but correct answer for an angle. Explain how the students would know that these different answers might both be correct answers.
2) There was a shipwreck. As the Coast Guard vessel approached the wreck, a rescue helicopter was spotted at an angle of elevation of 15^o and 280m from the Coast Guard vessel. The helicopter dropped its rescue rope. The wind was blowing quite hard, so the rope had to be let out further (25m altogether) in order to reach the wreck. How far was the wreck from the Coast Guard vessel if the vessel, wreck, and helicopter were in the same plane?
3) Matthew enjoys swimming in the ocean. One day Matthew decided to swim 9.2km from island A to island B; then, after resting a few moments, he swam 8.6km to island C. If island C to island A to island B forms a 52^o angle, determine how far Matthew has to swim to return to island A.

Suggested Resources

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

B6 **derive and analyse the Law of Sines, the Law of Cosines, and the formula "area of a triangle $ABC = \frac{1}{2} bc \sin A$"**

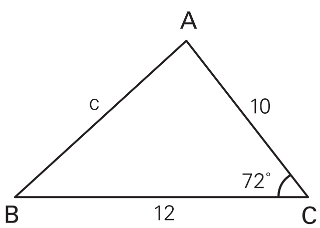
D3 **apply sine and cosine ratios to situations involving non-acute angles**

D5 **apply the Law of Sines, the Law of Cosines, and the formula "area of a triangle $ABC = \frac{1}{2} bc \sin A$" to solve problems**

Elaboration – Instructional Strategies/Suggestions

B6/D3 Students will interact with the teacher to develop a procedure for obtaining measurements of triangles that are not right-angled (oblique triangles). They should develop through teacher-led discussion, or a directed activity, the Law of Cosines and the Law of Sines.

Teachers should help students discover or develop the Law of Cosines which, stated symbolically, in one form, is $c^2 = a^2 + b^2 - 2ab \cos C$. Students should also be able to state this Law beginning with a^2 or b^2.



D3/D5 Applied to the given diagram to find AB, the Law of Cosines would be written as follows:

$AB^2 = BC^2 + AC^2 - 2(BC)(AC) \cos C$

B6 To develop this formula, students might begin by dropping a perpendicular from A to $\overline{BC}$. This creates some right triangles and splits $\overline{BC}$ into two lengths x and $12 - x$. An activity needs to encourage students to use right triangle trigonometry and the Pythagorean Theorem to express certain lengths and angle measures in terms of BC, AC, and the cosine of $\angle C$.

The activity might begin by asking students to express c^2 using the Pythagorean Theorem. Students should notice that	$c^2 = AR^2 + BR^2$ ①
$BC = BR + RC$ and try to use the value for BC in the formula. Since ...	$BC = 12$, then $RC = x$ and $BR = 12 - x$
Now, substituting into ① and expanding BR^2	$c^2 = AR^2 + (12 - x)^2$ $= AR^2 + 12^2 - 2(12)x + x^2$
rearranging gives ...	
$AR^2 + x^2 = AC^2$, and $BC = 12$ ...	$= AR^2 + x^2 + 122 - 2(12)x$ $= AC^2 + BC^2 - 2(BC)x$ ②
RC, or x can be expressed (using trig) as ...	
$AC = 10$ and $PC = 72^o$, so ...	$10 \cos 72^o$ $AC \cos C = RC$
substituting into ② ...	$c^2 = AC^2 + BC^2 - 2(BC)\ AC \cos C$

This representation now allows students to find AB ("c"), using the given measurements, and without having to have right angles.

D5 Students should apply the Law of Cosines in various situations. Students should think about using this Law when they have a situation where there is no right angle, and they need a third side length when the other two sides of the triangle are given.

... continued

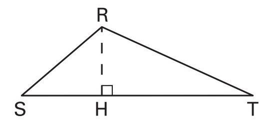
Applications of Trigonometry

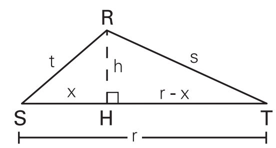
Worthwhile Tasks for Instruction and/or Assessment

B6

Performance

1) (Level 1 only) Using the diagram given, derive the Law of Cosines.





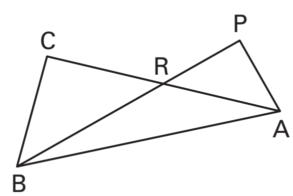
2) (Level 2 only) Given: $\triangle RST$ and altitude RH.
   Prove: $s^2 = t^2 + r^2 - 2rt \cos s$
   Hint:
   i) Use Pythagorean Theorem to express t^2 and s^2.
   ii) Express both of the above, beginning with "$h^2 =$".
   iii) What new equation can be formed now?
   iv) Expand binomials, if possible.
   v) What you're trying to prove has *s*, *t*, and *r* as the variables. Replace other variables in terms of *s*, *t*, and *r*.

B6/D5

Performance

3) Explain how you would know when to use the Law of Cosines in a problem.

D5/D3



Performance

4) a) Given
   BC = 87.4 m
   CA = 101.5m
   PA = 56.2 m
   $\angle CA = 112.4^{\circ}$
   $\angle PAC = 37.2^{\circ}$

   find the area $\triangle BPA$.
   b) Create a problem context for this given information.
   c) Create another problem like this, where you must first obtain a side, then determine an area.
5) What is the measure of the largest angle in the triangle with side lengths 16, 21, and 24cm?

Suggested Resources

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

D3 **apply sine and cosine ratios to situations involving non-acute angles**

D5 **apply the Law of Sines, the Law of Cosines, and the formula "area of a triangle $ABC = \frac{1}{2} bc \sin A$" to solve problems**

Elaboration – Instructional Strategies/Suggestions

D3/D5 Usually the formula for the Law of Cosines is used to find a missing side, given the other two sides and the angle between the given sides (the included angle). The formula can be restated as

$b^2 = a^2 + c^2 - 2ac \cos B$, or
$a^2 = b^2 + c^2 - 2bc \cos A$ to better set up the solution process.

The formula is structured so that students need not memorize. Instead, if the measure "*a*" is needed, then the other side of the formula is made up of all "*b*'s" and "*c*'s". If the measure "*b*" is needed, then the other side of the formula is all "*a*'s and *b*'s". The angle used in the formula is always the angle opposite the side needed.

Sometimes students will use the formula to find an angle measure when all 3 side lengths are given. They don't need to rearrange the formula first. For example, given $a = 12.0$ cm, $b = 9.5$ cm, and $c = 7.2$ cm, students could find any of the angle measures. If they wanted $\angle A$, they would begin the formula with a^2:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$12.0^2 = 9.5^2 + 7.2^2 - 2(9.5)(7.2) \cos A$$
$$144.0 = 90.25 + 51.84 - 136.8 \cos A$$

Next, they need to isolate the variable term:

$$144.0 - 90.25 - 51.84 = -136.8 \cos A$$

then isolate the variable:

$$\frac{1.91}{-136.8} = \frac{136.8 \cos A}{-136.8}$$
$$-0.01396 = \cos A$$
$$A = \cos^{-1}(-0.01396)$$
$$A = 90.8°$$

Students would conclude that the angle *A* measures 90.8°. Students need to think about using the Law of Cosines when they have a situation where there is no right angle and they need a third side length, given the other two, or an angle measure, given all the side lengths.

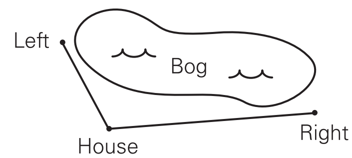
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

D3/D5

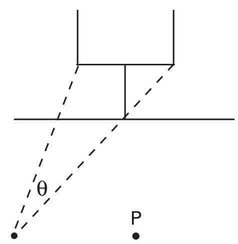
Performance

1) A farmer wants to find the length of the back of his property, which is mostly bog land. He knows that his left and right boundary lines connect near his house at an angle of 147°. The left boundary length is 90 m and the right is 110 m. Help the farmer.



2) Terry is building an A-frame cabin in the woods. The length of each of two rafters is 8.50m. If the angle of the apex of the frame is to be 46°, calculate the proposed width of the cabin at the base.

3) a) A football player is attempting a field goal. His position on the field is such that the ball is 7.5m to the left upright of the goal post and 10.0m to the right upright of the goal post. The goal posts are 4.3m apart. Find the angle marked θ.
   b) If the ball is moved to the middle of the field, position P, then the ball is equidistant to both uprights, approximately 8.5 m each. Find the angle corresponding to θ from this position.



D3/D5

Journal

4) a) Explain why the Law of Cosines might be a useful relationship to try to remember.
   b) How can you help yourself remember it? (Hint: It is the Pythagorean theorem, plus or minus some adjustment factor. How can I remember the adjustment factor?)

Suggested Resources

Brueningsen, Chris et al, *Real-World Math with the CBLTM System*, Texas Instruments

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992.

Applications of Trigonometry

Outcomes

SCO: In this course, students will be expected to

D3 apply sine and cosine ratios to situations involving non-acute angles

D5 apply the Law of Sines, the Law of Cosines, and the formula "area of a triangle $ABC = \frac{1}{2} bc \sin A$" to solve problems

Elaboration – Instructional Strategies/Suggestions

D3/D5 Teachers may want to discuss with students that the Law of Cosines 'looks' like the Pythagorean Theorem with an adjustment factor to make up for the lack of a right angle.

$$c^2 = a^2 + b^2 \text{ - adjustment factor}$$

They may want to have students examine several situations where the measure of $\angle C$ has various values, say 40°, 50°, 70°, 85°, 90°, 100°, 110°, 120°, ... Students would evaluate the 'adjustment factor' 2ab cos c and note that, as m $\angle C \to 90°$, 2ab cos $c \to 0$. As m $\angle C$ gets larger and larger beyond 90°, 2ab cos c gets smaller and smaller (the absolute value gets larger).

Talk to students now about the following:

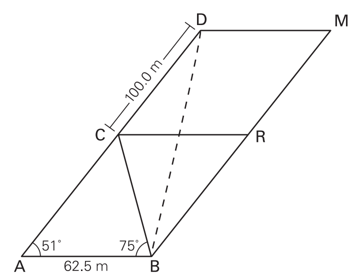
If $m\angle C = 90°$, then (Pythagorean Theorem)

If $m\angle C = 90°$, then c should get smaller. $c^2 = a^2 + b^2$ - the adjustment factor

if ???????, then c should get larger. $c^2 = a^2 + b^2$ - a larger and larger negative value.

Students should be prepared to combine both the Laws of Sines and Cosines in the same question when required.

For example, farmer Jones' property (ABRC) is shaped like a parallelogram (see diagram). He is given more land (DMRC). He needs to know the distance from D to B. He would first have to use the Law of Sines to get BC, then the Law of Cosines to BD.



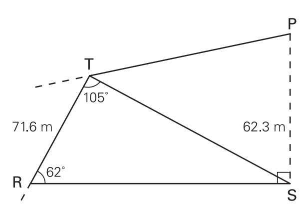
Applications of Trigonometry

Worthwhile Tasks for Instruction and/or Assessment

D3/D5

Activity

1) Given $\triangle ABC$ with b = 4.0 cm and c = 3.0 cm,
   a) ask students to determine "a" if $m\angle A = 90°$
   b) ask students to evaluate $2bc \cos A$, if $m\angle A = 90°$
   c) ask students to determine "a" again using $a^2 = b^2 + c^2 - 2bc \cos A$, given $m\angle A = 90°$. Make a conjecture about this new formula
   d) ask students to construct an accurate diagram of $\triangle ABC$ with b = 4.0 cm, c = 3.0 cm, and $m\angle A = 80°$
   e) ask students to
      i) predict if a < 5.0 cm, a = 5.0 cm, or a > 5.0 cm and explain their choice.
      ii) measure with a ruler the length "a" and record it.
      iii) calculate the length "a" using $a^2 = b^2 + c^2 - 2bc \cos 80°$.
   f) ask students to repeat step (e) given $m\angle A$ =60°, 65°, 85°, 89°.
   g) ask students to describe the pattern and how it fits with their conjecture in (c).
   h) ask students to repeat step (e) given $m\angle A$ =91°, 95°, 100°, 120° .
   i) ask students to describe the pattern and how it fits with their conjecture in (c).
   j) ask students to make a statement about how the formula $a^2 = b^2 + c^2 - 2bc$ cos A can be used.
2) Glen conjectured that if he knew the side measures, he could determine the $m\angle A$, using this new formula. Do you agree or disagree? Explain.
3) Colin conjectured that, if he knew the length of 'b' and 'a' and the $m\angle C$, he could determine the length C. Explain what Colin must be thinking.
4) In the town where Simon lives some roads were constructed as in the diagram. Simon needs to know how much longer the road from P to T is than the road from R to T.



Suggested Resources

Brueningsen, Chris et al, *Real-World Math with the CBLTM System*, Texas Instruments

Meiring, Steven P., "A Core Curriculum," *Addenda Series 9-12*, NCTM, 1992

Unit 2
Quadratics

(30 hours)

Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C4 **demonstrate an understanding of patterns that are arithmetic, power, and geometric and relate them to corresponding functions**

C29 **analyse tables and graphs to distinguish between linear, quadratic, and exponential relationships**

C3 **sketch graphs from descriptions, tables, and collected data**

Elaboration—Instructional Strategies/Suggestions

Students should extend previous knowledge to describe and reason in a variety of contexts using the mathematical constructs of function and relation and the symbolic language of algebra. Students should have had experience in creating and using symbolic and graphical representations of patterns, especially those tied to linear and quadratic growth. In this course these experiences will be extended to arithmetic, power, and geometric sequences with particular focus on quadratic and exponential relations.

C4/C29 To begin this unit, students should extend their work with patterns to include investigations of sequences of numbers that fall into two categories:

1) arithmetic sequences (a sequence where consecutive terms present a common difference)
2) power sequences, (a sequence made up of consecutive terms found by raising consecutive counting numbers to the same power) with a particular focus on quadratic relations

1st term	(t_1)	is	$2+3\cdot\_\_=\_\_=3n-1$
2nd term	(t_2)	is	$2+3\cdot\_\_=5$
3rd term	(t_3)	is	$2+3\cdot\_\_=8$
5th term	(t_5)	is	$\_\_\_\_=14$
100th term	(t_{100})	is	$2+3\cdot\_\_=\_\_$
nth term	(t_n)	is	$2+3\cdot\_\_=\_\_=3n-1$

Students should understand that an arithmetic sequence leads to a relationship that is linear and can be described symbolically. They might develop this in the following way: if given the sequence 2, 5, 8, 11, 14, find the 100th term. They can see that there is a common difference of three, and by completing an organized list (like that above) they might be able to predict the 100th term, then the nth term.

C4/C29/C3 Students should examine sequences through interplay between various representations, (verbal, symbolic, contextual, pictorial and concrete). For example, students might be given the diagram of towers (which represents the sequence above) and be asked to construct the towers with cubes and record the number of cubes in each tower as a sequence. They could graph the number of cubes in each tower against the tower number and examine this relationship. Students could talk about the constant growth rate, since the height increases by three with each additional tower. They should relate the constant growth rate to the slope of the line that would pass through the plotted points. Students might be asked to describe, in words and symbolically, the relationship between the tower number in the sequence and the corresponding height. From their description they might predict the 10th term in the sequence.

As a way of checking their equation, students could use graphing technology to obtain the equation. To do this, for example, they could enter the sequence of tower numbers in one list, and the sequence that represents the height of each tower into a second list, and use linear regression capabilities to obtain the equation $y = 3x - 1$.

... continued

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C4/C29/C3

Pencil and Paper

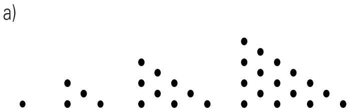
1) Complete each sequence and find the n^{th} term.
   a) 2, 4, 6, 8, 10, _, _, _, ... nth.
   b) 3, 6, 9, 12, _, _, _, ... nth.
   c) 17, 12, 7, 2, _, _, _, ... nth.

2) Explain why each of the above is called an arithmetic sequence.

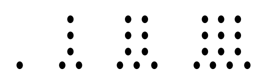
C4/C29/C3

Performance

a)



b)

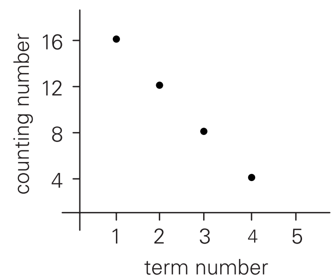


Number of dots on one side:	n	1	2	3	4	...	10	...	20
Number of dots in array: s									

Number of dots on one side:	n	1	2	3	4	...	10	...	20
Number of dots in array: t									

3) Explore the following dot patterns and determine if they form an arithmetic sequence or note: If the pattern is arithmetic, complete the table given to find the 10th and 20th term.

4) If this graph represents a sequence of numbers:
   a) Is the sequence arithmetic?
   b) What would be the value of the 8th term?
   c) Describe the sequence in words
   d) Describe the n^{th} term
   e) Write the equation for the graph.



5) Create a problem situation that can be represented with an arithmetic sequence. Have the problem solver display the sequence graphically, concretely, and symbolically for a purpose.

Suggested Resources

Quadratics

Outcomes

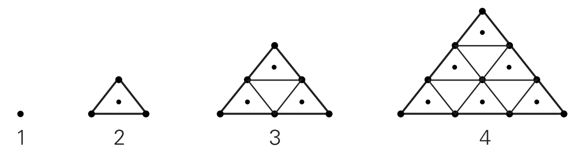
SCO: In this ourse, students will be expected to

C4 **demonstrate an understanding of patterns that are arithmetic, power, and geometric, and relate them to corresponding functions**

C29 **analyse tables and graphs to distinguish between linear, quadratic, and exponential relationships**

Elaboration—Instructional Strategies/Suggestions

C4/C29 As students examine sequences they will sometimes notice that the difference between the terms in the sequence does not increase/decrease at the same rate. For example, when students examine the dot pattern given below and create the sequence 1, 4, 9, 25, 36, ... to represent the number of dots in each figure, they should notice that each term in the sequence is the square of the number of dots on one side. They could use this to determine an algebraic expression ($t_n = n^2$) from which they could predict other terms in the sequence.



They should be able to describe the difference between the above pattern and the arithmetic patterns looked at previously. In the sequence above, there is no common difference between successive terms. The differences are 3, 5, and 7. However, if the consecutive differences are subtracted, students would see a common difference occur at the second level. When a common difference occurs at the second level, a second degree (quadratic) equation will result. The equation describes the quadratic relationship between the term and the term number. Sequences with a common difference that does not occur at the first level are called power sequences. The power sequence described above is quadratic.

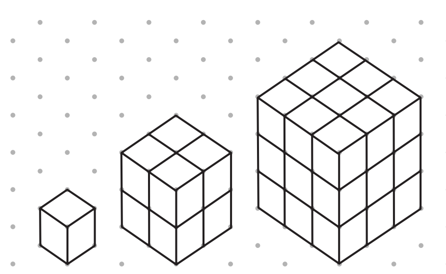
$t_n \rightarrow$ 1 4 9 16 25 36

$D_1 \rightarrow$ 3 5 7 9 11

$D_2 \rightarrow$ 2 2 2 2

Students might use cube-a-links to build towers and compare the growth rate visually between a quadratic sequence and that of an arithmetic sequence.

Ask students to extend the pattern below by drawing a sketch of the next figure, then have them build a table like the one below.



# cubes along one edge	Total # of cubes	D_1	D_2	D_3
1	1			
		> 7		
2	8		> 12	
		> 19		> 6
3	27		> 18	
		> 37		> 6
4	64		> 24	
		> 67		
5	125			

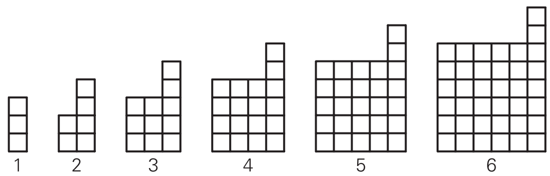
When a common difference occurs at the third level, a third degree (cubic) equation will result, another example of a power sequence.

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C4/C29

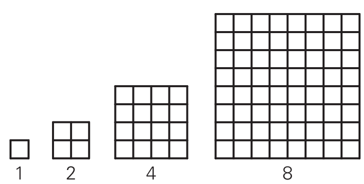
Performance



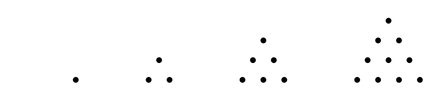
1) The area of a shape is the number of units inside. The figures below illustrate the areas of a shape that continues to grow.
   a) Copy and complete the following number sequence for the areas of the shapes above.

      3, 6, ___ , ___, ___, ___, ...

   b) What kind of sequence do the areas form? Describe the relationship between the number of square tiles along the bottom edge and the numbers in the sequence.
   c) Copy and complete the number sequence of the perimeters
      8, 12, ___, ___, ___, ...
   d) What kind of sequence do the perimeters form? Explain.
   e) Predict the area when there are 25 units along the bottom edge. What would be the perimeter for this same shape?
2) These figures illustrate a sequence of squares in which the length of the side is successively doubled.



   a) What are the perimeters of these four squares?
   b) What happens to the perimeter of a square if the length of its side is doubled?
   c) What are the areas of these four squares?
   d) What happens to the area of a square if the length of its side is doubled?
3) The first terms in the sequence of triangular numbers are illustrated by the figures below.



   a) Write the five numbers illustrated and continue the sequence to show the next five terms.
   b) Find the differences between the terms. Is there a common difference?
   c) Find the difference between the terms in the sequence from b).
   d) Are triangular numbers terms in an arithmetic sequence? Explain.
   e) Are they terms in a power sequence?

C4/C29

Journal

4) Ask students to explain how finding common differences in patterns helps determine what power equation to use.

Suggested Resources

Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C1 **model real-world phenomena using quadratic functions**

C8 **describe and translate between graphical, tabular, written, and symbolic representations of quadratic relationships**

C29 **analyse tables and graphs to distinguish between linear, quadratic, and exponential relationships**

Elaboration—Instructional Strategies/Suggestions

C1 Throughout this course, students will use mathematics to solve practical problems. Often, students will need to simplify the problem, determine irrelevant information, and express the problem with mathematical language or symbols. This is called making a mathematical model.

