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Atlantic Canada Mathematics Curriculum

New Brunswick
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New Nouveau Brunswick

Geometry and Applications in Mathematics 111/112

(Revised Statistics Unit)

CURRICULUM

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- The teachers and other educators and stakeholders across Atlantic Canada who contributed to the development of the *Geometry and Applications in Mathematics 111/112* curriculum guide.

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I. Background and Rationale

A. Background

Mathematics curriculum reform in Atlantic Canada is shaped by a vision which fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in an increasingly technological society. Curriculum reform has been motivated by a desire to ensure that students in Atlantic Canada benefit from world-class curriculum and instruction in mathematics as a significant part of their learning experiences.

The *Foundation for the Atlantic Canada Mathematics Curriculum* (1996) firmly establishes the *Curriculum and Evaluation Standards for School Mathematics* (1989) of the National Council of Teachers of Mathematics (NCTM) as a guiding beacon for pursuing this vision, which embraces the principles of students learning to value and become active "doers" of mathematics and advocates a curriculum which focusses on the unifying ideas of mathematical problem solving, communication, reasoning, and connections. These principles and unifying ideas are reaffirmed with the publication of NCTM's *Principles and Standards for School Mathematics* (2000). The *Foundation for the Atlantic Canada Mathematics Curriculum* establishes a framework for the development of detailed grade-level documents describing mathematics curriculum and guiding instruction.

Mathematics curriculum development has taken place under the auspices of the Atlantic Provinces Education Foundation (APEF), an organization sponsored and managed by the governments of the four Atlantic Provinces. APEF has brought together teachers with department of education officials to co-operatively plan and execute the development of curricula in mathematics, science, language arts, and other curricular areas. Each of these curriculum efforts has been aimed at producing a program that would ultimately support the Essential Graduation Learnings (EGLs), also developed regionally. These EGLs and the contribution of the mathematics curriculum to their achievement are presented in the "Outcomes" section of the mathematics foundation document.

B. Rationale

The *Foundation for the Atlantic Canada Mathematics Curriculum* provides an overview of the philosophy and goals of the mathematics curriculum, presenting broad curriculum outcomes and addressing a variety of issues with respect to the learning and teaching of mathematics. This curriculum guide is one of several which provide greater specificity and clarity for the classroom teacher. The *Foundation for the Atlantic Canada Mathematics Curriculum* describes the mathematics curriculum in terms of a series of outcomes—General Curriculum Outcomes (GCOs), which relate to subject strands, and Key-Stage Curriculum Outcomes (KSCOs), which articulate the GCOs further for the end of grades 3, 6, 9, and 12. This guide builds on the structure introduced in the foundation document, by relating Specific Curriculum Outcomes (SCOs) to KSCOs for *Geometry and Applications in Mathematics 111/112*. Figure 1 further clarifies the outcome structure.

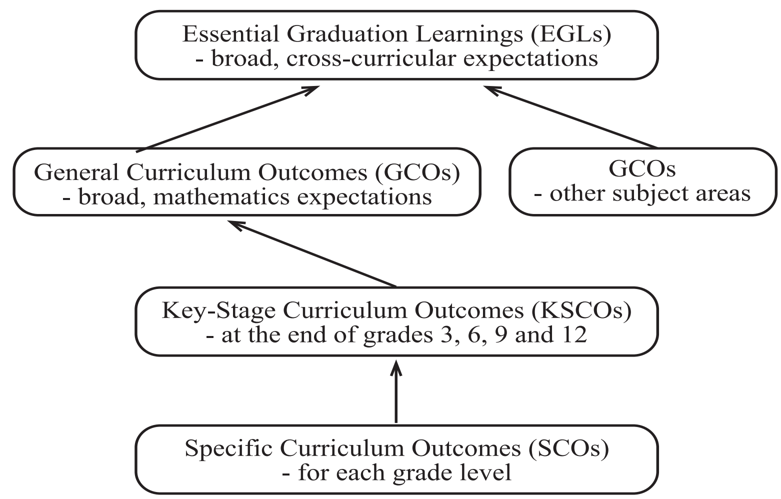


Figure 1: Outcome Framework

This mathematics guide is based upon several key assumptions or beliefs about mathematics learning which have grown out of research and practice, including the following: (i) mathematics learning is an active and constructive process; (ii) learners are individuals who bring a wide range of prior knowledge and experiences, and who learn via various styles and at different rates; (iii) learning is most likely when placed in meaningful contexts and in an environment that supports exploration, risk taking, and critical thinking, and nurtures positive attitudes and sustained effort; (iv) learning is most effective when standards of expectation are made clear and assessment and feedback are ongoing; and (v) learners benefit, both socially and intellectually, from a variety of learning experiences, both independent and in collaboration with others.