For any given problem, a variety of different mathematical models is possible. For example, students might draw a scale diagram, create a table, or draw a graph to show the relationship. Finally, they may describe the relationship algebraically. Each of these is a mathematical model of the problem.

In previous studies students have analysed and applied linear functions and have come to understand that a linear relation represents a constant growth rate. For example, in a snow board rental scheme, for every hour of rental, the cost increases by $3.50. When students graph this relationship, they can see that as the points increase from left to right on the graph, the cost increases by $3.50, and because this growth is constant, it represents a discrete linear relationship.

Students might remember from previous activities that, with a fixed perimeter, as the dimensions of a rectangle approach a square the area increases, maximizing with the area of the square. The campsite problem which follows, however, provides a variation on maximizing area. The campers are working with string for three side lengths of a rectangluar campsite, the fourth side being a river bank.

The campsite problem asks campers to stake out their campsite with 50 metres of string. They are to create a rectangular boundary, but since one side is along a river bank no string for that side is necessary. Students must find the length and width measurements to maximize the rectangular area of their campsite.

width	length	area	D_1	D_2
5		200	> 28	
6		228	> 24	> -4
7		252	> 20	> -4
8		272	> 16	> -4
9		288	> 12	> -4
10		300	> 8	> -4
11		308	> 4	> -4
12		312	> 0	> -4
13		312		

C8/C29 For the campsite problem, students might be asked to create a table of width versus area, beginning with a width of 5 m. As they record the different area calculations, they should notice that even though the width is increasing at a constant rate, the rate of increase is slowing down until it reaches 312 m^2. These decreasing differences are evident in the first set of differences (D_1), and this set of differences changes at a constant rate (evident in the second set of differences (D_2)). This constant in D_2 occurs when the relationship is quadratic.

Spreadsheets and/or table features on calculators might be used to generate the values in the table.

Formulas like "$50 - 2x$" for length and "$x\,(50 - 2x)$" for area would be necessary in this example.

Students should be encouraged to describe how the width/area pattern in this table differs from a table that shows a linear relationship.

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C29

Pencil and Paper

1	2	3	4	5	6	7
90	75	60	45	30	15	0

5	10	20	30	40	50
9	11	16	20	30	25

0	1	2	3	4	5	6	7	8	9	10
0	10	18	24	28	30	30	28	24	18	10

1) Compare the data in these tables. Which table(s) do you think represent(s) a quadratic relationship and explain your thinking.

C1/C8/C29

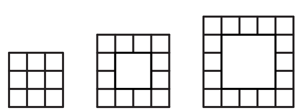
Performance

2) Murray and his friends were sitting in a booth at The Ol' Pizza Parlour that specializes in pizza pies à la 1960. Their menu advertises the following prices for plain cheese pizza:

Small:	(8" diameter)	\$0.85
	(10" diameter)	\$1.15
Medium:	(12" diameter)	\$1.55
	(13" diameter)	\$1.75
Large:	(15" diameter)	\$2.25
	(18" diameter)	\$3.15
	(24" diameter)	\$5.50

   a) Murray wonders if there is a mathematical relationship between the diameter and the price. Create a table and explain to Murray what the relationship is.
   b) Create a graph for the relationship.
   c) If The Ol' Pizza Parlour offered 20" pizzas, what do you predict would be the price.
   d) If the parlour wanted to introduce a gigantic size and sell it for \$7.50, what would be the diameter of the gigantic?
   e) Based on your model, what does the price-intercept represent?

3) Swimming Pool Problem: In the design of a particular swimming pool and surrounding patio (which we will call "square pools") the water surface of each pool is in the shape of a square. Around the pool there is a border of square-shaped patio tiles. The pools come in many sizes. Here are pictures of the three smallest pools available.



   a) Using the two-sided square tiles (such as unit tiles for algebra), build models of the above pools, using red for the water surface and white for the patio tiles.
      i) Organize data into the table below:
      ii) Plot graphs of
         a) pool # vs. # of red tiles
         b) pool # vs. # of white tiles

Pool #	# of red tiles	# of white tiles	total # of tiles

      iii) Describe the shapes of the two graphs. How are they the same? How are they different?
         a) Using cubes build towers to model how the growth in columns 2 and 3 changes.
         b) When will the number of red tiles and the number of white tiles be the same? Explain

Suggested Resources

Barnes, Mary, *Investigating Change: Functions and Modelling. Unit 1.* Melbourne: Key Curriculum Corporation, 1992.

Swan, Malcom. *The Language of Functions and Graphs*. Nottingham, UK: Shell Centre for Mathematical Education, 1985.

Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C1 **model real-world phenomena using quadratic functions**

C3 **sketch graphs from descriptions, tables, and collected data**

C8 **describe and translate between graphical, tabular, written, and symbolic representations of quadratic relationships**

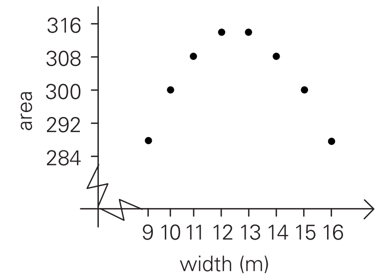
C29 **analyse tables and graphs to distinguish between linear, quadratic, and exponential relationships**

C31 **analyse and describe the characteristics of quadratic functions**

A7 **describe and interpret domains and ranges using set notation**

Elaboration—Instructional Strategies/Suggestions

C1/C3/C8/C29 Continuing with the campsite problem from the last two-page spread, students should produce a graph from the table of data they have calculated. As they plot the data points, they should notice that the points do not fall along a straight line. They should discuss how this shows that the area does not increase at a constant rate as it approaches the vertex. They will see as they continue to plot points that the graph takes on a nonlinear, parabolic shape. They will need to extend their previous table, to include larger values for the width that will result in the area decreasing. Students should discuss why the area begins to decrease.



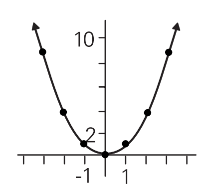
In analysing the graph, students should discuss whether or not it is correct to join the data points and, if they did, what these additional points would represent.

C31 Students should notice the symmetrical shape of the graph and understand that this symmetry is expected in a quadratic function. They should also note how the symmetry is visible in the table. They should explore, using the trace feature, the maximum value for the function with respect to the problem.

C31/A7 For the graph above, if a line were drawn through the maximum point (the vertex) parallel to the vertical axis, this line would be the axis of symmetry for the graph of the quadratic relationship. The equation is symbolized as x = "x-coordinate of vertex." The y-coordinate of the vertex is called the maximum value for the function and represents the maximum area for the campsite. The x-coordinate of the vertex represents the width that results in the maximum area. When asked for an appropriate domain for the graph in this situation, students might respond by saying "x-values from 1 to 20." They should symbolize this as $\{1 \le x \le 20, x \in R\}$. Students may use interval notation, $x \in [1,20]$, to represent the same thing. Students should discuss how the maximum (in this case) and minimum values determine restrictions on the range and how these are written symbolically. For example, in this problem the maximum value is a little above 312 m and no value below zero would make sense, so the range is $\{0 \prec y \le 315 | y \in R\}$ or $y \in (0,315)$.

Students should be reminded that the model quadratic function is $y = x^2$ whose domain is , and whose range is $\{y \ge 0 | y \in R\}$. These can both be written in interval rotation: domain: $x \in R(-\infty,\infty)$, range: $y \in [0,\infty)$. The graph of this quadratic function is symmetric about a line drawn through the vertex parallel to the vertical axis, and this symmetry can be seen in a table of values.

right/left	up
1	1
2	4
3	9



Students should be reminded that the following pattern occurs in the graph of $y = x^2$:

From the vertex, over 1, up 1; from the vertex, over 2, up 4; from the vertex, over 3, up 9. This pattern is useful for sketching graphs and their transformations.

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

C8/C29/C31

Journal

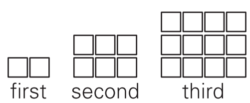
x	-2	-1	0	1	2	3	4
y	-1	-10	-15	-16	-13	-6	5

1) Explain how you can tell by examining this table of values that the equation that contains these points will be quadratic. Describe how the graph of this data will look.

C1/C3/C8/C29/C31/A7

Performance

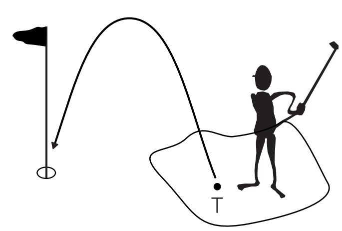
2) Begin with two cubes snapped together to form a rectangular prism (□□). Add cubes to model the following pattern.



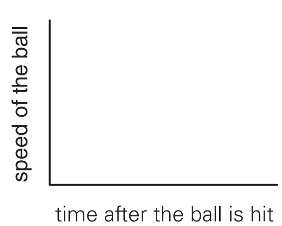
n	total cubes
1	2
2	6
3	
4	
5	
⋮	
10	
⋮	
25	

a) Count the total number of cubes used for each pattern. Complete the chart. Describe an appropriate domain and range for this relationship.
b) What kind of a relationship comes from the pattern?
c) Graph the relationship. Should you join the points? Explain.
d) Is there a maximum or minimum point? What is the axis of symmetry equation? Explain.

3) How does the speed of the ball change as it flies through the air in this golf shot?
a) Discuss this situation with your neighbour, and write down a clear description stating how you both think the speed of the golf ball changes.



b) Sketch a rough graph to illustrate the relationship between the height of the ball and time in seconds. Compare your graph with the given drawing. How is it the same? How is it different? State and interpret an appropriate domain and range for this relationship.



Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C1 **model real-world phenomena using quadratic functions**

F1 **analyse, determine, and apply scatter plots and determine the equation for curves of best fit, using appropriate technology**

$B4_2$ **use the calculator correctly and efficiently**

C3 **sketch graphs from descriptions, tables, and collected data**

C23 **solve problems involving quadratic equations**

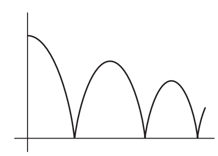
Elaboration—Instructional Strategies/Suggestions

C1 Students should, through a variety of experiences with functions, come to recognize the elements in a real-world problem that suggest a particular model. For example, area, accelerated motion, and trajectory suggest quadratic functions. There are two types of situations that students will encounter. One is the issue of finding the maximum or minimum value, that does not necessarily require the solving of an equation (discussed on this page). The other involves finding the intercepts, or zeros, as in the type of problem where students need to determine a value for a dimension that results in the doubling of an area. This does require the solving of an equation (discussion beginning on p. 57).

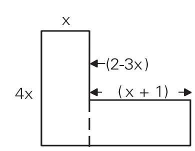
F1/$B4_2$/C3 In Year 10 students conducted experiments and explored data that was best fit by linear models. Sometimes the data was nonlinear and curved. Students explored it with power, exponential, and quadratic regression using technology.

In this course, students might conduct experiments and gather and plot data that will result in quadratic equations. For example, they might roll a pulley along a wire with a downward slope that has been marked in 10-cm intervals. Students would time the pulley from the starting position to each marked interval. They would use this data to determine an equation.

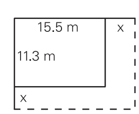
Teachers may also want to use a calculator-based laboratory (CBL) unit that connects to a graphing calculator or a computer-based laboratory to conduct experiments and collect data. Once the data is collected and displayed, it can be analysed to produce a mathematical model. For example, a motion detector can be used to gather data on a bouncing ball. The graph produced might resemble the one shown. Each of the bounces is a parabola. Students can use the technology to find the equation that best represents the data for each separate parabolic part of the graph and can compare the coefficients in each of the equations.



C23 There are geometric structures where the relationship between the dimensions and the area is quadratic. For example, students might explore irregular rectangular shapes, as in the diagram, and find the values for x and y that would maximize the area. They should determine that the area is $A = 4x^2 + (-9x^2 + 3x + 2)$.



When solving problems involving quadratic equations, students should observe that different strategies are required for solving the problem depending on the different situations described in the context. Students should solve problems that can be modeled with quadratic equations. For example, an addition of uniform width is being attached to both the front and right side of a one-storey building. If the building measures 15.5 m by 11.3 m, determine the width of the addition that results in doubling the floor area. Students should see that the strategy of solving an equation is required. In the campsite problem, seen earlier, a maximum value is required. This can be determined without solving an equation. Another problem might involve trajectory paths. Objects shot or thrown into the air often follow a path that can be described with a quadratic equation. For example, an arrow is shot into the sky along the trajectory $16t^2 + 60t + 1.5 = h$, where "*h*" is the height in metres and "*t*" is the time in seconds. Students could be asked to find the maximum height, where manipulating the equation into transformational form could be helpful, or they could be asked to determine how long the arrow was in the air, which might require them to solve an equation. Students might also be asked to evaluate (rather than solve) the trajectory equation for various heights given time, or for various times, given heights.



Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C1/F1/B4$_2$/C3/C23

Performance

1) Tell students to practise rolling a large marble or ball bearing on a propped-up clipboard from the lower left so that it rolls up the board and back down, leaving at the lower right. When they are ready, tell them to do the following]:
   a) Tape a sheet of carbon paper over a sheet of graph paper so that the marble will trace a path onto the graph paper.
   b) Roll the marble one more time, remove the carbon paper, and draw the X- and Y- axis on the graph paper in an appropriate place.
   c) Carefully locate several points on the curve and find the equation that best fits the data.
   d) Interpret the data.
   e) Determine the equation.
   f) Analyse the graph or use the equation to predict how far up the board the marble was after 0.05 seconds, 0.62 seconds.
   g) Explain how long the marble stayed on the board.
   h) Find two times when the marble was the same height up the board.

2) Chantal pulled the plug in her bathtub and watched closely as the water drained. As the water drained she made marks on her tub and used them later to determine the quantity of water remaining in the tub at various time intervals. The table contains the data she determined.

time in seconds	15	25	48	60	71	100	120	130	150	180	190
water (L) remaining	55	51.1	42.6	38.6	35	26.5	21.4	18.8	14.6	9.5	7.9

   Have students
   a) sketch a graph of the situation
   b) from the graph, predict how long it takes to empty the tub
   c) determine what function might model this situation and explain

C3/F1/B4$_2$

Journal

3) When students conducted the Ball Bounce activity in the classroom they observed (with a motion detector) a ball bouncing straight up and down beneath the detector. Yet the graph produced seemed to depict a ball bouncing sideways. Explain why this is so.
4) From the above activity, the graph produced appeared to be made up of many adjoined parabolic shapes. Focus on any one of those parabolic shapes and explain why it can be modeled by an equation in this form $y = a\,(x - h)^2 + k$. Explain how you would use this to determine the actual equation.

Suggested Resources

Brueningsen, Chris et al. *Real-World Math with the CBL System*. Dallas: Texas Instruments, 1994.

Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C1 **model real-world phenomena using quadratic functions**

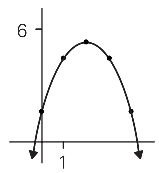
C8 **describe and translate between graphical, tabular, written, and symbolic representations of quadratic relationships**

C31 **analyse and describe the characteristics of quadratic functions**

C32 **demonstrate an understanding of how the parameter changes affect the graphs of quadratic functions**

Elaboration—Instructional Strategies/Suggestions

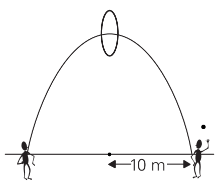
C1/C8/C31/C32 In previous studies students studied the transformations of a quadratic function and how those transformations are visible in the equation. Thus, students should be able to determine the equation of a quadratic using reverse logic. For example:



When students see the graph of a quadratic relation the maximum point tells them the vertex coordinates, which gives them the horizontal and vertical translation values. These values are placed in the equation of $y = x^2$, which then becomes $y - 5 = (x - 2)^2$.

Students must then decide if there is a vertical stretch or not, and if the parabolic shape is a reflection of $y = x^2$ in the x-axis. Because there is a maximum value, there is a reflection, and the equation becomes $-(y - 5) = (x - 2)^2$. It seems most natural to deal with the translation first, then the reflection, and then investigate whether there is a stretch or not. The pattern between the points on the graph should be from the vertex over 1 down 1 (which appears to be true); from the vertex over 2 down 4 (the square of 2). This is true (based on the y-intercept), so the quadratic relation is not stretched, and the final equation for this graph is $-(y - 5) = (x - 2)^2$.

Sometimes the stretch factor is hard to predict. Students can calculate the stretch factor if they know the vertex and one other ordered pair. For example, a baseball is thrown into the air to pass through a loop 20 m above the ground. If it passes through the loop at the top of its path and is caught 10 m from a point directly below the loop, determine the equation that describes its path. Since the students know about symmetry, the point directly below the loop must be halfway between where the ball was thrown and where it was caught, so the vertex point has a value (10, 20), assuming (0, 0) to be the point where the ball is thrown, and the equation $k(y - 20) = (x - 10)^2$.



Another point on the curve would be (20, 0) . Thus $k(0 - 20) = (20 - 10)^2$

$$-20k = 10^2$$

$$k = \frac{100}{-20} = -5.$$

Students can now solve for k ...

Students should conclude that since the value for k is –5, the stretch is $\frac{1}{5}$, in a negative direction. Students will write the equation of the path as $-5(y - 20) = (x - 10)^2$.

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C1/C8/C32

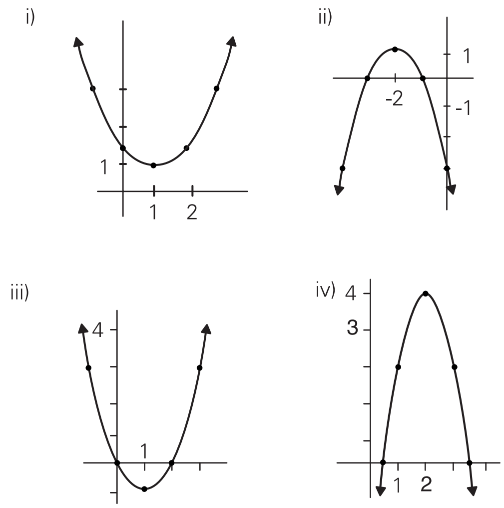
Pencil and Paper

1) An experiment was conducted and the following data was collected and put in a table.

Time (seconds)	2.0	2.5	3.0	3.5	4.0
Height (metres)	8.0	23.0	32.0	35.0	32.0

Your graphing calculator is broken. Use the table to determine the equation that best represents this data.

2) State the equations of the following graphs.



C8/C31/C32

Performance

3) How does the following equation allow you to visualize its graphical representation?

$$-\tfrac{1}{2}(y-1)=(x+5)^2$$

4) A rocket attains a height of 250 m when it is fired at a target that is 200 m distant on the horizon. There is a building, 85 m high, 20 m from the target and on direct line of fire. Will the rocket reach the target, and if so, by how much will it clear the building?

Suggested Resources

McKillip, David. *Pre-Calculus Mathematics One.* Toronto: Nelson, 1992.

Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C10 (111) **determine the equation of a quadratic function using finite differences**

C29 analyse tables and graphs to distinguish between linear, quadratic, and exponential relationships

Elaboration—Instructional Strategies/Suggestions

C10(111)/C29 Sometimes the coordinates of the vertex cannot be directly read from a given set of data. For example, from the given table students would know that the data is not linear (no common difference (D_1) between the y-values and the x- values increase at a constant rate). However, there is a common difference at the second level (D_2). This implies that this relationship can be modelled using a second degree equation (quadratic). Students can tell from the table that the D_1 values are decreasing in moving to the left. To approximate the vertex, students will need to examine where D_1 approaches zero.

x	1	2	3	4	5	6
y	5	19	43	77	121	175
D_1		14	24	34	44	54
D_2			10	10	10	10

Students should be able to find the equation of the quadratic relation using the finite difference method. For example, when students compare the table above with the table for the general quadratic equation $y = ax^2 + bx + c$ below, they can create equations with variables *a*, *b*, and *c*.

x	1	2	3
y	a + b + c	4a + 2b + c	9a + 3b + c
D_1	3a + b		5a + b
D_2		2a	

(Note: These general expressions for Y, D^1, and D^2 only apply for x-values of 1, 2 and 3. For other x-values, other general expressions will need to be determined.) Students should use the fact that the 2a is in the place of the 10 from the first table. Students would then solve $2a = 10$, to get a value for "*a*": a = 5.

Using the first level differences, (D_1).

The $3a + b$ aligns with the 14 →
$$\begin{aligned} 3a+b&=14 \\ 15+b&=14 \\ b&=-1 \end{aligned}$$

The $a + b + c$ aligns with the 5
$$\begin{aligned} a+b+c&=5 \\ 5-1+c&=5 \\ c&=1 \end{aligned}$$

Substituting the values $a = 5$, $b = -1$, and $c = 1$ into the model $y = ax^2 + bx + c$ gives the equation $y = 5x^2 - x + 1$. For various reasons, students might choose an alternative method to get the equation. They have observed that the constant in the second level of differences (D_2) is twice the value of the coefficient *a* in the related quadratic equation. (Again, note that this is only true for x-values of 1, 2 and 3.) Substituting the calculated value for *a* into the general form of the equation and using two of the known coordinate pairs, students can establish a "2 by 2" system of equations and solve for the *b* and *c* values. Values for *a*, *b*, and *c* can then be substituted into the general form to obtain the quadratic equation needed. This method is most often used because the difference table for the general quadratic (middle of page) is hard to remember and, more importantly, can be used only when independent variable values increase by one.

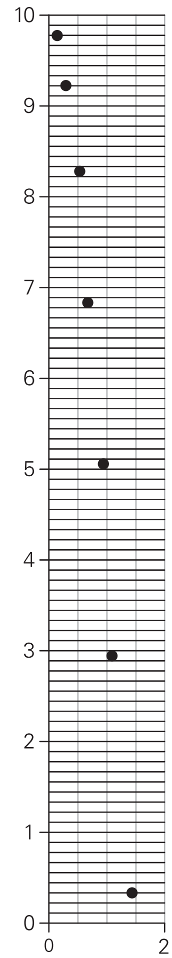
> Given in the above that $2a = 10$, then $a = 5$, so substituting:
>
> $5 = 5(1)^2 + b(1) + c$
>
> $19 = 5(2)^2 + b(2) + c$

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C10(111)/C29

Performance



1) The picture to the right simulates an object dropping from a height of 10 m. The height is recorded every 0.20 sec. Carefully determine, and record in a table, the heights of the ball at each interval, accurate to the nearest 0.01 m.
   a) Find the equation that describes the height of the object versus time.
   b) After how much time will the object hit the ground?
   c) How high will the object be 0.5 seconds after being dropped?

2) Find an equation that expresses the relationship between the number of sides of a polygon and the number of diagonals formed by connecting the non-consecutive vertices.
   a) Find the number of diagonals for a polygon with 20 sides.

3) A corner store charges $2 for a pack of AAA batteries. On the average, 200 packs are sold each day. A survey indicates that the sales will decrease by an average of 5 packs per day for each 10-cent increase in price.
   a) Use the information given to complete the row labelled "number sold."

Selling Price	2.00	2.10	2.20	2.30	2.40
Number sold	200				
Revenue					

D_1 D_2

   b) Calculate the revenue using the selling price and the number sold. Then calculate the first and second differences.
   c) Write an equation that describes the relationship between the revenue, y, and the selling price, x, charged per pack.
   d) Graph the equation and find the maximum revenue. What selling price provides maximum revenue?

Suggested Resources

Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C9 **translate between different forms of quadratic equations**

C31 **analyse and describe the characteristics of quadratic functions**

A7 **describe and interpret domains and ranges using set notation**

C32 **demonstrate an understanding of how the parameter changes affect the graphs of quadratic functions**

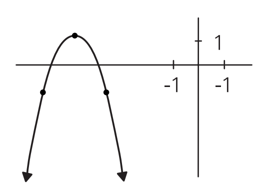
Elaboration—Instructional Strategies/Suggestions

C9 Quadratic equations are written in a variety of forms. These include the general form $y = ax^2 + bx + c, a \neq 0$; a transformational form $\left(\frac{1}{k}(y - v) = (x - h)^2, k \neq 0\right)$; and standard form $y = k(x - h)^2 + v$). In context, quadratic equations are usually written in general form. For example, students may be told that a football, when kicked, follows the path $h = -t^2 + 15t + 3$, or in another context, they may obtain an equation using quadratic regression that looks like $y = 20.1x^2 - 331.5\ x + 121.5$. However, obtaining equations from graphs or tables, students will often state the equation in transformational form (e.g., $-\frac{1}{2}(y - 1) = (x + 5)^2$) or in standard form ($y = -2(x + 5)^2 + 1$). Students should be able to move from one form to the other and appreciate why they need the equation in a particular form. For example, students may want the equation in transformational form to get the maximum value:

❒ Assuming that $f(x) = -2x^2 + 16x$ describes the path of a projectile, where "x" is the time in seconds and $f(x)$ is the height in metres, students are asked to find its maximum height.

To put an equation into transformational form or standard form requires an algebraic procedure called "completing the square." (See the elaboration on page 66.)

C31/A7/C32 When the equation is in transformational form or standard form, students can not only see the maximum or minimum value, but can give a complete analysis of the function since each of the transformations visible in the equation affects the shape and position of the parabola. The maximum/minimum value comes from the vertical translation value, which, in turn, comes from the vertex. The vertex helps inform students about an appropriate window when using graphing technology. The vertex also defines the range (since the axis of symmetry passes through the vertex). For example, $-\frac{1}{2}(y - 1) = (x + 5)^2$ has as its graph:



<u>Analysis:</u>

vertex: (–5, 1)

axis of symmetry: $x = -5$

domain: $\{x \in R\}$

range: $\{y \leq 1 | y \in R\}$

maximum value: 1

y-intercept: $-\frac{1}{2}(y-1) - 5^2$

$y - 1 = -50$

$y = -49$

zeros: $\frac{1}{2}(-1) = (x+5)^2$

$\frac{1}{2} = (x+5)^2$

$x = -5 \pm \frac{\sqrt{2}}{2}$ (the exact roots)

$\therefore x \doteq -5.7$, and -4.3 (the approximate roots)

Students will often want the equation in standard form, especially when using technology. They could easily express $-\frac{1}{2}(y - 1) = (x + 5)^2$ as $y = -2(x + 5)^2 + 1$.

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C9/C31/A7/C32
Performance

1) Remind students about the various properties (vertex, axis of symmetry, domain, and range and maximum value, the y-intercept, and number and values of zeros) of a quadratic function and that these properties are visible in an equation. Ask them to begin with $f(x) = -2x^2 + 12x + 3$ and determine the various properties.
2) Given the following properties, have students determine the equation and draw the graph of the function.

   Properties:

   vertex: (–0.8333, 14.08333)
   axis of symmetry: $x = -0.8333$
   domain: $\{x \in R\}$
   range: $\{y \in R | y \leq 14.08333\}$
   y-intercept: 12
   zeros: $\{-3, 1.333\}$

3) Create questions like 1) and 2) above and the corresponding answers. Interchange with your group members and have them answer the questions.
4) Given the following equations in transformational form:

   a) $3(y-2)=(x+1)^2$ b) $\frac{1}{2}(y+3)=(x-5)^2$

   c) $-5(y-8)=(x+3)^2$ d) $-\frac{2}{3}y=(x-1)$

   i) Describe the transformations of $y = x^2$ for each that are visible in the equation.
   ii) Sketch a graph on grid-paper that approximates the location and shape of each.
   iii) Change each equation into standard form.
   iv) Describe how the transformations of $y = x^2$ are visible in the standard form of the equation.
   v) Enter each equation into your graphing calculator and graph to check your approximations in part b).

C9/C31/A7/C32
Journal

5) Describe how the properties of a quadratic relationship can help you to determine the equation for that relationship. What are the minimum number of properties required to determine the equation. Which properties are they?

C32
Journal

6) Explain why the graph of $y = 3(x–2)^2$ looks like there could be a horizontal stretch even though it is not visible in $y = 3(x - 2)^2$. Show how you could write a quadratic equation in transformational or standard form to show the horizontal stretch.
7) A quadratic equation has a horizontal stretch of $\frac{1}{2}$, and no vertical stretch. Ben does not like horizontal stretches. Show how he could write the same equation to show the corresponding vertical stretch so that the graph would look the same as that drawn by a student using a horizontal stretch of $\frac{1}{2}$.

Suggested Resources

Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C9 **translate between different forms of quadratic equations**

C23 **solve problems involving quadratic equations**

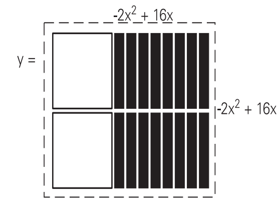
B1 **demonstrate an understanding of the relationships that exist between the arithmetic operations and the operations used when solving equations**

C32 **demonstrate an understanding of how the parameter changes affect the graphs of quadratic functions**

Elaboration—Instructional Strategies/Suggestions

C9/C23/B1 For graph sketching, or for finding maximum/minimum values in a problem, students should translate a quadratic equation into transformational form or standard form. In these forms the right-hand side of the equation is expressed as a factored perfect-square trinomial. Students should use concrete materials to develop an understanding of completing the square and to help make sense of the symbol manipulation.

They would begin with the given form:



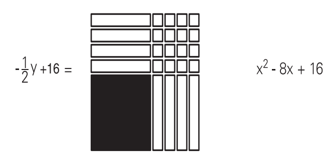
Using algebra tiles, students would construct a rectangle from the expression $-2x^2+16x$.

To create a square: first, students should use the strategy "make it simpler," so that the coefficient of $x = 1$, and remove the upper half of the rectangle (divide by two).

Secondly, they should change the white x^2 tile to a positive value and the x tiles to negative values (multiply all terms by –1).

Thirdly, to facilitate completing the square, students should move four –x tiles to a position above the square tile

and "complete the square" by filling in the space with the appropriate number of units. Students should notice the 16 is added to both sides.



It is important for students to understand that the "16" is obtained by taking half the number of x's to form the square, requiring a "4 by 4" set of units to complete the square.

To complete the symbol work, students should record the perfect square trinomial in factored form, $(x-4)^2$. Presently, the "16" is being added to $-\frac{1}{2}y$. To see the vertical

$$-\tfrac{1}{2}y+16=(x-4)^2$$

translation, students should factor $-\frac{1}{2}$ from the left side of the equation, resulting in

$$-\tfrac{1}{2}(y-32)=(x-4)^2$$

From this form they can now state that the maximum value is 32 metres, and it occurs four seconds after the projectile begins its flight (refer back to discussion of C9 on p. 60).

vertex $\rightarrow (4,32)$

B1/C32 Students might want to enter this equation into their calculators. This requires standard form, so they would multiply both sides by (–2) (inverse operation), then add 32 to both sides (inverse operation). Students should interpret the transformations in this form. Just the negative means that there is a reflection in the x-axis. The "$\frac{1}{2}$" means a vertical stretch of 2. The $(x-4)$ means an horizontal translation of +4, and the "+32", means a vertical translation of +32.

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

Suggested Resources

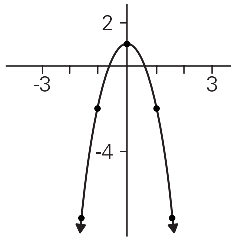
C9/B1/C32

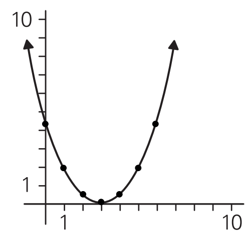
Pencil and Paper

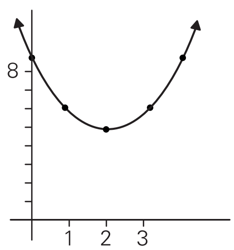
1) Match the following equations with their graphs and explain how you matched them.

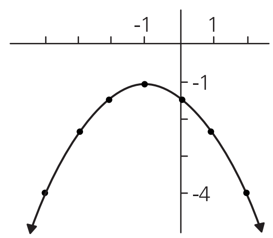
   a) $y = 0.5x^2 - 3x + 4.5$ b) $y = -3x^2 + 1$

   c) $y = \frac{1}{3}x^2 - \frac{2}{3}x - \frac{4}{3}$ d) $y = x^2 + 4x + 9$









C9/B1

Pencil and Paper

2) Explain the student's thinking from step 2 to step 3. Was the student correct? If so, explain why if not, explain what they should have done.

$$y = -3x^2 + 12x - 7 \text{ ... step 1}$$

$$y + 7 = -3\left(x^2 - 4x\right) \text{ ... step 2}$$

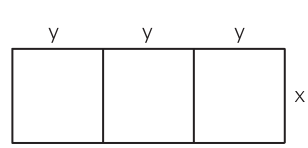
$$y + 7 = -3\left(x^2 = 4x + 4\right) - 4 \text{ ... step 3}$$

C9/C23

Performance

3) An environmental group wants to designate an area as a natural habitat. One side of the rectangular-shaped area is adjacent to a large lake and does not require fencing. The group is allowed to fence the remaining three sides using 48 km of fencing. What dimensions will ensure a maximum area for this natural habitat? What is the maximum area?

4) A rectangular playing field is fenced into three sections as shown. If the total amount of fencing used is 800 m, what are the values of x and y so that the area is a maximum?



Quadratics

Outcomes

SCO: In this ourse, students will be expected to

C22 **solve quadratic equations**

B10 **derive and apply the quadratic formula**

B1 **demonstrate an understanding of the relationships that exist between the arithmetic operations and the operations used when solving equations**

Elaboration—Instructional Strategies/Suggestions

C22 In Year 10, students related the factors of quadratic expressions with the x-intercepts of the graphs of their corresponding equations to understand that the solutions of a quadratic could be found by factoring. Students also learned to trace a graph using technology to solve simple quadratic equations that could not be factored.

C22/B10/B1 In this course, students will study a method that can be used to solve all quadratic equations. When factoring is difficult, students can use a formula to solve the quadratic equation. All students should see a derivation of the quadratic formula and understand why it can be used to solve any quadratic equation. Students in the Level 1 course should be able to produce and explain the derivation.

❒ Ask students to solve the equation $x^2 - 16 = 0$, then $x^2 - 15 = 0$, then $x^2 + 4x + 4 = 0$. For the first one, students will add 16 to both sides and solve $x^2 = 16$ to get $x = \pm 4$. For the second, they will use the same procedure and get $x = \pm\sqrt{15}$. They will need to factor the third one: $(x + 2)^2 = 0$, ultimately, $x = -2$.

Students should notice that a perfect square trinomial will lead to one factor squared, and by square root both sides will be able to solve for x. This is the "completing the square" procedure used previously to transform a quadratic equation into its transformational form (previous two-page spread). Ask students to solve a quadratic equation where the a value is greater than 1. By dividing by coefficient of x^2, they can simplify the equation and then use "completing the square." From their experience with tiles, students should realize how to obtain the constant term that will produce a perfect square trinomial. Then, by factoring, finding the square root, and isolating the variable, students will solve the equation.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ask students to solve the general quadratic equation. The result represents the solution to any quadratic equation and so can be used as a formula by students when needed. They may need some practice identifying the value for a, b, and c.

Students should solve problems that require the use of the quadratic formula. Many students will use the formula all the time, others will continue to use the factoring method when the expression is factorable. Still others will use technology and the features that are available to find the solutions.

Often one of the two solutions does not fit within the context of the problem. This solution is called inadmissable.

Quadratics

Worthwhile Tasks for Instruction and/or Assessment

C22/B10

Performance

1) A football is kicked into the air and follows the path $h = -4x^2 + 20x$, where x is the time in seconds and h is the height in metres.
   a) What is the maximum height of the football?
   b) How long does the ball stay in the air?
   c) How high is the ball after 6 seconds?
   d) How long does it take the ball to reach a height of 15 m?

2) An ice cream specialty shop currently sells 240 ice cream cones per day at a price of \$3.50 each. Based on results from a survey, for each \$0.25 decrease in price sales will increase 60 cones per day. If the shop pays \$2.00 for each ice cream cone, what price will maximize the revenue?
3) A rectangular office building surrounded with a uniformly wide parking strip is to be built on a rectangular lot of 4125 m^2. The building and the parking strip together, when completed, will measure 40.0 m by 60.0 m and take up 4/5 of the building lot. What will be the width of the parking strip?
4) Is it possible to bend a wire 15 cm in length to form the legs of a right triangle that has an area of 30 cm^2? If so, find the length of each side.

5) The following students were practising deriving the quadratic formula. Examine their attempts, locate all errors, and suggest why they may have made those error(s).

Nate

$$mx^2 + nx + d = 0$$

$$x^2 + \frac{n}{m}x + \frac{d}{m} = 0$$

$$x^2 + \frac{n}{m}x = -\frac{d}{m} + \left(\frac{n}{m}\right)^2$$

$$\left(x + \frac{n}{m}\right)^2 = \frac{-dm^2 + n^2m}{m^2}$$

$$x + \frac{n}{m} = \frac{1}{m}\sqrt{m\left(n^2 - dm\right)}$$

$$x = -\frac{n}{m} \pm \frac{\sqrt{m\left(n^2 - dm\right)}}{m}$$

Kate

$$px^2 + qx + 4 = 0$$

$$x^2 + \frac{p}{q}x = -\frac{r}{q}$$

$$x^2 + \frac{p}{q}x + \frac{1}{2}\left(\frac{p}{q}\right)^2 = \frac{-r}{q} + \frac{1}{2}\left(\frac{p^2}{q^2}\right)$$

$$\left(x + \frac{p}{2q}\right)^2 = \frac{-r\left(2q\right) + p^2}{2q^2}$$

$$x + \frac{p}{2q} = \frac{\pm\sqrt{p^2 - 2rq}}{\sqrt{2}q}$$

$$x = \frac{-p \pm \sqrt{p^2 - 2rq}}{2q}$$

C22/B10

Journal

6) Pretend that you are on the phone with a friend who missed the class about how the quadratic formula was developed, and what it is used for. Explain to your friend from where this formula comes, and how and when to apply it.

Suggested Resources

Quadratics

Outcomes

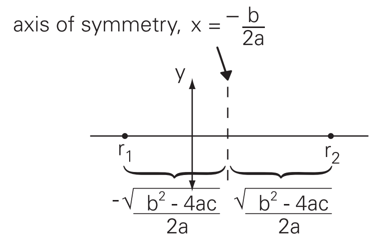
SCO: In this ourse, students will be expected to

B11 (111) **analyse the quadratic formula to connect its components to the graphs of quadratic functions**

C22 solve quadratic equations

Elaboration—Instructional Strategies/Suggestions

B11(111) Students should explore the quadratic formula to help them understand the ever-present symmetry in the graph of the quadratic equation. For example, students can express the formula in this form $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$ which should imply to them that the equation of the axis of symmetry passes through $-\frac{b}{2a}$ and that the roots are removed from the axis of symmetry by this value $\pm\frac{\sqrt{b^2-4ac}}{2a}$.

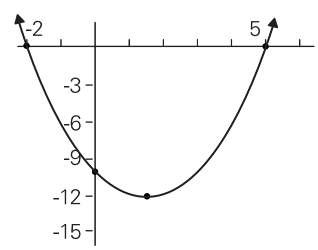


This, along with the transformational form

$\frac{1}{a}\left(y + \frac{b^2-4ac}{4a}\right) = \left(x + \frac{b}{2a}\right)^2$ might be useful to students who might want to express the vertex of any quadratic equation as

$\left(-\frac{b}{2a}, -\frac{b^2-4ac}{4a}\right)$.

C22 As students continue to solve quadratic equations using various methods, they should be encouraged to remember that the solutions to the equations ($ax + bx + c = 0$) are the x-intercepts on the related graph. From previous work, students will understand that the roots, factors, and zeros are connected. For example, if they solve the quadratic $x^2 - 3x - 10 = 0$ by factoring, they would get factors of $(x - 5)(x + 2) = 0$. Since the product is zero, then each factor may be zero giving roots of 5 and –2. Looking at the graph of $f(x) = x^2 - 3x - 10$, students should see that the x-intercepts are 5 and –2. To solve $x^2 - 3x - 10 = -2$ using the graph, they could translate the image of the x-axis down two units and read the new intercepts. They should realize they are solving $f(x) = -2$.



Students should understand that when solving equations by graphing, the solution to the equation, when equal to zero, can be read from x-intercepts values. This is because the given equation intersects the line $y = 0$ (the x-axis) at these points. When using the graph to solve the equation $x^2 - 3x - 10 = -12$, students could slide the x-axis down to $y = -12$, or draw the line $y = -12$, and read the horizontal intercepts as $x = 1$ and $x = 2$.

Quadratics

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Worthwhile Tasks for Instruction and/or Assessment

C22

Pencil and Paper

1) Explain which method would be best to use to solve each of the following quadratic equations. Then use the method, and one other, to determine the solution(s) for each. Which equation has roots whose difference is the greatest?
   a) $6x^2 - x - 2 = 0$ b) $x^2 - 4x - 5 = 3$
   c) $7x^2 = 1 - 2x$ d) $14x^2 = 5x$

Performance

2) Mary, David, and Ron are students in group 1. They have been solving quadratic equations together. Instead of co-operating, each has solved the equation on their own. Here are their solutions.
   a) Locate any errors, explain why they are errors, and decide which solution attempt is the best.
   The equation $A = x^2 + 3x - 110$ represents the area of a field. Students are asked to find the width (x) in metres if the area is 100 m^2.

Mary

$$100 = x^2 + 3x - 110$$
$$x^2 + 3x - 210 = 0$$
$$x = \frac{3\sqrt{9 - 4(1)(-210)}}{2}$$
$$= \frac{3 \pm \sqrt{-831}}{2}$$
$$\left\{3 \pm \frac{\sqrt{28.827}}{2}\right\}$$

David

$$x^2 + 3x - 110 = 100$$
$$x^2 + x - 10 = 0$$
$$(x-5)(x+3) = 0$$
$$x = 5, -3$$
$$\{5, 3\}$$

Ron

$$x^2 + 3x - 100 = 100$$
$$(x+55)(x-2) = 100$$
$$x = -55.2$$
$$\{55, 2\}$$

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* **B11(111)**

Journal

3) Pretend that you are on the phone with a friend who missed the class about how the quadratic formula helps explain the symmetry in the graph of a quadratic equation. Explain to your friend about this symmetry aspect.

4) Why does $-\frac{b}{2a}$ determine the x-reading of the vertex?

5) a) Sharon was telling Anthony that she could find the x-coordinate for the vertex of the quadratic function $y = 5x^2 + 13x - 21$ mentally. Sharon said it would be 13 divided by 5. Was she correct? Explain how you know.
   b) What would Sharon have to do next to get the y-coordinate for the vertex? *
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C22

Journal

6) Explain why it is important to make a quadratic equation equal zero before trying to solve it by factoring. Is this also true when trying to solve it by using the quadratic formula? Explain how you know you are correct.
7) Explain how you would use a graph to solve this quadratic equation:
   $3x^2 - 12x - 7 = -12.25$

Suggested Resources

Quadratics

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Outcomes

SCO: In this ourse, students will be expected to

A4 **demonstrate an understanding of the nature of the roots of quadratic equations**

C15 **relate the nature of the roots of quadratic equations and the x-intercepts of the graphs of the corresponding functions**

A9 **represent non-real roots of quadratic equations as complex numbers**

A3 **demonstrate an understanding of the role of irrational numbers in applications**

Elaboration—Instructional Strategies/Suggestions

A4/C15 When students solve the equation $4x^2 - 12x + 9 = 0$ using the quadratic formula, they will get only one value for x. This value "3/2" is called a double root (two equal real roots). When solving by factoring they would get $(x - 3/2)(x - 3/2) = 0$. So, $x = 3/2$ and $3/2$.

Students should note that when the discriminant ($b^2 - 4ac$) is zero a double root will result, and that it shows as a single x-intercept on the graph (the graph approaches the x-intercept and "bounces back," rather than "passes through" at the x-axis).

Students need to explore what happens when a parabola does not intersect the horizontal axis. Ask students to continue to explore using the equation $f(x) = x^2 - 3x - 10$. Have students solve the equation when $f(x) = -14$ both on the graph and with the quadratic formula. The quadratic formula produces a negative value for the discriminant, and the graph does not intersect the horizontal line $y = -14$. After exploring more examples, students should conclude that there are no real roots when the discriminant is negative.