II. Program Design and Components

A. Program Organization

As indicated previously, the mathematics curriculum is designed to support the Atlantic Canada Essential Graduation Learnings (EGLs). The curriculum is designed to significantly contribute to students meeting each of the six EGLs, with the communication and problem-solving EGLs relating particularly well with the curriculum's unifying ideas. (See the "Outcomes" section of the *Foundation for the Atlantic Canada Mathematics Curriculum*.) The foundation document then goes on to present student outcomes at key stages of the student's school experience.

This curriculum guide presents specific curriculum outcomes for *Geometry and Applications in Mathematics 111/112*. As illustrated in Figure 2, these outcomes represent the step-by-step means by which students work toward accomplishing the key-stage curriculum outcomes, the general curriculum outcomes, and, ultimately, the essential graduation learnings.

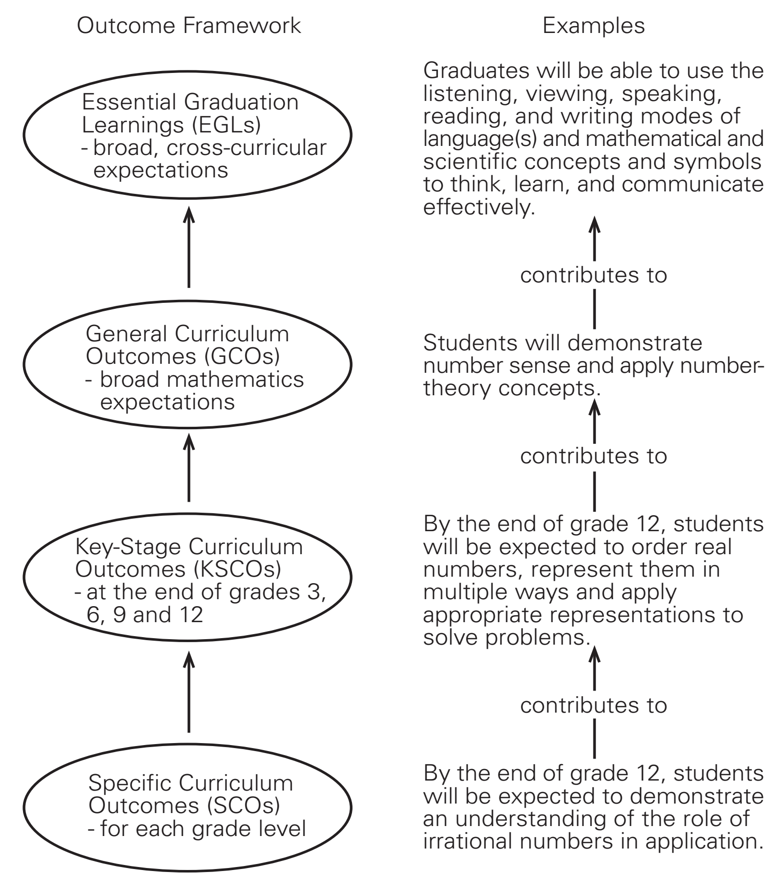


Figure 2: Examples of Outcomes

It is important to emphasize that the initial presentation of the specific curriculum outcomes for this course (pp. 17-32) follows the outcome structure established in the *Foundation for the Atlantic Canada Mathematics Curriculum* and does not represent a natural teaching sequence. In *Geometry and Applications in Mathematics 111/112*, however, a suggested teaching order for specific curriculum outcomes has been given within a sequence of four topics or units (i.e., Statistics; Independent Study; Probability; and Circle Geometry). While the units are presented with a specific teaching sequence in mind, some flexibility exists as to the ordering of units within the course. It is expected that teachers will make individual decisions as to what sequence of topics will best suit their classes. In most instances, this will occur in consultation with fellow staff members, department heads, and/or district level personnel.