A9 When the discriminant is negative, there are two imaginary roots, or two roots in the complex number system. The roots for $f(x) = -14$ would be $\frac{3 \pm i\sqrt{7}}{2}$. Using the quadratic formula, students would find the discriminant to be negative: $x = \frac{-3}{2} \pm \frac{\sqrt{-7}}{2}$. The $\sqrt{-7}$ is written $i\sqrt{7}$, since $i^2 = -1$.

A4/C15 In summary then, students can have a better sense about the roots of a quadratic by examining the value of the discriminant. When positive, there will be two real different roots, when zero, there will be a double root (two equal real roots), and when negative, there will be no real roots, but two complex roots (one being the conjugate of the other).

A3 Since irrational numbers arise when using the quadratic formula, discussion should centre around whether an exact or an approximate solution is appropriate. Students will recognize that irrational numbers can be written in exact form or by decimal approximations (± 14.4568). In applications based on measurements (area and trajectory problems), students will understand that approximations are, in fact, desirable. However, if exact answers are called for, students should be able to express them in simplified form. The simplified form may be more meaningful for comparison purposes. (If one side of a rectangle is $\sqrt{2}$, and the other side $\sqrt{18}$, then, since $\sqrt{18} = 3\sqrt{2}$, students can describe one side as being three times the other.)

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* **A4/C15** Students in the Level 1 course should benefit from discovering that if r_1 and r_2 are the two roots of a quadratic equation $ax^2 + bx + c = 0$, then $(r_1 + r_2) = \frac{-b}{a}$, and $(r_1 \bullet r_2) = \frac{c}{a}$. They may find this helpful when looking for an equation. For, example, if the roots of an equation are 3 and –4.

$(3 + (-4)) = (--1)$, so, $-b/a = -1$, so $b/a = 1$

and $((3)(-4)) = -12$, so, $c/a = -12$

So, the equation could be $x^2 + x - 12 = 0$, or $2x^2 + 2x - 24 = 0$, and so on.

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Quadratics

Worthwhile Tasks for Instruction and/or Assessment

A4/A9

Pencil and Paper

1) a) If one of the roots of a quadratic equation is $4i$, what is the other root?
   b) State a quadratic equation that has these two roots. Describe the appearance of its graph.
2) If two roots of a quadratic equation are 1/2 and 2/3, what might the equation be?
3) Solve for x:
   a) $\sqrt{2}\,x^2 - x - 3\sqrt{2} = 0$
   b) $\sqrt{2}\,x^2 = \sqrt{3}\,x + \sqrt{2}$

A4/A3

Pencil and Paper

4) Triangle PQR has vertices P (–4, 1), Q (2, –1), and R (1, k). Find all possible values of k such that $\triangle PQR$ is isosceles.
5) For what values of t does $x^2 + tx + t + 3 = 0$ have one real root?
6) Prove that if the quadratic equation $px^2 + (2p + 1)x + p = 0$ has equal roots, then $4p + 1 = 0$.
7) Prove that there is no real value of k for which the quadratic equation $x^2 - 10 = k(x - 2)$ has equal roots.
8) Prove that for all real values of m, the given quadratic equation has no equal roots $(m + 1)x^2 + 2m\,(2x + 1) = 1$
9) Given the general quadratic equation $ax^2 + bx + c = 0$, what relationship must be true among the coefficients a, b, and c for the equation to have
   a) two distinct real roots?
   b) two equal real roots?
   c) no real roots?

A4/A9

Performance

10) When Chantal was asked to describe the roots of the equation $14x^2 - 5x = -5$, she rearranged the equation so that it would equal zero, then used the quadratic formula to find the roots (see her work below left). Edward said that she didn't have to do all that, then showed the class his work. (See below right.) Are they both correct? Which method do you prefer? Explain.

A4/C15

Chantal	Edward
$x = \dfrac{5 \pm \sqrt{25+280}}{28}$	$b^2 - 4ac$
	$= 25 - 280$
$x = \dfrac{5 \pm \sqrt{305}}{28}$	$= -255$
so, there is one root	$\therefore$ no real roots

Journal

11) Create quadratic equations that have two distinct roots, two equal roots, and no real roots. Explain how you know.

A4/A9

Journal

12) Explain why you know that if a quadratic equation has one imaginary root, there has to be another.

Suggested Resources

UNIT 3:
RATE OF CHANGE

Unit 3
(10 Hours)
Rate of Change

Rate of Change

Outcomes

SCO: In this course, students will be expected to

C17 demonstrate an understanding of the concept of rate of change in a variety of situations

$B4_3$ calculate average rates of change

Elaboration—Instructional Strategies/Suggestions

C17 Students will explore “” in a variety of practical situations such as trajectory, travel, and economic and population growth. They should begin with simple models represented by linear functions whose average rate of change is constant and progress to variable rate of change, which implies some sort of curve. Students should be able to discern different rates of change visually (e.g., on a parabola, as the independent value increases and the dependent value approaches the maximum, the rate of change of the dependent value means zero as the independent approaches the x-coordinate of the vertex.

C17/$B4_3$ Students should explore some situations like heart rate, the rate at which runs are scored in a ball game, birth rate, population growth rate, and employment rate and learn how to calculate these rates. For example, heart rate

$$\frac{\text{Heart rate = Number of pulse beats ina time interval}}{\text{Time interval}}$$

can be calculated by counting the number of beats in a time interval, divided by the time interval.

Rates are important because they tell students how one thing is changing in relation to another. For example, the rate at which runs are scored in a baseball game can be thought of as

$$\frac{\text{Change in the number of runs scored}}{\text{Change in the number of innings played}}$$

Students might read about an oil spill in the newspaper and how pollution control authorities monitor the spread by estimating the area covered by the oil at different times, then calculating the rate of spread as

$$\frac{\text{Change in area}}{\text{Change in time}}$$

This would tell them how fast the area of the oil slick is changing.

Rate of Change

Worthwhile Tasks for Instruction and/or Assessment

C17/B4$_3$
Pencil and Paper

1) The high school girls basketball team is selling T-shirts to raise money for the season. They set the price at \$20 a shirt, but were selling an average of only 60 per week. By experimenting, they found on average, for each extra \$5 they charge, they sell 10 fewer T-shirts.
   a) If they increase the price from \$20 to \$25, how does the number sold change? What is the average rate of change of sales with respect to price?
   b) If they increase the price from \$25 to \$30, what is the average rate of change of sales with respect to price?
   c) Explain your findings in a) and b)

B4$_3$
Pencil and Paper

2) David took his pulse when he was resting and counted 20 beats in 15 seconds. Then he did 10 minutes of vigorous exercise and took his pulse again. This time he counted 52 beats in 15 seconds. How much did his heart rate increase as a result of the exercise?

C17/B4$_3$
Journal

3) If a linear function is defined by $y = mx + c$,
   a) what is its rate of change?
   b) what is the significance of a negative rate of change?
4) Describe how you think statisticians calculate the birth rate in a country.
5) In many sports, data are collected to help coaches and athletes improve performance. Very often they are reported as rates. Choose a sport you are interested in and see if you can find any rates that are used in analysing performance.

Suggested Resources

Barnes, Mary, "Investigating Change," Unit 1, Curriculum Corp., 1992.

Rate of Change

Outcomes

SCO: In this course, students will be expected to

C17 **demonstrate an understanding of the concept of rate if change in a variety of situations**

C16 **demonstrate an understanding that slope depicts**

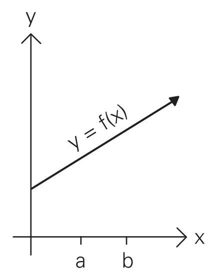
C30 **describe and apply rates of change by analysing graphs, equations, and descriptions of linear and quadratic functions**

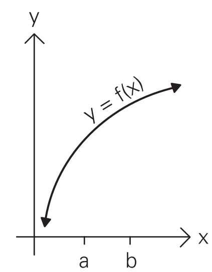
B4$_3$ **calculate average rates of change**

Elaboration—Instructional Strategies/Suggestions

C17/C16/C30 Students should be able to look at a graph and tell how fast the dependent is changing with respect to the independent value and determine if it is changing at the same rate all the time, or if the rate of change varies over time. If a function is given by $y = f(x)$, then students should talk about "the average of y with respect to x". This would mean the change in y per unit change in x.

Average rate of change of y with respect to $x = \dfrac{\text{change in } y}{\text{change in } x} = \dfrac{\Delta y}{\Delta x}$.





Continuing, if x changes from a to b then $f(x)$ changes from $f(a)$ to $f(b)$. In this case, the change in x is $(b - a)$ and the change in y is $f(b) - f(a)$. If f is linear, and students were to calculate the rate of change between a and b, they would get the same result no matter where a and b were located along the x-axis. They should conclude that the rates of change don't change along a straight line. If f is curved, then students should expect a variation in the rate of change for different intervals along the curve.

B4$_3$ The average rate of chagne of y with respect to x over the interval a to b, is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(b) - f(a)}{b - a}$$

For example, if b = 5 and a = 2, and $\frac{1}{2}$, then

$$\begin{aligned}
\frac{f(b) - f(a)}{b - a} &\Rightarrow \frac{\left(\frac{1}{2}(5) + 3\right) - \left(\frac{1}{2}(2) + 3\right)}{2 - 5} \\
&= \frac{4 - 5.5}{-3} \\
&= \frac{-1.5}{-2} \\
&= 0.5 \text{ or } \frac{1}{2}
\end{aligned}$$

Some students may choose the points in a different order and, thus, express the formula in a different format. The same answer will result.

$$\begin{aligned}
\frac{f(a) - f(b)}{a - b} &\Rightarrow \frac{\left(\frac{1}{2}(2) + 3\right) - \left(\frac{1}{2}(5) + 3\right)}{2 - 5} \\
&= \frac{4 - 5.5}{-3} \\
&= \frac{-1.5}{-2} \\
&= 0.5 \text{ or } \frac{1}{2}
\end{aligned}$$

Rate of Change

Worthwhile Tasks for Instruction and/or Assessment

C17/C16/C30/B4$_3$

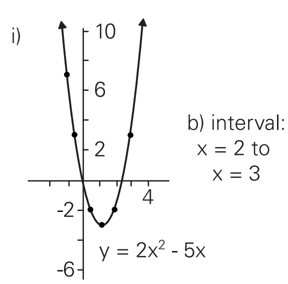
Pencil and Paper

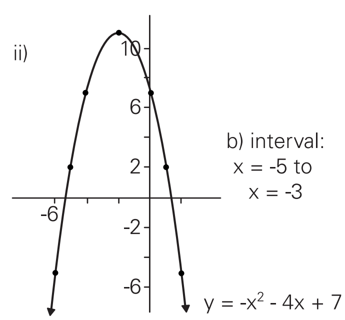
1) When tested for diabetes an individual was asked to consume a sweetened drink. The blood sugar level decreased slowly as time passed. We can say that blood sugar concentration C is a function of time t and write $C = f(t)$. Write an expression for the average of a person's blood sugar concentration over the period from one hour to three hours after they have stopped drinking.
2) You are travelling in an air balloon for the first time. As you increase your height above sea level, the air temperature changes. Suppose that the relationship between air temperature *TEC* and height above sea level h metres is given by $T = f(h)$.
   a) Use function notation to write an expression for the rate of change of temperature with respect to height for the interval from h = 200 to h = 500.
   b) In clear air, the air temperature decreases by about 1E*C* for each 100 metres above sea level. If the temperature at sea level is 25E*C*, find an algebraic formula for $f(h)$.
   c) What is the rate of change of T with respect to h experienced by a hot air balloonist as he rises from 200 m to 500 m?
   d) Would a glider pilot experience the same rate of change of temperature as she climbs from 1000 m to 2000 m?

C30/B4$_3$

Pencil and Paper

3) a) Describe in words the average rate of change depicted in the following graphs.
   b) Calculate the average rate of change in each.





Journal

4) a) Describe what Naomi is doing in the first step of the calculation for the average rate of change, and why.
   b) Has she made any errors in finding the average rate of change. If so, correct them.

for $y = 0.3x^2$

from $x = 1$ to $x = 2$

$$\frac{f(b) - f(a)}{b - a} = \frac{.3(2)^2 - .3(1)^2 - 5(1)}{2 - 1} \quad \text{step 1}$$

$$= -8.8 - (-4.7)$$

$$= 4.1$$

so the rate of change is 4.1

Suggested Resources

Rate of Change

Outcomes

SCO: In this course, students will be expected to

$B4_3$ calculate average rates of change

C17 demonstrate an understanding of the concept of rate if change in a variety of situations

C16 demonstrate an understanding that slope depicts

Elaboration—Instructional Strategies/Suggestions

$B4_3$/C17/C16 Student should understand that finding the rate of change of a linear function is very easy, because it is the same over any interval.

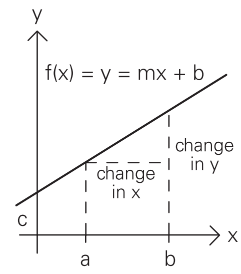


fig. 1

The graph of the function is a straight line, and the rate of change is just the slope of the line, $\frac{\text{change in } y}{\text{change in } x}$.

If the function is given by $y = mx + b$, its rate of change is m.

Students should be able to explain that a positive slope indicates a positive rate of change. Where a function's rate of change is positive, it is said to be increasing. For example, if you work longer hours you earn more money if you're being paid at an hourly rate. The linear function that has a positive rate of change is called an increasing function. Similarly, a negative slope would indicate a negative rate of change. For example, as the height above sealevel increases, air temperature decreases. The linear function that has a negative rate of change is called a decreasing function. If there is zero slope, there would be a zero rate of change. For example, as the number of hours of work increase, the amount of pay remains the same, since the pay is a salary instead of an hourly rate. For example, if the number of hours of work increases and the amount of pay remains the same, there would be a zero rate of change. The linear function with a zero rate of change is called a constant function.

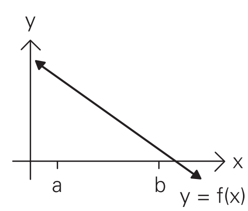


fig. 2

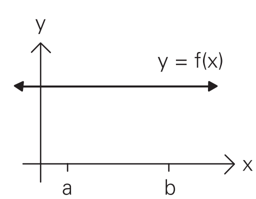
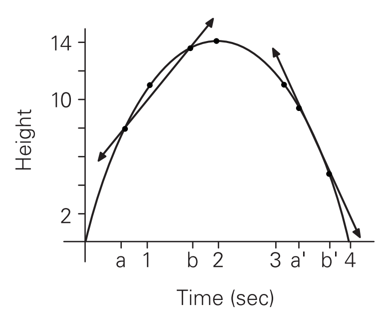


fig. 3

For nonlinear functions the rate if change is not the same everywhere. For example, motion is represented often as a curve. If the graph slopes upward to the right, it means the rate if change is positive. For example, for each second that passes as a soccer ball is kicked into the air, its height continues to increase until the ball reaches its maximum. At the maximum, the rate of change is zero. As the ball begins to fall to the ground, the rate of change is negative. Where a function's rate if change is negative, it is said to be decreasing.



Rate of Change

Worthwhile Tasks for Instruction and/or Assessment

$B4_3$/C17/C16

Performance

1) Phuong has a casual job in which she earns $10.60 per hour. If she works more than 8 hours in a day, she is paid "time and a half," that is, $15.90 per hour, for every hour over 8 hours. And if she is required to work more than 11 hours in a day, she is paid "double time," that is, $21.20 for every additional hour.
   a) Draw a graph to show Phuong's earnings as a function of the number of hours she has worked.
   b) Explain how the graph helps you to tell when Phuong is earning money at the fastest rate.
   c) Explain why the graph appears to change steepness after 8 hours.

2) Kim had a job last Saturday. His employer paid him $50 to cut his lawn. Five hours later Kim was finished and walked home. On his way home the mower sprung a leak and left a trail of gas. The tank was leaking 0.1 litres per minute, and the cost of gas is 53.9 cents a litre. If it took him one hour to get home, draw a graph that shows the relationship between time and net income for Kim's mowing.
3) Harry thinks that when finding the average rate of change there is no difference no matter whether the function is linear or nonlinear. Would you agree with Harry, or not? Explain your thinking.

Suggested Resources

Rate of Change

Outcomes

SCO: In this course, students will be expected to

C17 demonstrate an understanding of the concept of rate if change in a variety of situations

C30 describe and apply rates of change by analysing graphs, equations, and descriptions of linear and quadratic functions

$B4_3$ calculate average rates of change

Elaboration—Instructional Strategies/Suggestions

C17/C30 Students should conduct experiments and gather data that when organized and graphed will result in situations where the rate of growth or decay changes between each set of points. The graphs of these situations will result in curved relationships.

For example, in the rectangular campsite problem (in the quadratic unit, p. 46), the campers had 50 metres of string to place along three edges of a rectangular camp lot to maximize the area. From their table of collected values, students observed that, even though they used widths that increased at a constant rate, the area of the campsite increased, but at a varying rate. The graph that resulted had a parabolic shape, concave downward. Students should also be asked to observe when the rate if change was zero. A connection should be made between zero and the maximum or minimum value for a function.

$B4_3$ Some students were asked to calculate the average rate if change for the area of the campsite when the width changed from 5 m to 6 m. Their work would look like this:

$$\begin{aligned}\frac{f(b) - f(a)}{b - a} &= \frac{f(6)-f(5)}{6-5}\\ &= \frac{228-200}{1}\\ &= 28 \text{ m}^2 \text{ of area per 1 m change in width}\end{aligned}$$

Others calculated the rate if change for the area in the interval from 9 m to 10 m, and still others in the interval 11m to 12 m.

$$\text{from } 9 \rightarrow 10 \Rightarrow \frac{f(10)-f(a)}{10-9} \Rightarrow \frac{300-288}{1} = 12 \text{ m}^2 \text{ per 1 m change in width.}$$

$$\text{from } 11 \rightarrow 12 \Rightarrow \frac{f(12)-f(11)}{12-11} \Rightarrow \frac{312-308}{1} = 4 \text{ m}^2 \text{ per 1 m change in width.}$$

C30 Clearly, students can see that the rate if change is different on different intervals, which is typical of nonlinear situations.

Rate of Change

Worthwhile Tasks for Instruction and/or Assessment

C17/C30/B4$_3$

Performance

1) Ask students to go back and revisit a problem (see reference below) worked on in the quadratic unit. Construct a graph to represent the situation and in each discuss how the average rate of change (of the phenomena studied) changes: e.g., increasing, decreasing, speed, etc. at various intervals along the curve.
   a) punting a football-hangtime problem (p. 61)
   b) the campsite problem (p. 46)
   c) Chantal's bathtub problem (p. 51)
   d) the golf shot problem (p. 49)

2) Again, referring to the above question, calculate the average rate of change for each using the following information:
   a) Find the average rate of change of the height of the ball after it has been in the air between two and three seconds.
   b) Explain how you would use the average rate of change in the area to determine the dimensions that result in the campsite having a maximum area.

   For parts c) and d), create a problem you would ask your partner to see if he/she understands average rate of change.

Suggested Resources

Rate of Change

Outcomes

SCO: In this course, students will be expected to

$C28_3$ **solve problems involving instantaneous rate of change**

C30 **describe and apply rates of change by analysing graphs, equations and descriptions of linear and quadratic functions**

C18 **demonstrate an understanding that the slope of a line tangent to a curve at a point is the instantaneous rate if change of the curve at the point of tangency**

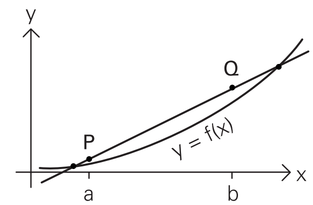
C27 **approximate and interpret slopes of tangents to curves at various points on the curves, with and without technology**

$B4_2$ **use the calculator correctly and efficiently**

C17 **demonstrate an understanding of the concept of rate if change in a variety of situations**

Elaboration—Instructional Strategies/Suggestions

$C28_3$/C30 People often talk about the rate at which the population or the cost of living is increasing or decreasing. These rates can be found by determining the change in population over a time interval and dividing by the change in time. For example if the population of the nation increased by 2.4 million over a six-year period, students could say that the average rate of increase over six years is 0.4 million per year. Sometimes it makes sense to talk about rates of change over very short periods; for example, credit card interest rates, heart beats per five seconds, typing words per minute, car speed in kilometres per hour, speed of sound in metres per second. By analogy, it should make sense that invested money can earn interest as fast as it can be calculated (so much per second). Students should agree that it does make sense to talk about someone's speed at any instant, even if that speed is changing. For example, when a student rides a bicycle or drives a car, the speedometer shows the speed at every moment.



$C28_3$/C18/C27/$B4_3$/C17 Students can find the average rate of change of a function from its graph. The slope of the straight line joining the two points that represent the interval is the average rate of change over that interval. What if those two points come from an interval across which the graph is curved? For example, the average rate of change of *f* from $x = a$ to $x = b$ is equal to the slope of the secant line *PQ*. This is not the same thing as the instantaneous rate of change at *P* or at *Q*, but, instead, is the average rate of change across the interval from *P* to *Q*.

To approximate the instantaneous rate of change, have students choose a point such as *Q* on the graph and zoom in on that part of the graph using graphing technology, continuing to do so several times. Eventually that part of the curve will appear to look straight even though students will know it is not. This suggests a way for students to measure the slope of a curve at a point. First, they would magnify (zoom in) the graph around the point *Q*, until it seems like a straight line. Then they would choose another point P on the curve and find the slope of *PQ* (*PQ* looks straight and is actually a secant to the curve) in the usual way.

Students should then imagine what will happen if the graph is "de-zoomed" to its original size. The points *P* and *Q* will be very close together, in fact, difficult to distinguish. Thus, the line *PQ* appears to be tangent to the curve at *Q*, e.g., it appears to be touching at only one point. This slope would approximate the instantaneous rate of change at the point *Q* on the curve. Students can increase the accuracy by taking the two points close, and closer to one another. For example, using $f(x) = x^2$ and letting *Q* be located at $x = 2$ (*b*), and selecting *P* at $x = 1.75$ (*a*), then

$\frac{f(b)-f(a)}{b-a} \Rightarrow \frac{f(2)-f(1.75)}{2-1.75} \Rightarrow \frac{4-3.06}{0.25} \Rightarrow 3.7$. As *P* moves closer to *Q*, the 0.25 becomes smaller. Select *P* (1.99) to be very close to *Q*. Then

$$\Rightarrow \frac{f(2)-f(1.99)}{0.01} \Rightarrow \frac{4-3.9601}{0.01} \Rightarrow \frac{0.0399}{0.01} \Rightarrow 3.99.$$

From this, students should understand that as they make the interval between the two points smaller and smaller, they will be able to estimate the slope of the tangent line at *Q* (2, 0) to be 4. In fact, the graphing calculator algorithm uses a similar process where the slope at a point $(a, f(a))$ is $\frac{f(a\_0.0001)-f(a-0.0001)}{(a+0.0001)-(a-0.0001)}$.

Rate of Change

Worthwhile Tasks for Instruction and/or Assessment

$C28_3$/C30/C18/C27/$B4_2$/C17

Performance

1) A projectile was propelled upward into the air from a height of 15 m at a velocity of 25 m/sec. What was its speed at the instant it hit the ground?
   a) Recall that acceleration due to gravity is approximately –9.8m/sec^2; in free fall the height is $-1/2\ gt^2$. Use this fact and the model $h = -1/2\ gt^2 + v_it + H$ where v_i is initial velocity and H is initial height, to write the function for this situation.
   b) Graph the function using a graphing calculator.
      i) When did the projectile reach its maximum height?
      ii) What was the maximum height?
      iii) How many seconds was the projectile in the air?

   c) Recall that average velocity is change in height divided by change in time.
      i) During what time periods did the projectile have positive velocity?
      ii) When did it have negative velocity?
      iii) Was the velocity ever zero? Explain.
      iv) Find the average velocity from the moment of projection to the moment the projectile reaches maximum height.
      v) Approximate the velocity at time 2 seconds; at time 4 seconds.

      (Part vi) is only to be done if this is an activity that develops a formula. Omit if an assessment item.)
      vi) Determine a general formula for the average velocity between $t = a$ and $t = b$, assuming $a + b$. Use function notation for the height.
   d) Find the following average velocities, and from those predict the instantaneous velocity of the projectile at 5.5 seconds:
      i) between $t = 3$ and $t = 5.5$ seconds
      ii) between $t = 4$ and $t = 5.5$ seconds
      iii) between $t = 5$ and $t = 5.5$ seconds
      iv) between $t = 5.25$ and $t = 5.5$ seconds

   e) Solve the original problem, "What was its velocity at the instant it hit the ground?"

Suggested Resources

Meiring, Steven P. et al. *A Core Curriculum: MAking Mathematics Count for Everyone* Addenda Series Grades 9–12. Reston, VA: NCTM, 1992.

Rate of Change

Outcomes

SCO: In this course, students will be expected to

C18 demonstrate an understanding that the slope of a line tangent to a curve (at a point) is the instantaneous rate of change of the curve at the point of tangency

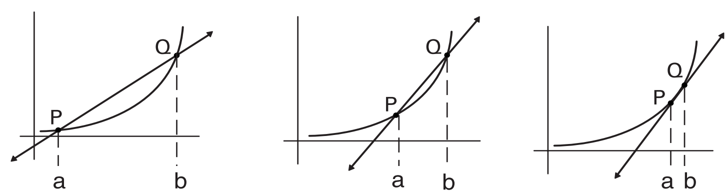
C17 demonstrate an understanding of the concept of rate if change in a variety of situations

C27 approximate and interpret the slopes of tangents to curves at various points on the curves, with and without technology

$B4_2$ use the calculator correctly and efficiently

Elaboration—Instructional Strategies/Suggestions

C18/C17 The previous pages describe how students should come to understand that an instantaneous rate of change can be approximated by examining average rates of change (slopes of secant lines) over smaller and smaller intervals (two points on the curve that determine the secant moving closer and closer together). That is, the instantaneous rate of chagne at *Q* is approximated by the slope of *PQ*. As *P* moves close and closer to *Q*, they should recognize that a calculation of $\frac{f(b)-f(a)}{b-a}$ will generate values for the average rate of change, and that by choosing values for "*a*"progressively closer to "*b*"the resulting values get closer and closer to the instantaneous rate of change. The instantaneous rate if change then would be the slope of the tangent line at that point on the curve.



C27/$B4_2$ Technology allows students to confirm their estimates for instantaneous rates of change. It should be noted that since students have not studied limits in this course, they do not have a method for calculating the exact value of the instantaneous rate of change. This will be developed in a later course.

The table feature on the graphing calculator can be used to facilitate hand calculations for the approximate value of the instantaneous rate of change at points on a curve. For example, to find the slope of the tangent line at $x = 3$ on $f(x) = x^2$, students would enter $y = x^2$ at Y_1 = on the function screen. They would enter the expression $\frac{(y_1(3)-y_1(x))}{(3-x)}$ at Y_2 = and explore the table for y values as x approaches 3, 3.1, 3.01, 3.001.

From this they could approximate the slope of the tangent line at $x = 3$ to be 6. They would interpret this to mean that the function $y = x^2$ has an instantaneous rate of change of 6 at $x = 3$.

Once students demonstrate understanding for the instantaneous rate of change at any point on a curve being the slope of the tangent line at that point on the curve, they might want to use the graphing calculator to add tangent lines to the functions being studied at specific points, and to approximate the instantaneous rate of change at those points. For example, by selecting *5: tangent* (in the Draw menu of the TI-83 or TI-83+, typing in the x-value for the point of tangency, and pressing Enter, students will see the tangent line drawn to the curve. The technology records, at the bottom of the screen, the x-value of the point of tangency and the equation for the tangent line (in $y = mx + b$ form). Knowing that the slope of the tangent line is the instantaneous rate if change at that point, students can read its approximate value from the given equation.

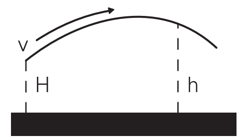
Rate of Change

Worthwhile Tasks for Instruction and/or Assessment

C18/C17/C27/B4$_2$

Performance

1) In order to determine a projectile's displacement h from the ground, in metres, at any particular time, t, in seconds, the formula $h = H + v_i\, t - 4.9t^2$ is used, where v_i is the initial velocity, in metres per second, and H is the initial height, in metres, from which the projectile is launched.



   a) Babe Ruth, the all-time Yankee slugger, was renowned for hitting high fly balls. One day he made contact with a ball 1 m above the ground and hit the ball at a speed of 29.4 m/s. State the equation. Determine the average rate if change of the height of the ball during the interval between 2 and 2.5 seconds. Determine the maximum height of the fly ball.
   b) How long did the ball remain in the air?
   c) What was its speed the instant it hit the ground?
   d) Draw a tangent to the curve at $x = 2$ and explain how it can be used to estimate the speed of the ball 2 seconds after it was hit.

2) An arrow is released with a initial speed of 39.2 m/s. It travels according to the path $h\ (t) = 39.2t + 1.3 - 4.9t^2$, where h is the height reached, in metres, and t is the time taken, in seconds.
   a) Determine where the arrow's speed is the least. Explain.
   b) Determine the speed of the arrow 1 second before impact.

3) The table shows the precipitation data for a tropical region. As the amount of precipitation goes up to a maximum and drops again later in the year, the situation may be modeled with a quadratic function.
   a) Assuming that the model is reasonable, find the quadratic function for the amount of precipitation in terms of the number for the month of the year.
   b) Using the equation and what you know about slopes of tangents, produce a series of tangent lines on a graph from which the graph of the equation can be determined.

Month	Precipitation
Jan	5 mm
Feb	60 mm
Apr	140 mm
Jul	185 mm
Aug	180 mm
Nov	105 mm

Suggested Resources

Unit 4
Exponential Growth
(30 Hours)

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C2 model real-world phenomena using exponential functions

B2 demonstrate an intuitive understanding of the recursive nature of exponential growth

Elaboration—Instructional Strategies/Suggestions

C2 Exponential functions (like $y = ab^x$) form another important family of functions. Exponential functions are extremely useful in applications that include population growth, compound interest, radioactive decay, and value depreciation. Consider a very common classroom growth example: The teacher has some important data saved on his graphing calculator. He wants to share it with all 32 students. He asks the class to consider which method would be the best to share the data and how much time would be saved (it takes one minute to link and transmit the data).

Method 1: Teacher links with each student and transmits the data one person at a time.
Method 2: Teacher links with one student, then each of them links with a student, and so on.

Solution:

Students may try to find a common difference with the table in Method 2. The fact that there is no common difference when sequences of differences between successive terms is examined at D_1 and D_2 level signifies that the relationship is not linear or quadratic.

Method 1

Time period	# of people with data
0	1
1	2
2	3
3	4
.	.
31	32
32	33
35	36

Method 2

Time period	# of people with data
0	1
1	2
2	4
3	8
4	16
5	32
6	64

B2 Upon further study of the pattern in Method 2 (as seen below) students might determine a relationship that they could describe in words as "the number of people (t) with the data during any time period (t_n) can be determined by doubling the number in the previous time period."

Time Period	# of people with data
0	$1 = 1 \rightarrow 2^0$
1	$2 = 2 \rightarrow 2^1$
2	$2{\bullet}2 = 4 \rightarrow 2^2$
3	$2{\bullet}2{\bullet}2 = 8 \rightarrow 2^3$
4	$2{\bullet}2{\bullet}2{\bullet}2 = 16 \rightarrow 2^4$
5	$= 32 \rightarrow 2^5$
6	$= 64 \rightarrow 2^6$
.	$= \bullet \rightarrow$
.	$= \bullet \rightarrow$
n	$= \bullet \rightarrow 2^n$

Using symbols: $t_2 = t_1 \cdot 2$, $t_3 = t_2 \cdot 2$ and so on, leading to $t_n = t_{n-1} \bullet 2$. Some students will continue looking at the pattern to see that it can be symbolized by 2^n. Students should state that each term in the sequence is a multiple of each previous term, and that this multiple is the common ratio between consecutive terms (e.g., $t_n \div t_{n-1}$).

Another example might be to consider the following pay schedule: Students might describe this pattern as "You get paid one dollar on day one, then each day after you get three times as much as the previous day."Thus the amount of money earned (M) on any given day can be determined by tripling the amount earned on the previous day. Using symbols:

$M_2 = M_1 \cdot 3$

$M_5 = M_4 \cdot 3$

and so on, leading to $M_n = M_{n-1} \cdot 3$

continued...

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C2/B2

Pencil and Paper

1) This is an old story that is found in many similar versions. According to one version, King Shirham of India wanted to reward his Grand Vizier, Sissa Ben Dahir, for inventing and introducing him to the game of chess. The Grand Vizier requested as his reward that the king have one grain of rice placed on the first square of the board, two on the second, four on the third, eight on the fourth, and so on, until the board was filled according to this pattern. The king tried to talk him out of this silliness because he was willing to give him jewels or money, but to everyone's amazement, the Grand Visier stood his ground. Did he do the right thing?

Square	Rice on the Square	Rice on the Board	Pattern?
1	1	1	
2	2	1 + 2 = 3	
3	4	3 + 4 = 7	
4			
5			
6			
.			
10			
.			
25			
.			
64			

Complete the chart. Look for a pattern and describe it. Determine the amount of rice on the 64th square. Determine the amount of rice on the whole board.

2) Imagine you have won the Publishers' Clearance House Sweepstakes. You can chose between a lump sum of \$2 000 000 or \$10 000/mo for the rest of your life. You decide that in either case, you will deposit half the money in a savings account that earns interest every month at a rate of 0.5% per month. Investigate to determine which is the better deal? Explain.

B2

Journal

3) Given the two sequences below. Explain why one of them will produce terms that depict exponential growth. Explain why the other will not.

   i) $t_n = 2t_{n-1} + t_{n-2}, t_1 = 5$
   and $t_2 = 9$

   ii) $t_n = 0.5t_{n-1}$, t_1=0.25

4) List the first five terms of this sequence:

   $t_1 = 4,\ t_2 = 3,\ t_n = 3t + 2t_{n-2},\ n > 2 \in N$

   Describe the sequence. Do you think you could predict the 10th term? Explain.

5) Steve said that $f(x) = 3^{x+2} - 2(x + 1)$ defines an exponential function. When the domain is $x > 0 | x \in R$. Is he correct?

... continued

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C2 **model real-world phenomena using exponential functions**

B2 **demonstrate an intuitive understanding of the recursive nature of exponential growth**

C25 **solve problems involving exponential and logarithmic equations**

C11 **describe and translate between graphical, tabular, written, and symbolic representations of exponential and logarithmic relationships**

C4 **demonstrate an understanding of patterns that are arithmetic, power, and geometric and relate them to corresponding functions**

Elaboration—Instructional Strategies/Suggestions

... continued

B2 Students should be able to see that each term in the sequence is triple the previous term, and could be symbolized using 3^{n-1}.

Students might consider how to symbolize getting paid $2 on the first day, then tripling each day thereafter. As they develop the table to the left, each term is multiplied by 3 to get the next term leading to the amount paid on the nth day.

day	Money ($)
1	2
2	$2\cdot 3$
3	$2\cdot 3\cdot 3$
4	$2\cdot 3\cdot 3\cdot 3$
.	.
n	$2\cdot 3^{n-1}$

It is often easier for students to work with exponential situations by using the relationship between successive function values; in fact, their first recourse in evaluating exponential functions is often through calculating the next value from the previous value.

Sequences where determining the *n*th term requires knowledge of all preceding terms are called recursive rules.

The replay capabilities on graphing calculators permit recursive patterns to be easily evaluated. It does not take long, however, for students to realize that even with instant-replay capability, recursive patterns do not efficiently determine output for very large input values. For example, students would need many term values to find the money earned for day 50 of the money tripling function if they know only the recursive definition. Once students understand this difficulty, they will be ready to appreciate that function rules for recursive patterns can be stated in an alternative and arguably more usable way. However, the sequence editor and table feature on the graphing calculator can make this much more efficient.

This outcome (B2) will be formalized in a later course.

C2/C25 During their study of exponential growth, students should examine some issues that have become important in today's society, like world population growth, ozone layer depletion, pollution control, etc. The tendency of populations to grow at an exponential rate was pointed out in 1798 by an English economist Thomas Malthus, in his book *An Essay on the Principle of Population*. Malthus suggested that unchecked exponential growth would outstrip the supply of food and other resources and lead to wars and disease. He wrote "Population, when unchecked, increases in a geometrical ratio. Subsistence increases only in an arithmetic ratio."

C11/C4 Students should discuss this Thomas Malthus quote and the consequences. During their discussion they will gain a better understanding of geometric and arithmetic ratio.

Students should re-establish that functions whose first term is C and whose other terms are generated by multiplying the preceding term by a constant, *a*, can be expressed in the form $f(n) = Ca^{n-1}$, where $n, n \geq 1$ is the number of the term and the variable *a* in this situation is the common ratio. By contrast, students should remember that terms in arithmetic sequence are generated by a linear function where each term results by adding a constant to the previous term.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

continued ...

C4

Pencil and Paper

1) Which of the following sequences are arithmetic, which are power, and which are exponential?
   a) 3, 5, 7, 9, 11, ...
   b) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}$
   c) –6, –4, –2, 0, 2, 4, 6
   d) –2.988, –1.963, –0.8889, 0.3333, 2, 5
   e) 8, 4, 0, -4, -8
   f) 2, 4, 8, 32
   g) 0.01235, 0.03704, 0.1111, 0.3333, 1, 3
   h) $\frac{1}{2}, \frac{1}{6}, \frac{1}{10}, \frac{1}{14}$
   i) $\sqrt{2}, 2, \sqrt{8}$
   j) 4, 5, 12, 25, 44

C2/C25/C4

Activity

2) Ask students to read the following excerpt from the book *The Origin of Species* by Charles Darwin, published in 1859. It discusses exponential growth. "A struggle for existence inevitably follows from the high rate at which all organic beings tend to increase. Every being, which during its natural lifetime produces eggs or seeds, must suffer destruction during some period of its life, ... otherwise, on the principle of geometrical increase, its numbers would quickly become so inordinately great that no country could support the product. Hence, as more individuals are produced than can possibly survive, there must in every case be a struggle for existence ... It is the doctrine of Malthus applied with manifold force to the whole animal and vegetable kingdoms ... There is no exception to the rule that every organic being naturally increases at so high a rate, that if not destroyed, the earth would soon be covered by the progeny of a single pair."

   Ask students to discuss this in their groups and respond to "Do you think the ideas are still relevant?"

C2/C25/C11/C4

Performance

3) A certain bacteria have a doubling time of 20 minutes. If you started with a single organism, with a mass of about 10^{-12}gm, and it grew exponentially for one day, what would be the total mass produced? What would it be after two days? How does this compare with the mass of the earth (approximately 6×10^{21} tonnes)? Is this possible? Explain?
4) a) You have two parents, four grandparents, and eight great-grandparents. If you go further back in your family tree, how many ancestors would you have *n* generations ago?
   b) Decide approximately how many years equals one generation and work out roughly how many of your ancestors would have been alive 2000 years ago. Is this possible?

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C33 **analyse and describe the characteristics of exponential and logarithmic functions**

C29 **analyse tables and graphs to distinguish between linear, quadratic, and exponential relationships**

A7 **describe and interpret domains and ranges using set notation**

C25 **solve problems involving exponential and logarithmic equations**

Elaboration—Instructional Strategies/Suggestions

C33/C29/A7 Students should examine the characteristics of the curves of exponential graphs. Ask students to graph the relationship explored on p. 84, e.g., graph $y = 2^x$ and explain why the graph (from left to right) curves slowly at first, then much more quickly. Then graph $y = 3^x$ and compare its growth rate with the $y = 2^x$.

Ask students to use technology to explore the behaviour of these two graphs using the window settings $-10 \le x \le 2$, and range $-1 \le y \le 10$. Have them describe the curving behaviour and explore the behaviour at the extreme left and extreme right. Have them predict the behaviour if they explored using x-values in the domain smaller than –10, or y-values greater than 1. Then have students explore the behaviour of these two graphs using the window settings $-10 \le x \le -1$ and $-0.1 \le y \le 10$. Have them change the range to $-0.1 \le y \le .01$ and the domain to $-15 \le x \le -5$. Have students discuss the behaviour of the graph as it approaches the x-axis. Have them explain why the graphs on this screen look so similar to the graph on the previous screen.

Ask the students if they think either of the two graphs will ever intersect the x-axis and to explain their answers.

Ask students to graph $y = 50(\frac{1}{2})^x$ using $0 \le x \le 10$, and $0 \le y \le 60$. Explore whether this curve intersects the x-axis. Have students confirm that these will not intersect. Ask students why this function is decreasing. Ask students to describe a situation that could be represented by this graph.

This graph could represent the story of the grasshopper jumping towards a fence. It starts 50 metres away and jumps halfway to the fence with each jump. If it continues to jump halfway to the fence with each jump, will it ever hit the fence? The x-axis is a horizontal asymptote—a line to which a curve approaches but will never intersect where $|x|$ is very large, but will never intersect at these extreme values.

All exponential curves have a horizontal asymptote. Ask students: "In all the explorations above was the x-axis a horizontal asymptote?"

Functions whose first term is a, $a \ne 0$ and whose other terms are generated by multiplying the preceding term by a constant, *h*, where $b \ne 1$, can be expressed in the form $f(x) = ab^{x-1}$, where x is the number of the term. Moreover, for any function of the form $f(x) = ab^{x-1}$ with positive base a and for any integers x and $x + 1$ in the domain of the function, it is true that $f(x+1) = b \cdot f(x)$.

The family of functions used to describe the patterns of change in the wage-tripling situation or the data-transfer situation are called exponential functions. Properties of exponential growth and decay can be studied by examining exponential functions of the form $f(x) = ab^x$.

C25 Exponential functions have another important property. No matter what the current value of the function is, it takes the same length of time to double or triple in size. By extending the table, students could find the number of hours it takes for the bacteria to triple, or quadruple, etc. In the table below, students can see that the number of bacteria always doubles in three hours. This property provides a quick way to estimate the future size of a population.

Time in hours	0	1	2	3	4	5	6	7
Number of bacteria	100	126	160	200	250	316	400	500

When numbers are small, doubling does not cause a very big increase, so the graph has a fairly gentle slope. But when the numbers are large, doubling makes an enormous difference, and the graph tends to become very steep. This is the typical shape of a growth curve.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C33/C29/A7/C25

Performance

1) Use your calculator to draw a graph of the function defined by $y = 2^x - 1$. Set the window, $0 \le x \le 64$ and $0 \le y \le 100$. Describe how your graph looks.
   a) Does this graph model the rice on a checker board problem from p. 85? Explain.
   b) Note that even though we would like to see what happens all the way up to 64 squares, as soon as x gets to be 5 or 6, the value of y increases very rapidly, almost in a vertical line on a graph. One way to help this problem is to change the range for the y-value to something much bigger, say 100 000. Make this change and then redraw the graph of this function.
   c) Describe how your graph looks.
   d) How would you describe the asymptote on this graph?

C29/C25

Performance

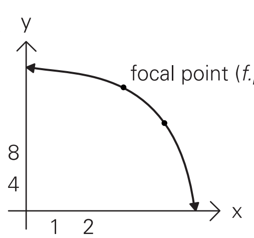
2) Kareem is a car enthusiast. He owns a new Mercedes, worth \$60 000, and a vintage Corvette, worth \$30 000. The new Mercedes will appreciate at 5% per year, and the Corvette will appreciate in value at 10% per year because both are collectors' items.
   a) How long will it take the value of the Mercedes to double?
   b) How long will it take for the value of the Corvette to double?
   c) Establish a formula for the value of the Mercedes after t years.
   d) Establish a formula for the value of the Corvette after t years.
   e) Will the two cars ever have the same value? If so, when? If not, explain why not.

C33/C29/A7

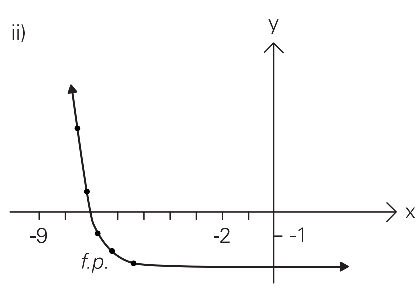
Pencil and Paper

3) Given the following representations of exponential functions, describe the relationship in words, then write the domain and range for each.

i)



ii)



iii)

D	1	2	3
1000	1166.40	1259.71	1360.49

iii)

D	1	2	3
3 000 000	1 500 000	750 000	187 500

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C34 demonstrate an understanding of how the parameter changes affect the graphs of exponential functions

C25 solve problems involving exponential and logarithmic equations

Elaboration—Instructional Strategies/Suggestions

C34/C25 Consider questions like: "What could I expect to earn on day 22 (given the wage-tripling scheme described earlier) if I were to earn \$10 on day one?" "Which car would be more valuable in three years—a \$15,000 car that depreciates 30% every year or a \$12,000 car with a 20% depreciation rate?"