Decisions on sequencing will depend on a number of factors, including the nature and interests of the students themselves. For instance, what might serve well as a "kickoff" topic for one group of students might be less effective in that role with a second group. Another consideration with respect to sequencing will be co-ordinating the mathematics program with other aspects of the students' school experience. An example of such co-ordination would be studying aspects of measurement in connection with appropriate topics in science. As well, sequencing could be influenced by events outside of the school, such as elections, special community celebrations, or natural occurrences.

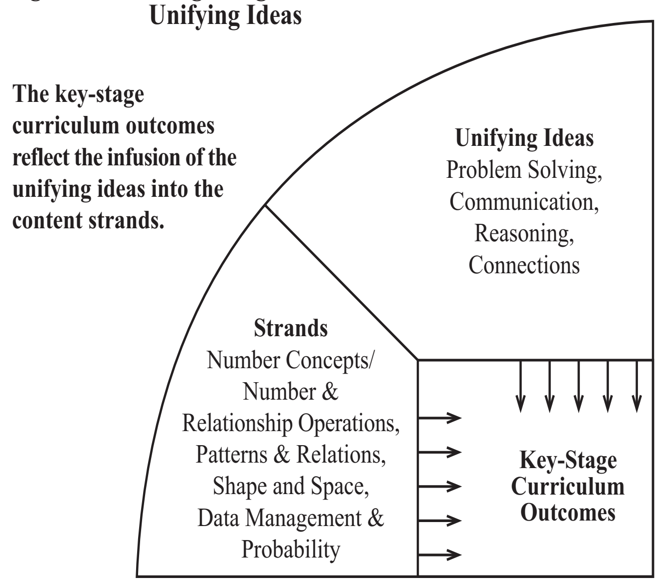
B. Unifying Ideas

The NCTM *Curriculum and Evaluation Standards* (1989) and *Principles and Standards* (2000) establishes mathematical problem solving, communication, reasoning, and connections as central elements of the mathematics curriculum. The *Foundation for the Atlantic Canada Mathematics Curriculum* (pp. 7-11) further emphasizes these unifying ideas and presents them as being integral to all aspects of the curriculum. Indeed, while the general curriculum outcomes are organized around content strands, every opportunity has been taken to infuse the key-stage curriculum outcomes with one or more of the unifying ideas. This is illustrated in Figure 3.

These unifying concepts serve to link the content to methodology. They make it clear that mathematics is to be taught in a problem-solving mode; classroom activities and student assignments must be structured so as to provide opportunities for students to communicate mathematically; via teacher encouragement and questioning, students must explain and clarify their mathematical reasoning; and mathematics with which students are involved on a day-to-day basis must be connected to other mathematics, other disciplines, and/or the world around them.

Students will be expected to address routine and/or non-routine

Figure 3 - Integrating Strands and Unifying Ideas



mathematical problems on a daily basis. Over time, numerous problem-solving strategies should be modelled for students, and students should be encouraged to employ various strategies in many problem-solving situations. While choices with respect to the timing of the introduction of any given strategy will vary, strategies such as try-and-adjust, look for a pattern, draw a picture, act it out, use models, make a table or chart, and make an organized list should all become familiar to students during their early years of schooling, whereas working backward, logical reasoning, trying a simpler problem, changing point of view, and writing an open sentence or equation would be part of a student's repertoire in the later elementary years. During middle school and the 9/10 years, this repertoire will be extended to include such strategies as interpreting formulas, checking for hidden assumptions, examining systematic or critical cases, and solving algebraically. *Geometry and Applications in Mathematics 111/112* will continue to develop students' problem-solving reportoires.

Opportunities should be created frequently to link mathematics and career opportunities. Students need to be aware of the importance of mathematics and the need for mathematics in so many career paths. This realization will help maximize the number of students who strive to develop and maintain the mathematical abilities required for success in higher-level mathematics programming in senior high mathematics and beyond.

C. Learning and Teaching Mathematics

The unifying ideas of the mathematics curriculum suggest quite clearly that the mathematics classroom needs to be one in which students are actively engaged each day in the doing of mathematics. No longer is it sufficient or proper to view mathematics as a set of concepts and algorithms for the teacher to transmit to students. Instead, students must come to see mathematics as a vibrant and useful tool for helping them understand their world, and as a discipline which lends itself to multiple strategies, student innovation, and, quite often, multiple solutions. (See the "Contexts for Learning and Teaching" section of the foundation document.)