Using exponential functions of the form $f(x) = Ca^x$ to answer questions like these amounts to changing the values of C and a. An activity like Exploring the Graphs for Exponential Functions (see below) engages students in exploring the effects of changing the values of the parameters C and a on the graph of $f(x) = Ca^x$.

In the activity Exploring the graphs of Exponential Functions (NCTM Addenda—Algebra) students are given equations like $f(x) = 0.6^x$, $f(x) = 4(2.3)^x$, etc. and asked to enter them into a graphing utility, look at their graphs and their tables of values, and to respond to:

1) Describe the trends (growth and decay) in the relation between x and y values.
2) Describe any x-values that result in the same y-value.
3) Describe the overall affect of C on the tables and graphs for $f(x) = Ca^x$.
4) Consider two exponential functions $f(x) = a^x$ and $g(x) = b^x$ where a and b are distinct positive numbers.
   i) What's the effect if $a < b$ and each is greater than 1; less than 1?
   ii) What's the effect if a > b and each is < 1 less than 1; greater than 1?

From this investigation students should make conjectures about the effect that C and a have on the function $f(x) = Ca^x$. It is important that they test and verify their conjectures.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C34

Pencil and Paper

1) Describe what happens to the curve of the graph of the function $f(x) = b^{Ax}$ (where $A > 0$) as x increases without limit. What happens to the curves as x decreases through the negative real numbers?
2) Describe what happens to the curve of the graph of the function $f(x) = b^{Ax}$ (where $A < 0$) as x increases without limit. What happens to the curve as x decreases through the negative real numbers?

Performance

3) Graph the function $f(x) = 8^{Ax}$ for $A = 1$, 0.5, and 0.25.
   a) Show algebraically how the y-coordinate of a point on the graph of $f(x) = 8^{Ax}$ is related to the y-coordinate of the point vertically above or below it on the graph of $f(x) = 8^{0.5x}$.
   b) Describe how the graph of $f(x) = 8^{Ax}$ changes as A approaches 0.
4) Graph the function $f(x) = 8^{Ax}$ for $A = -1, -0.5, -0.25$. Describe how the graph of $f(x) = 8^{Ax}$ changes as A approaches 0. What transformation would map the graph of $f(x) = 8^{Ax}$ onto the graph of $f(x) = 8^{-Ax}$? Explain.
5) a) How are the graphs of $f(x) = 5^x$ and $g(x) = -5^x$ related?
   b) How are the graphs of $f(x) = 5^x$ and $h(x) = 5^{-x}$ related?
   c) Do the graphs f and h intersect? If so, at what point(s)?

C34

Pencil and Paper

For students in the Level 1 course:

6) Given the function f, defined by $f(x) = b^{Ax}$ where $b > 1$, $A > 0$ and where .
   a) Prove that there is a *doubling time*, d such that $f(x + d) = 2f(x)$ for all x.
   b) Write an equation that relates A, b, and d and is independent of x.
7) Describe in words the transformations of $y = 3^x$ and $y = 2^x$ that are visible in the equations given:

   i) $-\frac{1}{2}(y-3) = 2^{-3x}$

   ii) $y+5 = 3^{2(x-1)}$

8) Describe the approximate location and position of graphs of the following equations:

   i) $-y = 3^{\frac{1}{2}(x+4)}$

   ii) $\frac{1}{2}(y+2) = 2^{-(x-1)}$

9) Create a problem where this equation would need to be solved for the variable C. $10\ 000 = C \cdot 2^{0.5x}$

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C2 **model real-world phenomena using exponential functions**

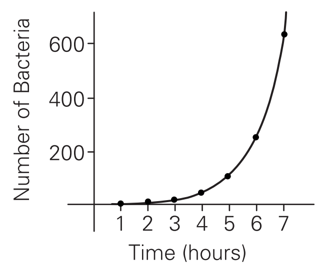
C25 **solve problems involving exponential and logarithmic equations**

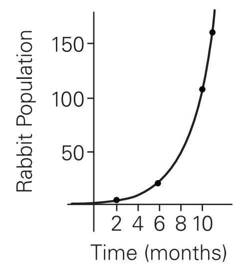
C11 **describe and translate between graphical, tabular, written and symbolic representations of exponential and logarithmic relationships**

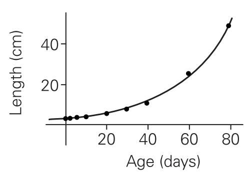
Elaboration—Instructional Strategies/Suggestions

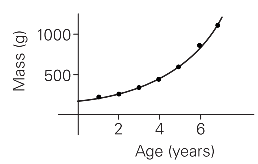
C2/C11/C25 Students will investigate typical growth patterns (population growth, economic growth, etc.) and learn about the exponential functions used to model them. For example, students might be asked to study graphs that deal with various types of growth and answer questions like the following:

1) How are they the same?
2) How are they different?
3) Describe the growth represented here.









Students might be asked to look at tables of values to find patterns:

1) Number of bacteria present at different times:

Time in hours	0	1	2	3	4	5	6	7
Number	100	126	160	200	250	316	400	500

2) The table gives more detail of population increases from 1955 to 1990. The numbers are in billions.

Year	1995	1960	1965	1970	1975	1980	1985	1990
Population	2.70	2.98	3.29	3.63	4.01	4.43	4.84	5.29

Students should calculate the ratio of each entry in the second row to the entry in the previous column. Ask students what they notice.

From these tables that contain collected data, students should notice that the growth ratios are more or less constant, e.g., in the first table $\frac{160}{126} = 1.26 = \frac{126}{100}$. In general, students should learn that if growth ratios are constant, then they can model the growth with a function in the form $f(x) = Ca^x$, where C is the size of the starting population and a is the growth rate. Functions of this type are known as exponential function.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C2/C25/C11

Performance

1) The table below shows the actual world population figures according to a census bureau for the period from 1800 to the late 1900s, as well as projections to the year 2020.

Year	1801	1925	1959	1974	1986	1997
Population in billions	1	2	3	4	5	6

a) How long did it take the world population to double from 1 billion to 2 billion? Is this more or less than the doubling time associated with the exponential function above?

b) How long did it take the population to double from 2 billion to 4 billion? How close is this to the doubling time computed above? How long will it take to double from 3 billion to 6 billion? Is this more or less than the doubling time predicted by the exponential equation? Is the world population growing exponentially? Is its growth exceeding exponential growth? Explain your answer.

2) If you deposit your money in the bank, you will receive interest on this money. Typically, it is compounded monthly. This means that after the first month goes by you will get interest on the original amount deposited and then the next month you will get interest on the original amount as well as interest on this first interest. This is the principle of *compound interest*. Normally, when the bank quotes the rate of interest they give it on an annual basis, along with how often it is compounded. For example, 6% per annum compounded monthly. It's important to remember that 6% per annum is 0.5% per month (0.005 in decimal form).

a) Complete the table from the headings given, for months 1, 2, 3, 4, 12, 24, 120, 240 and 480, and finally n.

Month	Expression for the amount at the start of the month	Expression for the amount at the end of the month	Amount
	\$100	\$100 (1.005)	\$100.50

b) What is the expression for the amount at the end of the nth month?

c) Plot a graph, and from it, determine the amount after the 200th month.

d) i) Does your money increase in value quickly at the start?
   ii) Does your money increase at the end of the 40 year period?
   iii) By how much has your money increased over the 40 year period?
   iv) Think of a situation in which this kind of long-term financial planning would be appropriate.

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C2 model real-world phenomena using exponential functions

C11 describe and translate between graphical, tabular, written and symbolic representations of exponential and logarithmic relationships

Elaboration—Instructional Strategies/Suggestions

C2/C11 As stated on the previous pages, students will investigate typical exponential patterns. Some exponential patterns shows a decay process rather than growth. When studying the graph of temperature dropping as a cup of coffee cools and the graphs of the decay of a radioactive element, students will notice that the graphs are decreasing throughout the domain and approach a horizontal asymptote as the values in the domain increase. Ask students to explain what is different about an equation that results in a graph that depicts a decay process. (See C34.) When studying the graphs of various types of decay (the decay of a swinging pendulum, the cooling of a cup of coffee, the decay of a radioactive particle), students should

1) notice how the graphs are alike and how they are different
2) be able to describe the decay represented—how quickly the decay occurs, what is the half-life, what does the asymptote represent

In their continued study of tables of graphs they should be able to describe all the characteristics of an exponential relationship, only now as a decay, rather than a growth process.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C2/C11

Performance

1) Carbon 14 is an isotope of carbon found in plants, which have absorbed it from the atmosphere. From the time the carbon 14 is absorbed, it decays exponentially. The percentage, y, of carbon 14 in the plant x years after absorption is given by the following equation: $y = 10^{\,2 - 0.00005235x}$. Graph this equation and determine the half-life of carbon 14.
2) When one face of a cube is painted red and tossed, the probability that it will land redside up is $\frac{1}{6}$ because each cube has six sides, and only one of those sides is painted red. Tossing many cubes and knowing how many will show red faces is an unpredictable, random process. Rarely will $\frac{1}{6}$ of the cubes do this on any toss. However, if you repeat the toss many, many times, the average number that show red will approach $\frac{1}{6}$. So, if one were to toss 100 cubes and remove the red ones, and continue this, it would take about four tosses for approximately half of the cubes to be removed, so the half-life of a group of cubes is about four tosses. (After one toss $\frac{5}{6}$ remain, $\frac{1}{6}$ has decayed; after two tosses $\frac{5}{6}$ of $\frac{5}{6}$, or $\frac{25}{36}$ remain; and after four tosses $\frac{5}{6}$ of $\frac{5}{6}$ of $\frac{5}{6}$ or $\frac{625}{1296}$ of the cubes are left, 52 have been removed).

   In this model, assume that the removal of a cube corresponds to the decay of a radioactive nucleus. The chance that a particular radioactive nucleus in a sample of identical nuclei will decay in a second is the same for each second that passes, just as the chance that a cube would come up red was the same for each toss $\left(\frac{1}{6}\right)$. The smaller the chance of decay, the longer the half-life, (time for half of the sample to decay) of a particular radioactive isotope.

   Follow this procedure:
   a) Toss the 100 wooden cubes onto a table surface. Remove all the cubes that land red side up. Place to the side.
   b) Gather up the remaining cubes and toss them again. Again, remove all the cubes that land red side up.
   c) Repeat this experiment until there are 10 remaining cubes. (Given the small starting number, statistically numbers below 10 become insignificant. In actual situations, the number of atoms is much greater.)
   d) Preparing to graph:
      i) What does the number of trials represent in real-life?
      ii) What does the number of cubes remaining after each toss represent?
      iii) What is the dependent variable, and what is the independent variable?
   e) Use regression analysis to obtain a graph and an equation that best fits the data.
   f) Determine the half-life from your graph and equation. Show all your work.
   g) Approximately how many radioactive nucleii will remain after 10 time periods? Explain.
   h) According to your model how much time will pass for the 100 nucleii to decay to 15?

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

A5 demonstrate an understanding of the role of real numbers in exponential and logarithmic expressions and equations

Elaboration—Instructional Strategies/Suggestions

A5 As students model situations and solve problems that can be represented by functions like $y = Ca^{x},$ they need to be sure that the expression a^x has a meaning for all values of x. The way a^x is defined for rational and irrational values of x is explained below.

Rational exponents:

Students should already be familiar with the definition of a^x in the case where x is a positive or negative integer.

If m and n are both positive integers,

$a^{1/n} = \sqrt[n]{a}$, which is the number whose nth power is a

$a^{m/n} = \sqrt[n]{a^m}$, which is the number whose nth power is a^m

$a^{-\min} = (\frac{1}{a})^{\frac{m}{n}}$, which is the reciprocal of $a^{\frac{m}{n}}$

$a^0 = 1, a \neq 0$

These four definitions give a meaning to a^x for any rational number x.

Irrational exponents:

If x is irrational in $y = a^x$, then the value a^x will be an approximation. For example, given $y = 2^{\sqrt{3}}$, students might choose $\sqrt{3} \doteq 1.72$, and $y \doteq 2^{1.72}$. This results in $y \doteq 3.29436$ ... However if technology is used to evaluate $y = 2^{\sqrt{3}}$, the result will be $y \doteq 3.322$ ... Students will not need much emphasis on how ax is defined for irrational values of x in this course. The concept of a^x where x is irrational is studied in a future course.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

A5

Performance

1) a) Apply the definition of b^t, where t is rational, to evaluate the following without using a calculator: $8^{\frac{1}{3}}$, $8^{-\frac{1}{3}}$, $8^{\frac{2}{3}}$, $8^{-\frac{2}{3}}$, $16^{\frac{3}{4}}$, $27^{-\frac{2}{3}}$, $32^{\frac{6}{5}}$, $125^{\frac{2}{3}}$, $25^{-\frac{3}{2}}$

   b) Evaluate using your calculator.
   c) Check using a graphing utility.

2) Which of the following is of greatest value? Explain.

   a) $(3^3)^3$ b) 3^{33} c) 3^{3^3} d) 33^3

3) Prove that $(2^3)^4$ … 2^7.
4) Simplify each expression and evaluate if possible:

   a) $4^2 \cdot \left(2^3\right) \div 32$

   b) $\left(3xyz\right)^3 \cdot \left(9x^2y\right)^2 \div \left(3x\right)^5$

   c) $\dfrac{\left(x^m\right)^2}{x^n}\left(\dfrac{x^{n+1}}{x^n}\right)^2$

5) Simplify: i) $5^n + 5^n$ ii) $2^{x+5} + 2^{x+5}$
6) Explain how 3^{k+1} can be the same as $3(3^k)$. Explain how you might use this idea in a question like "... factor $1 - 2^{2x+1} + 4^{2x}$ ...".
7) Give the meaning of each power:

   a) x^{-5} b) $y^{\frac{1}{3}}$ c) $a^{\frac{2}{3}}$ d) n^{-y}

8) Evaluate:

   a) $16^{\frac{3}{4}}$ b) $64^{\frac{-1}{2}}$

   c) $\left(3^{-2} + 2^{-3}\right) \div \left(2^{-1} - 3^{-1}\right)$ d) $5^2 + 10^0 - 4^{\frac{1}{2}} - \left(\dfrac{1}{9}\right)^{\frac{-1}{2}}$

9) Explain how you could use a graph to evaluate each of the following:

   a) $7^{0.3333}$ b) $-2^{\frac{-7}{2}}$
   c) $3 \cdot 2^{x-5} = 2 \cdot 10^{-2x-10}$

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C2 **model real-world phenomena using exponential functions**

C25 **solve problems involving exponential and logarithmic equations**

C3 **sketch graphs from descriptions, tables, and collected data**

F1 **create and analyse scatter plots and determine the equations for curves of best fit, using appropriate technology**

$B4_2$ **use the calculator correctly and efficiently**

Elaboration—Instructional Strategies/Suggestions

C2/C25/C3/F1/$B4_2$ Students should realize that when they work with real data, there is always some variability due to measurement error. They can never measure exactly and may often have to measure indirectly, so they cannot expect to generate a model that will fit the data perfectly. No matter how accurate the data, they may have to simplify a complex situation in order to construct a model. Assuming the growth ratios are constant is a way of simplifying the situation. When students think they have found a model, they should always graph it along with the data, to check how well it fits. They need to ask themselves whether any assumptions they made were justified, and whether some other model might fit the data better.

Up to now, what students have been doing could be described as empirical curve fitting. They have looked for patterns in tables of data and determined a formula to describe them. But the tables were not very extensive, and the pattern was not always perfect. To justify the assumption that exponential functions are the best models for growth, they need to think about the processes by which living things grow, and perhaps conclude that it is only reasonable to assume that growth will be exponential under certain conditions. If growth is constant, however, linear is th best model. For bacteria growth to be exponential temperature and food supply need to be steady. For populations, the birth and death rates need to be more or less constant. Students should consider situations as described in the Instructional Activities on the next page. This will be their first attempt to give a theoretical explanation for the observed patterns.

Equations like $y = 2^x$, 3^x, $\left(\frac{1}{2}\right)^x$, and $\left(\frac{1}{3}\right)^x$, can be determined from tables. Some real situations will be more complicated, and finding the appropriate model would be very difficult to do by hand, so, it is expected that students will use technology.

Students should also consider conducting experiments using calculator- or computer-based laboratories. They should determine the equation using exponential regression and analysing how well the equation fits the data.

Students should also deal with situations in which the dependent variable decreases in a gradual way. Students should be encouraged to conduct an experiment of a situation such as a swinging pendulum to find out how a swing gradually slows down or measuring the temperature of a cup of coffee to determine how fast it cools. They could use the CBL or PSL probes to measure the temperature over time, or a motion detector (or their naked eye) to time or measure the decreasing amplitude of a pendulum.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C2/C25/C3/F1/B4$_2$

Performance

1) Suppose that a strain of bacteria has a doubling time of one hour at room temperature if a plentiful food supply is provided.
   a) If 10 organisms are present at the start, find how many there will be after 0, 1, 2, ... hours, and enter your results in a table.
   b) Write an algebraic expression for the number after t hours.
   c) Doubling times for bacteria vary according to the temperature and the strain of bacteria. Add more rows to your table to show the growth of bacteria with doubling times of i) 2 hours ii) 3 hours iii) 30 minutes iv) 20 minutes
   d) Find an algebraic expression for each of these growth functions.
   e) Generalize your formula to bacteria with a doubling time of d hours.
   f) Sketch the graphs of all your functions on the same axes and label each one. Will the domains and ranges of these functions result in graphs that are continuous or discrete?
   g) If C is the number at the start, and d the doubling time, write down a formula for N, the number of bacteria after t hours.
2) Suppose that, in each unit of time, t, the number of deaths is 5% of the population and the number of births is 15%, so that the increase in each time interval is 10% of the size at the beginning. The time interval may be a year, a month, or a week, depending on the species you are dealing with.
   a) If there are 100 individuals in the population when $t = 0$, how many will there be when t = 1, 2, 3, ...?
   b) What is the ratio of each of these numbers to the one before it?
   c) Use this growth ratio to work out a formula for the size of the population at any time t.
   d) Go through the same steps to find what the formula would be if the increase per unit of time was 5%.
   e) Generalize to the situation where there is an increase of r% per unit of time.
3) When Jose and Terri carried out a coffee cooling experiment, they obtained the results given in the table below.

t	0	2	4	6	8	10	12	14	16	18	20
T	89.0	83.2	78.0	73.7	70.4	67.4	64.8	62.6	60.0	58.0	55.8

   Here t stands for the time in minutes since the experiment began and T for the temperature of the water in degrees Celsius. Room temperature was 20°C.
   a) Graph the data. What do you think will happen to the water temperature if you wait long enough?
   b) Add another row to the table showing the difference between the water temperature and room temperature. How would you test whether an exponential model would fit these numbers?
   c) The first three numbers do not appear to fit the same pattern as the rest. Can you suggest a reason for this?
   d) Find an exponential function that fits the numbers from $t = 6$ to $t = 20$ as well as possible.
   e) According to this model, what temperature would you expect the water to be after 30 minutes?
4) Kate bought a computer for $2 000, to use in a business she is setting up. If it depreciates at a rate of 30% per year, what will the depreciated value be after one year, two years, ... five years? Find an expression for its value after n years and show this on a graph. Approximately how long does it take for the value of the computer to reduce to half the initial amount?

Suggested Resources

Barnes, Mary. *Investigating Change: Growth and Decay: Unit 7*. Melbourne: Curriculum Corporation, 1993.

Brueningsen, Chris, et al. *Real-World Math with the CBL System.* Dallas: Texas Instruments, 1994.

Exponential Growth

Outcomes

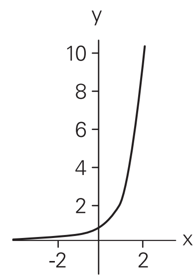
SCO: In this course, students will be expected to

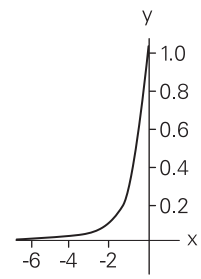
C33 **analyse and describe the characteristics of exponential and logarithmic relationships**

C29 **analyse tables and graphs to distinguish between linear, quadratic, and exponential relationships**

Elaboration—Instructional Strategies/Suggestions

C33/C29 Students should understand that there is a typical shape for an exponential graph. They should be able to talk about how quickly y-values become very large in growth situations. If students wanted to see the results of larger x-values, they would need to reduce the scale on the y-axis by a huge factor, otherwise the graph would run right off the page. If graph one represents $y = 2^x$, then what would be the value of y if $x = 10$ or 15. When x is negative the value for $y = 2x$ soon becomes so small that it is hard to distinguish the graph from the x-axis. To get a better idea of how the graph behaves for negative values for x, students should extend the x-axis and reduce the scale on the y-axis (see graph 2). Ask students to explain why the two graphs look so much alike.





Both graphs 1 and 2 are graphs of $y = 2^x$. Students should compare the graphs $y = 3^x$, $y = 4^x$, $y = 10^x$ with $y = 2^x$ and describe what point they all have in common. When x is positive, which graph lies above the other? When x is negative, which graph lies above the other? Students should make conjectures from the above questions and test them on $y = 1.5^x$ and $y = 2.5^x$. Students should compare graphs of $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$. They should try to graph $y = b^x$ when b is negative. What happens? Explain.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C33/C29

Journal/Paper and Pencil

1) Explain why all equations of the form $y = Ca^x$ where $a > 0$ and $C > 0$ and $a \neq 1$, pass through (0, 1).
2) Explain why the graph of $y = 2.5^{x}$ approaches the x-axis more quickly than the graph of $y = 1.5^{x}$.

Performance

1) a) Investigate the graphs of $y = 36^{\frac{1}{2}}$, $y = (6^2)^{\frac{1}{2}x}$, $y = (\sqrt{36})^x$, and $y = (\sqrt{6})^{2x}$. Explain what is happening.
   b) Based on what you have learned in a) rewrite $(5^2)^{\frac{1}{2}x}$ three different ways.
   c) Investigate the graph of $y = 2^{\frac{1}{3}x}$ and $y = (5^2)^{\frac{1}{2}x}$. Explain what is happening.
2) Graph $y_1 = 2^{-x}$, $y_2 = 2^{x}$. Harry said that if he graphs $y_1 = 2^{-x}$ for $x \leq 0$, $y_2 = 2^x$ for $x \geq 0$, on the same axes then he gets the graph $y_3 = x^2 + 1$. Do you think Harry is correct? Use mathematical reasoning to correct Harry or defend him.
3) Examine the following tables and indicate which one(s) are suggesting an exponential relationship. Explain your thinking.

X	Y
0	0.093
1	0.1875
2	0.375
3	0.75
4	1.5
5	3

X	Y
–7	20 000
–6	200
–5	2
–4	0.02
–3	0.0002
–2	0.000002

X	Y
–6	–3456
–4	–1504
–3	–828
–2	–352
1	–124

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C35 (111) **write exponential functions in transformational form, and as mapping rules to visualize and sketch graphs**

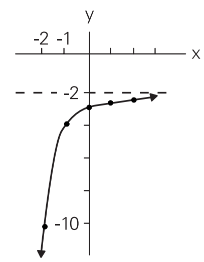
A7 **describe and interpret domains and ranges using set notation**

Elaboration—Instructional Strategies/Suggestions

C35(111)/A7 Students should explore exponential relationships so that when they represent the relation with an equation and/or mapping rule, they will be able to estimate the position of the graph, the orientation, and the behaviour on the coordinate system.

Based on the investigations of the transformations of $y = x^2$ in their previous study, students should explore transformations of the exponential equation $y = 2^{x}$.

Students might begin by graphing $y = 2^{x}$ and $y = 2^{-x}$. Have students explore whether they can position one of them onto the other for comparison. Ask them a series of questions that will lead them to discover that $y = 2^{-x}$ is the image of $y = 2^{x}$ after a reflection in the y-axis, and this transformation can be recorded in a mapping rule: (x, y) 6 $(-x, y)$.



Students should continue to investigate all the transformations as they did for quadratic and trigonometric functions, leading to an ability to take an equation like this $-\frac{1}{2}(y+2) = 2^{-2(x+1)}$ and form a mapping rule (x, y) 6 $\left(-\frac{1}{2}x - 1,\ -2y - 2\right)$. Most importantly they should be able to visualize the shape and location of the graph on the coordinate system.

Student thinking might go something like this:

- The graph of $y = 2^{x}$, $y = 3^{x}$, etc. all pass through the point (0, 1).
- This point (0, 1) is called a focal point because the graph looks like it behaves differently on one side than on the other side.
- For example on the graph $y = 2^{x}$ to the left of the focal point the graph curves slowly downwards toward the asymptote. On the right of the focal point the graph curves quickly upwards as the x values increase.
- Thus in graphing the equation above, first I notice a negative on both the x and the y which indicates that the graph of $y = 2^x$ will be reflected in both the y- and x-axis, resulting in the graph curving slowly upward towards the asymptote on the right of the focal point and downward quickly on the left of the focal point.
- The domains include all real numbers (x 0 R), but the ranges are restricted by the asymptotes $\left(y < 2 \middle| y \in R\right)$.

To find the image of the focal point (0, 1), students would first consider the reflections. Obviously, only the reflection in the x-axis will have an effect, moving (0, 1) to (0, –1). Then consider the stretches, and again only the vertical stretch has an effect since the stretch factors multiply the (0, –1) coordinates to locate the stretched image at (0, –2). Then students deal with the translations moving (0, –2) left 1 and down 2 to (–1, –4). Due to the double reflection and the two stretch values, the graph will be an increasing shape, from left to right, rising very sharply (steep slopes for the tangents to the curve) towards the focal point, then continuing to increase at a very slow rate as it approaches the horizontal asymptote, which is two units above the focal point (vertical stretch of 2).

To describe the x-values for which the graph ($y = 2^x$) increases, students would write $x \in R$ or, in interval notation, $x \in (-\infty, \infty)$. Y-values can be described as being restricted by the asymptotes. Thus, for exponential functions the y-values always fall entirely above or below the horizontal asymptote and can be represented using inequalities and set notation $\left(y \geq 0 \middle| y \in R\right)$.

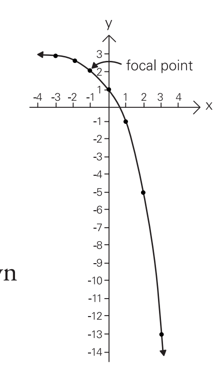
Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C35(111)/A7

Performance

1) An exponential function, *f*, is defined by $f(x) = y = 2^{-x+1} - 8$.
   a) Write the equation in transformational form.
   b) Describe the function as a transformation of $y = 2^x$ and graph it.
   c) State its domain and range and the equation of its asymptote.
   d) Find its zero(es).
   e) Solve for x if $y = 24$.
   f) Use a calculator to find $f(-0.5)$ and $f\left(\sqrt{3}\right)$ to two decimal places.
2) An exponential function, *g*, is defined by $g(x) = y = -3^{x+1} + 6$.
   a) Write the equation in transformational form.
   b) Describe this as a transformation of $y = 3^x$ and graph it.
   c) State the domain, the range and the equation of its asymptote.
   d) Use a calculator to approximate its zero(es) to two decimal places.
   e) Solve for x if $g(x) = -3$.
   f) Use a calculator to find $g(0.5)$, $g\left(\sqrt{2}\right)$, and $g(-2.6)$.
   g) What relation would describe the region above the graph of *g*? below the graph of *g*?
3) a) Draw the graph of $y = 2^x$.
   b) Use your knowledge of transformations to describe in words each of the following and graph them on the same coordinate system.
      A: $y = 2^{x+3}$ B: $y + 1 = 2^x$
      C: $-y = 2^x$ D: $y = 3(2^{x-2}) + 5$
   c) Write the mapping rule for each transformation.
   d) What transformation will change the asymptote of $y = 2^x$?
4) Find the equation of the image $y = 2^x$ under each of the following mappings.
   a) $(x, y) \rightarrow (-x, y)$
   b) $(x, y) \rightarrow (-x, y+1)$
   c) $(x, y) \rightarrow (3x-1, y+5)$
5) Given that $f(x) = y = (1.2)^{x-1} - 3$.
   a) What are the domain and the range of the function?
   b) For what values of x does f increase? decrease?
   c) What is the approximate zero of f?
   d) Describe f as a combination of transformations of $y = (1.2)^x$.
6) For $y = 2^x$, prove that a horizontal translation of -2 is equivalent to a vertical stretch of 4.
7) What is the equation for this graph? Students might think the following:
   a) from focal point—over 1, down 1 over 2, down 3 over 3, down 7 so its $y = 2^x$ reflected in x-axis, no stretches
   b) focal point has moved left 1, up 1
   c) equation: $-(y - 1) = 2^{x+1}$



Suggested Resources

Exponential Growth

Outcomes

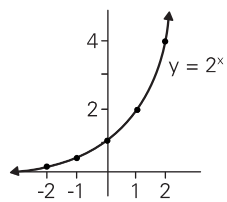
SCO: In this course, students will be expected to

C35 (111) **write exponential functions in transformational form and as mapping rules to visualize and sketch graphs**

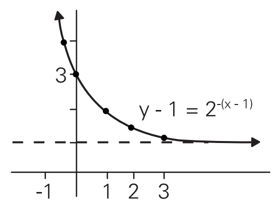
C33 **analyse and describe the characteristics of exponential and logarithmic relationships**

Elaboration—Instructional Strategies/Suggestions

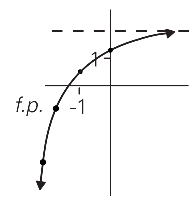
C35(111)/C33 It should be clear to students that as they explore equations, tables, and graphs, the exponential graph is always increasing or decreasing and always approaches a horizontal asymptote, e.g., students can see an increasing exponential graph in A and C, but decreasing in B. The horizontal asymptote in A is $y = 0$, and in B is $y = 1$, and C is $y = 2$ (always determined by the vertical translation). This ensures then that exponential equations will at the most have one real root, and this will occur only when the graph crosses the x-axis (as in C).



A



B



C

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

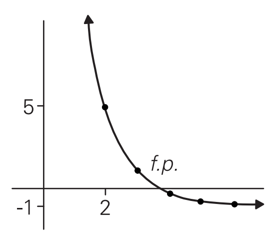
C35(111)/C33

Performance

1) Analyse the graph of the function $y = -3^{(x+2)}$ and answer the following questions:
   a) State the domain and range.
   b) Are there any zeros for the function? If so, what are they?
   c) Write the equation for the asymptote, if any.
   d) Does the function increase or decrease with x? Give reasons for your answers.

2) a) Explain why the following equation is not in transformational form $f(x) = -3 \cdot 2^{(2x-6)} + 5$
   b) Put the above equation in transformational form and describe the transformations of $y = 2^x$ in words and as a mapping rule.
   c) State the domain, range, zero(es), and equation for the asymptote.
   d) Describe the interval for which the graph decreases using symbols.
   e) If, for some reason, the domain is restricted to $-5 \le x \le 0$, describe the corresponding range using set notation.

3) Answer the following questions about the graph drawn on the right. Is the curve exponential? How do you know?
   a) Is it a transfromation of $y = 2^x$ or $y = 3^x$?
   b) Are there any reflections? Describe them.
   c) Are there any stretches? How do you know?
   d) State the equation for the curve, then check with your graphing calculator.



Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

B1 **demonstrate an understanding of the relationships that exist between arithmetic operations and the operations used when solving exponential equations**

B12 **apply real number exponents in expressions and equations**

C24 **solve exponential and logarithmic equations**

C25 **solve problems involving exponential and logarithmic equations**

Elaboration—Instructional Strategies/Suggestions

B1/B12/C24/C25 When asked to solve an equation in the form $4 = 2^{x}$ students should understand that the solution is located at the point where $y = 4$ and $y = 2^x$ intersect, and this can be found using a graph. Students should realize that they are looking for an x value that when written as an exponent with a base of 2 gives the result 4. However, this process may become complicated in equations such as $2^{3x+5} = 0.5$. Now it is not as clear what x value would be such that $2x$ would give the answer 0.5. However, students understand that $0.5 = \frac{1}{2}$ and that $2-1 = \frac{1}{2}$.

Therefore, they can rewrite $2^{3x+5} = 0.5$ as $2^{3x+5} = 2^{-1}$. Because the bases are equal, students realize $3x + 5=-1$ and solve this equation for x.

Many students will understand that a problem occurs when both sides of the equation cannot be expressed with the same base. Trial and error methods should be explored leading to the need for logarithms (see page 116). A few students would enjoy the puzzle-solving-like opportunity to unravel equations like:

1) $5^{2x+1} = \sqrt[3]{25}$ 4) $9^{x^2+1} = 27^x\left(3^x\right)^x$

2) $4^{2x-3} - 9 = 55$ 5) $5^{2x} - 26\left(5^x\right) + 25 = 0$

3) $9^{2x+1} = 81(27x)$ 6) $3^{x+1} + 3^x = 324$

(Note: 5) and 6) are for students in the Level 1 course only.)

Discussion should focus on multiple representations and their justification. For example, $d^{\frac{3}{2}} = 27$ can be expressed as $\left(d^3\right)^{\frac{1}{2}}$ or $\left(d^{\frac{1}{2}}\right)^3$. Students could use this strategy to solve

1) $d^{\frac{3}{2}} = 27$ and 2) $d^{\frac{3}{2}} = 8$

Ans:

$$d^{\frac{3}{2}} = 27 \qquad d^{\frac{3}{2}} = 8$$
$$(d^{\frac{1}{2}})^3 = 3^3 \qquad (d^3)^{\frac{1}{2}} = 64^{\frac{1}{2}}$$
$$d^{\frac{1}{2}} = 3 \qquad d^3 = 64$$
$$d = 9 \qquad d = 4$$

... continued

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

B1/B12/C24

Pencil and Paper

1) Solve the following algebraically:

a) $8^{2x} = 2^{4x+1}$

b) $4^x = 2^{x+7}$

c) $5^{2x+1} = \sqrt[3]{25}$

2) Solve for x

a) $3^{2x} - 10\left(3^x\right) + 9 = 0$

b) $3^{x+1} + 3^x = 324$

c) $9^{x^2+1} = 27^x\left(3^x\right)^x$

d) $x^{\sqrt{2}} = 81^{\frac{1}{\sqrt{8}}}$

3) Find the value for x which $\dfrac{\left(4^x\right)\left(2^{2x+2}\right)\left(4^{x+2}\right)}{32^x} = 512$.

4) Solve for the following equations:

a) $\left(\frac{1}{8}\right)^{2x-3} = \left(\frac{1}{2}\right)^{x+2}$

b) $(x-5)^{\frac{2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{9}}$

c) $9^{2x+1} = 81\left(27^x\right)$

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

B1 **demonstrate an understanding of the relationships that exist between arithmetic operations and the operations used when solving exponential equations**

B12 apply real number exponents in expressions and equations

C24 solve exponential and logarithmic equations

C25 solve problems involving exponential and logarithmic equations

Elaboration—Instructional Strategies/Suggestions

... continued

B1/B12/C24/C25 Discussion with students in the Level 1 course might also include recognizing that $2 \cdot 2^{2x} = 2^{2x+1}$. For example, when students are asked to solve $2^{2x+1} + 3 \cdot 2^x = 5$ algebraically they could rewrite as $2\left(2^{2x}\right) + 3 \cdot 2^x - 5 = 0$ and again as $2\left(2^x\right) + 3 \cdot 2^x - 5 = 0$ the factoring:

then $2{\bullet}2 = -5$ and $2^x = 1$

$2^x = -\frac{5}{2}$ and $2^x = 2^0$

$x = \phi$ or $x = 0$

$\therefore \{0\}$

or by substituting $a = 2^x$;

$$
\begin{aligned}
&2a^2 + 3a - 5 = 0 \\
&(2a+5)(a-1) = 0 \\
&2a = -5 \text{ or } a = 1 \\
&a = \tfrac{5}{2} \text{ or } a = 1,\ 50 \\
&2^x = -\tfrac{5}{2} \text{ or } 2^x = 1,\ 50 \\
&x = \phi \text{ or } x = 0 \\
&\therefore \{0\}
\end{aligned}
$$

... continued

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

B1

Pencil and Paper

1) What follows are some examples of student work. You are to explain what the student is doing or thinking between step 1 and step 2:

a)
$$(x-3)^4 = (x+3)^2 \quad \text{Step 1}$$
$$\left[(x-3)^2\right]^2 = (x+3)^2 \quad \text{Step 2}$$
$$(x-3) = x+3$$
$$x^2 - 6x + 9 = x + 3$$
$$x^2 - 7x + 6 = 0$$
$$\vdots$$

b)
$$4^{x+1} = 64 \quad \text{Step 1}$$
$$4^{x+1} = 4^3 \quad \text{Step 2}$$
$$\therefore x+1 = 3$$
$$\vdots$$

c)
$$\left(\frac{1}{32}\right)^{2x} < \frac{1}{6^4}$$
$$\left(\frac{1^5}{2^5}\right) < \frac{1}{2^6} \quad \text{Step 1}$$
$$\left(\frac{1}{2}\right)^{10x} < \left(\frac{1}{2}\right)^6 \quad \text{Step 2}$$
$$10x < 6$$
$$\vdots$$

C25

Performance

2) The table represents population increases from 1955 to 1990. The numbers are in billions.

Year	1955	1960	1965	1970	1975	1980	1985	1990
Population	2.7	2.98	3.29	2.67	4.01	4.43	4.84	5.29

a) Graph the relationship using the values in the above table and determine the equation that best represents the relationship.
b) Using the equation, determine the population in the year 2010.
c) Explain how you would use the equation to predict how long it would take for the population in year 2010 to double, assuming it continues to grow at this rate.

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

C19 demonstrate an understanding, algebraically and graphically, that the inverse of an exponential function is a logarithmic function

C11 describe and translate between graphical, tabular, written, and symbolic representations fo exponential and logarithmic relationships

A7 describe and interpret domains and ranges using set notations

Elaboration—Instructional Strategies/Suggestions

C19 As students explore and solve exponential equations in various forms, eventually they will come across a situation where they need to solve for the exponent where they are unable to find common base of all terms the same. When this has happened earlier in their learning, students simply read a value from the graph or used trial and error methods.

It is now time for students to explore this concept in more depth. Ask students to find x when $3^x = 9$. Students will quickly say that $x = 2$ because $3^2 = 9$. Now ask students to find x when $3^x = 7$ using a trial and error procedure. (The graphing calculator is a tool students could use here by drawing the graphs $y = 3^x$ and $y = 7$ and examining the interaction points.

C19/C11/A7 To symbolize what the calculator has done, students need to be introduced to the term 'logarithm.' This name is used to describe the inverse function of an exponential function with the same base.

- Ask students to plot $y = 10^x$ from a table where $-5 \le x \le 1$ and state the domain and range.
- Have them then to reflect $y = 10^x$ across $y = x$ and plot the resulting curve. Have them state the domain and range again and compare.
- Discuss with students why they should draw the inverse $y = 10^x$ by reversing the coordinates in the table for $y = 10^x$ and plotting them ($y = x$ is the line of symmetry between the graphs of a function and its inverse).
- Use the 10^x feature (2nd, Log on the TI-83), to enter 10^(2.1) and hit enter. Then have them press Log and ANS [answer: (–2nd, (–))] and hit Enter.
- Ask them to
  – describe what has happened
  – discuss how one (10^x or LOG) is a function that undoes what the other function did (the term for this is "inverse function")
  – explain how they can show the same process on the graphs they drew.
  – use the 'log' button on their calculator and check their table and graph by finding the logarithm of each of the x-values in the table.
  – create the same two graphs using technology, then examine the table in the TABLE feature

Students should develop the equation for this new graph as $y = \log x$ ($\log x$ and $\log_{10}x$ mean the same—the base 10 is assumed). They start with $y = 10^x$, switch the x and y to get $x = 10^y$, then solve for y by using the symbol for logarithm (log). Since y was the exponent, and *log* means exponent, then $y = \log x$.

Similarly if they were to begin with $y = 2^x$, the inverse function would be written $y = \log_2 x$. Students need to understand how to evaluate $\log_2 7$ using their calculator since the 'log' button assumes base 10, (see B13, next two-page spread).

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

C19/C11/A7

Performance

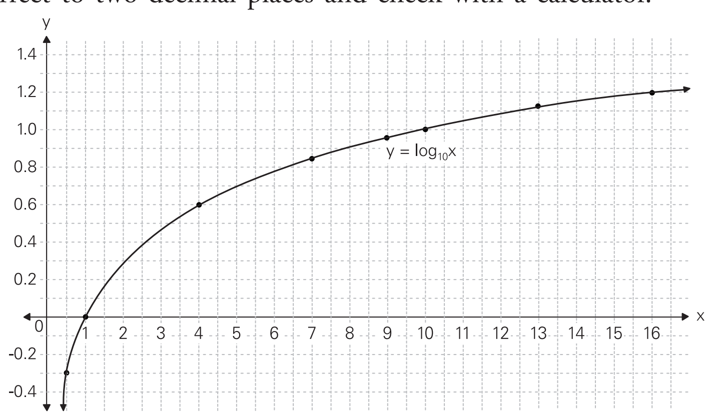
1) a) Find the equation of the inverse function, h, defined by $h(x) = 7^{2x}$ and graph it.
   b) Is this inverse a function?
   c) Find the zero(es) of h^{-1}.
   d) Could you graph h^{-1} by graphing $h(x)$ first? How?

2) a) Sketch the graph of $f(x) = 10^x$ for these values of x. x = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0.
   b) Estimate from the graph the values of x for y = 1, 2, 3, ... , 10.
   c) Construct a table of values using the values of y in (b) for x and the values of x in (b) for y. Plot the ordered pairs obtained and join them with a smooth curve.
   d) What relation would you use to describe the graph expressing y in terms of x?
   e) State the domain and the range of this graph.

C19/C11/A7

Pencil and Paper

For questions 3 and 4 below, use the graph of $y = \log_{10} x$ shown. Give your answers correct to two decimal places and check with a calculator.



3) Find an approximate value for each of the following.
   a) $\log_{10} 4$ b) $\log_{10} 7$
   c) $\log_{10} 8.5$ d) $\log_{10} 3.5$

4) What is the approximate value of x for each of the following?
   a) $\log_{10} x = 0$ b) $\log_{10} x = 1$ c) $\log_{10} x = 0.7$
   d) $\log_{10} x = 0.4$ e) $\log_{10} x = 0.65$ f) $\log_{10} x = 0.35$

5) Explain that the equation of the image of $y = \log_5 x$ under a vertical stretch of 2 followed by a horizontal translation of –1 and then a vertical translation of 2 is $y = 2\log_5 (x + 1) + 2$.

Suggested Resources

Exponential Growth

Outcomes

SCO: In this course, students will be expected to

B13 demonstrate an understanding of the properties of logarithms and apply them

C24 solve exponential and logarithmic equations

Elaboration—Instructional Strategies/Suggestions

B13/C24 Before calculators and computers were readily available, logarithms of numbers were an indispensable aid to scientific calculations. Although this role has been taken over by machines, logarithmic functions are still crucial models for many important scientific phenomena. It is still important to be comfortable with the algebraic properties of logarithms as well as exponential functions. Students should strive for mental math capabilities with the use of many of these properties. Three to five minutes a day should devoted to helping students develop strategies. For example, students should be able to calculate mentally $\log_2 32$ using the strategy of answering "2 to which power = 32?".

Properties to be developed (assume x and y > 0)

1) If $y = a^x$, then $x = \log_a y$

2) $\log_a 2 + \log_a 3 = \log_a 6$ then generalize to $\log x + \log y = \log xy$ and describe this in words.