The learning environment will be one in which students and teachers make regular use of manipulative materials and technology, and actively participate in discourse and conjecture, verify reasoning, and share solutions. This environment will be one in which respect is given to all ideas in which reasoning and sense making are valued above "getting the right answer." Students will have access to a variety of learning resources, will balance the acquisition of procedural skills with attaining conceptual understanding, will estimate routinely to verify the reasonableness of their work, will compute in a variety of ways while continuing to place emphasis on mental computation skills, and will engage in homework as a useful extension of their classroom experiences.

D. Meeting the Needs of All Learners

The *Foundation for the Atlantic Canada Mathematics Curriculum* stresses the need to deal successfully with a wide variety of equity and diversity issues. Not only must teachers be aware of, and adapt instruction to account for, differences in student readiness, but they must also remain aware of avoiding gender and cultural biases in their teaching. Ideally, every student should find his/her learning opportunities maximized in the mathematics classroom.

NCTM's *Principles and Standards* (2000) cites equity as a core element of its vision for mathematics education. "All students, regardless of their personal characteristics, backgrounds, or physical challenges, must have opportunities to study – and support to learn – mathematics. Equity does not mean that every student should receive identical instruction; instead, it demands that reasonable and appropriate accommodations be made as needed to promote access and attainment for all students"(p. 12).

At grade 11 in New Brunswick, variations in student readiness, aptitude, and post-secondary intentions are addressed in significant part by the provision of courses at levels 1, 2 and 3. Students at all levels will work toward achievement of the same key-stage and general curriculum outcomes, and many of the course-specific curriculum outcomes will also be the same or similar. As well, the instructional environment and philosophy should be the same at all levels, with high expectations maintained for all students. The significant difference between levels will be the depth, breadth, and degree of sophistication and formalism expected with respect to each general outcome. Similarities between courses should allow some students to move from one course level to another.

By and large, Level 3 courses will be characterized by a greater focus on concrete activities, models, and applications, with less emphasis given to formalism, symbolism, computational or symbol-manipulating facility, and mathematical structure. Level 1 and 2 courses will involve greater attention to abstraction and more sophisticated generalizations, while Level 3 courses would see less time spent on complex exercises and connections with advanced mathematical ideas. Level 1 courses, which are designed for particularly talented students of mathematics, will be characterized by both more sophisticated engagement with mathematical concepts and techniques, and the extension of some topics beyond the scope provided at Level 2. These extensions will be included in Level 2 curriculum guides and identified with a *** ** * * ** *** symbol.

By way of a brief illustration, students at all levels should develop an understanding of exponential relationships. Students taking Level 3 courses have as much need as others to understand the nature of exponential relationships, given the central place of these relationships in universal, everyday issues such as investment, personal and government debt, and world population dynamics. The nature of exponential relationships can be developed through concrete, hands-on experiments and data analysis that does not require a lot of formalism or symbol manipulation. The more formal and symbolic operations on exponential relationships will be much more prevalent in Level 1 and 2 courses.

Finally, within any given course at any level, teachers must understand, and design instruction to accommodate, differences in student learning styles. Different instructional modes are clearly appropriate, for example, for those students who are primarily visual learners versus those who learn best by doing. Further, the practice of designing classroom activities to support a variety of learning styles must be extended to the assessment realm; such an extension implies the use of a wide variety of assessment techniques, including journal writing, portfolios, projects, presentations, and structured interviews.

E. Support Resources

This curriculum guide represents the central resource for the teacher of *Geometry and Applications in Mathematics 111/112*. Other resources are ancillary to it. This guide should serve as the focal point for all daily, unit, and course-long planning, as well as a reference point to determine the extent to which the instructional outcomes should be met.

Nevertheless, other resources will be significant in the mathematics classroom. Textual and other print resources will be significant to the extent that they support the curriculum goals. Teachers will need professional resources as they seek to broaden their instructional and mathematical skills. Key among these are the NCTM publications, including the *Principles and Standards for School Mathematics*, *Assessment Standards for School Mathematics*, *Curriculum and Evaluation Standards for School Mathematics*, the *Addenda Series*, *Professional Standards for Teaching Mathematics*, and the various NCTM journals and yearbooks. As well, manipulative materials and appropriate access to technological resources (e.g., software, videos) should be available. Calculators will be an integral part of many learning activities.