3) $\log x - \log y = \log \frac{x}{y}$

4) $\log 9 = 2 \log 3$ and generalize this to $\log x^2 = 2\log x$

5) $\log_a x = \frac{\log x}{\log a}$

6) $\log \frac{1}{p} = -\log p$

7) $\log_a a = 1$

8) $\log_b b^x = x$

9) $\log_b a = \frac{1}{\log_a b}$

These properties are useful when simplifying logarithmic expressions and trying to solve equations requiring logarithms. Since solving logarithm equations algebraically requires students to use the strategy of simplifying the equation to a single 'log' term of the same base on both sides, students would use various properties above to simplify the expression.

Exponential Growth

Worthwhile Tasks for Instruction and/or Assessment

B13/B24

Pencil and Paper

1) Express each in logarithmic form.

a) $9 = 3^2$ b) $\frac{1}{9} = 3^{-2}$ c) $\sqrt[3]{5} = 5^{\frac{1}{3}}$ d) $27^{\frac{1}{3}} = 3$

2) Express each in exponential form.

a) $7 = \log_2 128$ b) $-6 = \log_2\left(\frac{1}{64}\right)$ c) $\log_3 4 = x$ d) $\frac{1}{3} = \log_2\left(\sqrt[3]{2}\right)$

3) Evaluate each expression.

a) $\log_9 81$ b) $\log_6 6$ c) $\log_2 1$ d) $\log_3\left(\frac{1}{9}\right)$

e) $\log_7 7$ f) $\log_{81} 9$ g) $\log_2 (2^3)$ h) $\log_a a^x$

i) $\log_a a^{\sqrt{2}}$ j) $\log_5 0$

4) If $\log_{10} 4 = 0.602$, find $\log_{10} 16$ without using a calculator.

5) If $\log_b x = p$ and $\log_b y = q$, find $\log_b (xy)$.

6) If $\log_a b = x$ and $\log_a c = y$, find $\log_a\left(\frac{b^2}{c}\right)$.

7) Prove that $\log_a x = \dfrac{\log_b x}{\log_b a}$

8) Express each of the following as a single logarithm in simplest form.

a) $8\log_4 2$ b) $\log_5 48 - \log_5 12 + \log_5 4$

c) $3\log_6 15 - \log_6 25$ d) $\log_a x + \log_a y - \log_a z$

e) $\log_b 2x + 3(\log_b x - \log_b y)$ f) $\frac{1}{2}\log_a x - \frac{2}{3}\log_a y$

9) Evaluate.

a) $\log_3 9 + \log_3 64 + \log_4\left(\frac{1}{16}\right) + \log_3\left(\frac{1}{9}\right) + \log_{10}\left(\frac{1}{10}\right)$

b) $\log_2 32 - \log_2\left(\frac{1}{32}\right) + \log_4 8 - \log_8 16$

c) $3\log_2 4 + 2\log_3 9 + \log_{10}(0.1) - \log_3\left(\frac{1}{9}\right)$

10) Express each logarithm in terms of m and n where $\log_b x = m$ and $\log_b y = n$.

a) $\log_b\left(\frac{x}{\sqrt{y}}\right)$ b) $\log_b (xy^2)$

c) $\log_b\left(\sqrt{xy^2}\right)$ d) $\log_b\left(\frac{x}{\sqrt{y}}\right)$

11) Solve for x:

a) $\log_{\sqrt{2}} 8 = x$ b) $x = \log_2 8 - \log\frac{1}{9} + 2\log_5 125$

Suggested Resources

Appendix A:
Assessing and Evaluating Student Learning

Assessing and Evaluating Student Learning

In recent years there have been calls for change in the practices used to assess and evaluate students' progress. Many factors have set the demands for change in motion, including the following:

- ***new expectations for mathematics education as outlined in*** **Curriculum and Evaluation Standards for School Mathematics** (NCTM 1989)

The *Curriculum Standards* provide educators with specific information about what students should be able to do in mathematics. These expectations go far beyond learning a list of mathematical facts; instead, they emphasize such competencies as creative and critical thinking, problem solving, working collaboratively, and the ability to manage one's own learning. Students are expected to be able to communicate mathematically, to solve and create problems, to use concepts to solve real-world applications, to integrate mathematics across disciplines, and to connect strands of mathematics. For the most part, assessments used in the past have not addressed these expectations. New approaches to assessment are needed if we are to address the expectations set out in *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989).

- *understanding the bonds linking teaching, learning, and assessment*

Much of our understanding of learning has been based on a theory that viewed learning as the accumulation of discrete skills. Cognitive views of learning call for an active, constructive approach in which learners gain understanding by building their own knowledge and developing connections between facts and concepts. Problem solving and reasoning become the emphases rather than the acquisition of isolated facts. Conventional testing, which includes multiple choice or having students answer questions to determine if they can recall the type of question and the procedure to be used, provides a window into only one aspect of what a student has learned. Assessments that require students to solve problems, demonstrate skills, create products, and create portfolios of work reveal more about the student's understanding and reasoning of mathematics. If students are expected to develop reasoning and problem-solving competencies, then teaching must reflect such, and in turn, assessment must reflect what is valued in teaching and learning. Feedback from assessment directly affects learning. The development of problem-solving, and higher, order thinking skills will become a realization only if assessment practices are in alignment with these expectations.

- ***limitations of the traditional methods used to determine student achievement***

Do traditional methods of assessment provide the student with information on how to improve performance? We need to develop methods of assessment that provide us with accurate information about students' academic achievement and information to guide teachers in decision making to improve both learning and teaching.

What Is Assessment?

Assessment is the systematic process of gathering information on student learning. Assessment allows teachers to communicate to students what is really valued—what is worth learning, how it should be learned, what elements of quality are considered most important, and how well students are expected to perform. In order for teachers to assess student learning in a mathematics curriculum that emphasizes applications and problem solving, they need to employ strategies that recognize the reasoning involved in the process as well as in the product. *Assessment Standards for School Mathematics* (NCTM 1995, p. 3) describes assessment practices that enable teachers to gather evidence about a student's knowledge of, ability to use, and disposition toward, mathematics and of making inferences from that evidence for a variety of purposes.

Assessment can be informal or formal. Informal assessment occurs while instruction is occurring. It is a mind-set, a daily activity that helps the teacher answer the question, Is what is taught being learned? Its primary purpose is to collect information about the instructional needs of students so that the teacher can make decisions to improve instructional strategies. For many teachers, the strategy of making annotated comments about a student's work is part of informal assessment. Assessment must do more than determine a score for the student. It should do more than portray a level of performance. It should direct teachers' communication and actions. Assessment must anticipate subsequent action.

Formal assessment requires the organization of an assessment event. In the past, mathematics teachers may have restricted these events to quizzes, tests, or exams. As the outcomes for mathematics education broaden, it becomes more obvious that these assessment methods become more limited. Some educators would argue that informal assessment provides better quality information because it is in a context that can be put to immediate use.

Why Should We Assess Student Learning?

We should assess student learning in order to

- improve instruction by identifying successful instructional strategies
- identify and address specific sources of the students' misunderstandings
- inform the students about their strengths in skills, knowledge, and learning strategies
- inform parents of their child's progress so that they can provide more effective support
- determine the level of achievement for each outcome

As an integral and ongoing part of the learning process, assessment must give each student optimal opportunity to demonstrate what he/

she knows and is able to do. It is essential, therefore, that teachers develop a repertoire of assessment strategies.

Assessment Strategies

Some assessment strategies that teachers may employ include the following.

Documenting classroom behaviours

In the past teachers have generally made observations of students' persistence, systematic working, organization, accuracy, conjecturing, modeling, creativity, and ability to communicate ideas, but often failed to document them. Certainly the ability to manage the documentation played a major part. Recording information signals to the student those behaviours that are truly valued. Teachers should focus on recording only significant events, which are those that represent a typical student's behaviour or a situation where the student demonstrates new understanding or a lack of understanding. Using a class list, teachers can expect to record comments on approximately four students per class. The use of an annotated class list allows the teacher to recognize where students are having difficulties and identify students who may be spectators in the classroom.

Using a portfolio or student journal

Having students assemble on a regular basis responses to various types of tasks is part of an effective assessment scheme. Responding to open-ended questions allows students to explore the bounds and the structure of mathematical categories. As an example, students are given a triangle in which they know two sides or an angle and a side and they are asked to find out everything they know about the triangle. This is preferable to asking students to find a particular side, because it is less prescriptive and allows students to explore the problem in many different ways and gives them the opportunity to use many different procedures and skills. Students should be monitoring their own learning by being asked to reflect and write about questions such as the following:

- What is the most interesting thing you learned in mathematics class this week?
- What do you find difficult to understand?
- How could the teacher improve mathematics instruction?
- Can you identify how the mathematics we are now studying is connected to the real world?

In the portfolio or in a journal, teachers can observe the development of the students' understanding and progress as a problem solver. Students should be doing problems that require varying lengths of time and represent both individual and group effort. What is most important is that teachers discuss with their peers what items are to be part of a meaningful portfolio, and that students also have some input into the assembling of a portfolio.

Projects and investigative reports

Students will have opportunities to do projects at various times throughout the year. For example, they may conduct a survey and do a statistical report, they may do a project by reporting on the contribution of a mathematician, or the project may involve building a complex three-dimensional shape or a set of three-dimensional shapes which relate to each other in some way. Students should also be given investigations in which they learn new mathematical concepts on their own. Excellent materials can be obtained from the National Council of Teachers of Mathematics, including the *Student Math Notes.* (These news bulletins can be downloaded from the Internet.)

Written tests, quizzes, and exams

Some critics allege that written tests are limited to assessing a student's ability to recall and replicate mathematical facts and procedures. Some educators would argue that asking students to solve contrived applications, usually within time limits, provides us with little knowledge of the students' understanding of mathematics.

How might we improve the use of written tests?

- Our challenge is to improve the nature of the questions being asked, so that we are gaining information about the students' understanding and comprehension.
- Tests must be designed so that questions being asked reflect the expectations of the outcomes being addressed.
- One way to do this is to have students construct assessment items for the test. Allowing students to contribute to the test permits them to reflect on what they were learning, and it is a most effective revision strategy.
- Teachers should reflect on the quality of the test being given to students. Are students being asked to evaluate, analyse, and synthesize information, or are they simply being asked to recall isolated facts from memory? Teachers should develop a table of specifications when planning their tests.
- In assessing students, teachers have a professional obligation to ensure that the assessment reflects those skills and behaviours that are truly valued. Good assessment goes hand-in-hand with good instruction and together they promote student achievement.

Appendix B:
SCOs for Grades 9 and 10

GCO A: Students will demonstrate number sense and apply number theory concepts.

Elaboration: Number sense includes understanding of number meanings, developing multiple relationships among numbers, recognizing the relative magnitudes of numbers, knowing the relative effect of operating on numbers, and developing referents for measurement. Number theory concepts include such number principles as laws (e.g., commutative and distributive), factors and primes, number system characteristics (e.g., density), etc.

By the end of grade 9, students will be expected to

A1 solve problems involving square root and principal square root

A2 graph, and write in symbols and in words, the solution set for equations and inequalities involving integers and other real numbers

A3 demonstrate an understanding of the meaning and uses of irrational numbers

A4 demonstrate an understanding of the inter-relationships of subsets of real numbers

A5 compare and order real numbers

A6 represent problem situations using matrices

By the end of grade 10, students will be expected to

A1 relate sets of numbers to solutions of inequalities

A2 analyse graphs or charts of situations to derive specific information

A3 demonstrate an understanding of the role of irrational numbers in applications

A4 approximate square roots

A5 demonstrate an understanding of the zero product property and its relationship to solving equations by factoring

A6 apply properties of numbers when operating upon expressions and equations

A7 demonstrate and apply an understanding of discrete and continuous number systems

A8 demonstrate an understanding of and apply properties to operations involving square roots

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration: Operation sense consists of recognizing situations in which a given operation would be useful, building awareness of models and the properties of an operation, seeing relationships among operations and acquiring insights into the effects of an operation on a pair of numbers. Operation principles and procedures would include such items as the effect of identity elements, computational strategies and mental mathematics.

By the end of grade 9, students will be expected to

B1 model, solve, and create problems involving real numbers

B2 add, subtract, multiply, and divide rational numbers in fractional and decimal forms, using the most appropriate method

B3 apply the order of operations in rational number computations

B4 demonstrate an understanding of, and apply the exponent laws, for integral exponents

B5 model, solve, and create problems involving numbers expressed in scientific notation

B6 judge the reasonableness of results in problem situations involving square roots, rational numbers, and numbers written in scientific notation

B7 model, solve, and create problems involving the matrix operations of addition, subtraction, and scalar multiplication

B8 add and subtract polynomial expressions symbolically to solve problems

B9 factor algebraic expressions with common monomial factors concretely, pictorially, and symbolically

B10 recognize that the dimensions of a rectangular model of a polynomial are its factors

B11 find products of two monomials, a monomial and a polynomial, and two binomials, concretely, pictorially, and symbolically

By the end of grade 10, students will be expected to

B1 model (with concrete materials and pictorial representations) and express the relationships between arithmetic operations and operations on algebraic expressions and equations

B2 develop algorithms and perform operations on irrational numbers

B3 use concrete materials, pictorial representations, and algebraic symbolism to perform operations on polynomials

B4 identify and calculate the maximum and/or minimum values in a linear programming model

B5 develop, analyse, and apply procedures for matrix multiplication

B6 solve network problems using matrices

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration: Operation sense consists of recognizing situations in which a given operation would be useful, building awareness of models and the properties of an operation, seeing relationships among operations and acquiring insights into the effects of an operation on a pair of numbers. Operation principles and procedures would include such items as the effect of identity elements, computational strategies and mental mathematics.

By the end of grade 9, students will be expected to

B12 find quotients of polynomials with monomial divisors

B13 evaluate polynomial expressions

B14 demonstrate an understanding of the applicability of commutative, associative, distributive, identity, and inverse properties to operations involving algebraic expressions

B15 select and use appropriate strategies in problem situations

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally.

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 9, students will be expected to

C1 represent patterns and relationships in a variety of formats and use these representations to predict and justify unknown values

C2 interpret graphs that represent linear and non - linear data

C3 construct and analyse tables and graphs to describe how changes in one quantity affect a related quantity

C4 determine the equations of lines by obtaining their slopes and y-intercepts from graphs, and sketch graphs of equations using y-intercepts and slopes

C5 explain the connections among different representations of patterns and relationships

C6 solve single-variable equations algebraically, and verify the solutions

C7 solve first-degree single-variable inequalities algebraically, verify the solutions, and display them on number lines

C8 solve, and create problems involving linear equations and inequalities

By the end of grade 10, students will be expected to

C1 express problems in terms of equations and vice versa

C2 model real-world phenomena with linear, quadratic, exponential, and power equations, and linear inequalities

C3 gather data, plot the data using appropriate scales, and demonstrate an understanding of independent and dependent variables, and domain and range

C4 create and analyse scatter plots using appropriate technology

C5 sketch graphs from words, tables, and collected data

C6 apply linear programming to find optimal solutions to real-world problems

C7 model real-world situations with networks and matrices

C8 identify, generalize, and apply patterns

C9 construct and analyse graphs and tables relating two variables

C10 describe real-world relationships depicted by graphs, tables of values, and written descriptions

C11 write an inequality to describe its graph

C12 express and interpret constraints using inequalities

C13 determine the slope and y-intercept of a line from a table of values or a graph

C14 determine the equation of a line using the slope and y-intercept

C15 develop and apply strategies for solving problems

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally. (continued)

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 10, students will be expected to

C16 interpret solutions to equations based on context

C17 solve problems using graphing technology

C18 investigate and find the solution to a problem by graphing two linear equations with and without technology

C19 solve systems of linear equations using substitution and graphing methods

C20 evaluate and interpret non-linear equations using graphing technology

C21 explore and apply functional relationships notation, both formally and informally

C22 analyse and describe transformations of quadratic functions and apply them to absolute value functions

C23 express transformations algebraically and with mapping rules

C24 rearrange equations

C25 solve equations using graphs

C26 solve quadratic equations by factoring

C27 solve linear and simple radical, exponential, and absolute value equations and linear inequalities

C28 explore and describe the dynamics of change depicted in tables and graphs

C29 investigate, and make and test conjectures concerning, the steepness and direction of a line

C30 compare regression models of linear and non-linear functions

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally. (continued)

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 10, students will be expected to

C31 graph equations and inequalities and analyse graphs both with and without graphing technology

C32 determine if a graph is linear by plotting points in a given situation

C33 graph by constructing a table of values, by using graphing technology, and when appropriate, by the slope y-intercept method

C34 investigate and make and test conjectures about the solution to equations and inequalities using graphing technology

C35 expand and factor polynomial expressions using perimeter and area models

C36 explore, determine, and apply relationships between perimeter and area, surface area, and volume

C37 represent network problems using matrices and vice versa

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

Elaboration: Concepts and skills associated with measurement include making direct measurements, using appropriate measurement units and using formulas (e.g., surface area, Pythagorean Theorem) and/or procedures (e.g., proportions) to determine measurements indirectly.

By the end of grade 9, students will be expected to

D1 solve indirect measurement problems by connecting rates and slopes

D2 solve measurement problems involving conversion among SI units

D3 relate the volumes of pyramids and cones to the volumes of corresponding prisms and cylinders

D4 estimate, measure, and calculate dimensions, volumes, and surface areas of pyramids, cones, and spheres in problem situations

D5 demonstrate an understanding of and apply proportions within similar triangles

By the end of grade 10, students will be expected to

D1 determine and apply formulas for perimeter, area, surface area, and volume

D2 apply the properties of similar triangles

D3 relate the trigonometric functions to the ratios in similar right triangles

D4 use calculators to find trigonometric values of angles and angles when trigonometric values are known

D5 apply trigonometric functions to solve problems involving right triangles, including the use of angles of elevation

D6 solve problems involving measurement using bearings and vectors

D7 determine the accuracy and precision of a measurement

D8 solve problems involving similar triangles and right triangles

D9 determine whether differences in repeated measurements are significant or accidental

D10 determine and apply relationships between the perimeters and areas of similar figures, and between the surface areas and volumes of similar solids

D11 explore, discover, and apply properties of maximum areas and volumes

D12 solve problems using the trigonometric ratios

D13 demonstrate an understanding of the concepts of surface area and volume

D14 apply the Pythagorean Theorem

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties, and relationships.

Elaboration: Spatial sense is an intuitive feel for one's surroundings and the objects in them and is characterized by such geometric relationships as (i) the direction, orientation, and perspectives of objects in space; (ii) the relative shapes and sizes of figures and objects; and (iii) how a change in shape relates to a change in size. Geometric concepts, properties, and relationships are illustrated by such examples as the concept of area, the property that a square maximizes area for rectangles of a given perimeter, and the relationships among angles formed by a transversal intersecting parallel lines.

By the end of grade 9, students will be expected to

E1 investigate, and demonstrate an understanding of, the minimum sufficient conditions to produce unique triangles

E2 investigate, and demonstrate an understanding of, the properties of, and the minimum sufficient conditions to guarantee, congruent triangles

E3 make informal deductions, using congruent triangle and angle properties

E4 demonstrate an understanding of and apply the properties of similar triangles

E5 relate congruence and similarity of triangles

E6 use mapping notation to represent transformations of geometric figures, and interpret such notations

E7 analyse and represent combinations of transformations, using mapping notation

E8 investigate, determine, and apply the effects of transformations of geometric figures, on congruence, similarity, and orientation

By the end of grade 10, students will be expected to

E1 explore properties of, and make and test conjectures about 2- and 3-dimensional figures

E2 solve problems involving polygons and polyhedra

E3 construct and apply altitudes, medians, angle bisectors, and perpendicular bisectors to examine their intersection points

E4 apply transformations when solving problems

E5 use transformations to draw graphs

E6 represent network problems as digraphs

E7 demonstrate an understanding of, and write a proof for, the Pythagorean Theorem

E8 use inductive and deductive reasoning when observing patterns, developing properties, and making conjectures

E9 use deductive reasoning and construct logical arguments and be able to determine, when given a logical argument, if it is valid

GCO F: Students will solve problems involving the collection, display and analysis of data.

Elaboration: The collection, display and analysis of data involves (i) attention to sampling procedures and issues, (ii) recording and organizing collected data, (iii) choosing and creating appropriate data displays, (iv) analysing data displays in terms of broad principles (e.g., display bias) and via statistical measures (e.g., mean), and (v) formulating and evaluating statistical arguments.

By the end of grade 9, students will be expected to

F1 determine characteristics of possible relationships shown in scatter plots

F2 sketch lines of best fit and determine their equations

F3 sketch curves of best fit for relationships that appear to be non-linear

F4 select, defend, and use the most appropriate methods for displaying data

F5 draw inferences and make predictions based on data analysis and data displays

F6 demonstrate an understanding of the role of data management in society

F7 evaluate arguments and interpretations that are based on data analysis

By the end of grade 10, students will be expected to

F1 design and conduct experiments using statistical methods and scientific inquiry

F2 demonstrate an understanding of the concerns and issues that pertain to the collection of data

F3 construct various displays of data

F4 calculate various statistics using appropriate technology, analyse and interpret data displays, and describe relationships

F5 analyse statistical summaries, draw conclusions, and communicate results about distributions of data

F6 solve problems by modeling real-world phenomena

F7 explore non-linear data using power and exponential regression to find a curve of best fit

F8 determine and apply the line of best fit using the least squares method and median-median method with and without technology, and describe the differences between the two methods

F9 demonstrate an intuitive understanding of correlation

F10 use interpolation, extrapolation and equations to predict and solve problems

F11 describe real-world relationships depicted by graphs and tables of values

F12 explore measurement issues using the normal curve

F13 calculate and apply mean and standard deviation using technology, to determine if a variation makes a difference

GCO G: Students will represent and solve problems involving uncertainty.

Elaboration: Representing and solving problems involving uncertainty entails (i) determining probabilities by conducting experiments and/or making theoretical calculations, (ii) designing simulations to determine probabilities in situations which do not lend themselves to direct experiment, and (iii) analysing problem situations to decide how best to determine probabilities.

In grade 9, students will be expected to

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G1 make predictions of probabilities involving dependent and independent events by designing and conducting experiments and simulations

G2 determine theoretical probabilities of independent and dependent events

G3 demonstrate an understanding of how experimental and theoretical probabilities are related

G4 recognize and explain why decisions based on probabilities may be combinations of theoretical calculations, experimental results, and subjective judgements