F. Role of Parents

Societal change dictates that students' mathematical needs today are in many ways different than were those of their parents. These differences are manifested not only with respect to mathematical content, but also with respect to instructional approach. As a consequence, it is important that educators take every opportunity to discuss with parents changes in mathematical pedagogy and why these changes are significant. Parents who understand the reasons for changes in instruction and assessment will be better able to support their children in mathematical endeavours by fostering positive attitudes towards mathematics, stressing the importance of mathematics in their children's lives, assisting children with mathematical activities at home, and, ultimately, helping to ensure that their children become confident, independent learners of mathematics.

G. Connections Across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating learning experiences—through teacher-directed activities, group or independent exploration, and other opportune learning situations. However, it should be remembered that certain aspects of mathematics are sequential, and need to be developed in the context of structured learning experiences.

The concepts and skills developed in mathematics are applied in many other disciplines. These include science, social studies, music, technology education, art, physical education, and home economics. Efforts should be made to make connections and use examples which apply across a variety of discipline areas.

In science, for example, the concepts and skills of measurement are applied in the context of scientific investigations. Statistical concepts and skills are applied as students collect, present, and analyse data. Examples and applications of many mathematical relations and functions abound.

In social studies, knowledge of confidence intervals is valuable in intrepreting polling data, and an understanding of exponential growth is necessary to appreciate the significance of government debt and population growth. As well, students read, interpret, and construct tables, charts, and graphs in a variety of contexts such as demography.

Opportunities for mathematical connections are also plentiful in physical educaiton, many technological courses and the fine arts.

III. Assessment and Evaluation

A. Assessing Student Learning

Assessment and evaluation are integral to the process of teaching and learning. Ongoing assessment and evaluation are critical, not only with respect to clarifying student achievement and thereby motivating student performance, but also for providing a basis upon which teachers make meaningful instructional decisions. (See "Assessing and Evaluating Student Learning" in the *Foundation for the Atlantic Canada Mathematics Curriculum*.)

Characteristics of good student assessment should include the following: i) using a wide variety of assessment strategies and tools; ii) aligning assessment strategies and tools with the curriculum and instructional techniques; and iii) ensuring fairness both in application and scoring. The *Principles for Fair Student Assessment Practices for Education in Canada* elaborate good assessment practice and serve as a guide with respect to student assessment for the mathematics foundation document. (See also, Appendix A, "Assessing and Evaluating Student Learning.")

B. Program Assessment

Program assessment will serve to provide information to educators as to the relative success of the mathematics curriculum and its implementation. It will address such questions as the following: Are students meeting the curriculum outcomes? Is the curriculum being equitably applied across the region? Does the curriculum reflect a proper balance between procedural knowledge and conceptual understanding? Is technology fulfilling a proper role?

IV. Designing an Instructional Plan

It is important to develop an instructional plan for the duration of the course. Without such a plan, it is easy to run out of time before all aspects of the curriculum have been addressed. A plan for instruction that is comprehensive enough to cover all outcomes and topics will help to highlight the need for time management.

It is often advisable to use pre-testing to determine what students have retained from previous grades relative to a given topic or set of outcomes. In some cases, pre-testing may also identify students who have already acquired skills relevant to the current course. Pre-testing is often most useful when it occurs one to two weeks prior to the start of a a topic or set of outcomes. When the pre-test is done early enough and exposes deficiencies in prerequisite knowledge/ skills for individual students, sufficient time is available to address these deficiencies prior to the start of the topic/unit. When the whole group is identified as having prerequisite deficiencies, it may point to a lack of adequate development or coverage in previous grades. This may imply that an adjustment is required to the starting point for instruction, as well as a meeting with other grade level teachers to address these concerns as necessary.

Many topics in mathematics are also addressed in other disciplines, even though the nature and focus of the desired outcome is different. Whenever possible, it is valuable to connect the related outcomes of various disciplines. This can result in an overall savings in time for both disciplines. The most obvious of these connections relate to the use of measurement in science and the use of a variety of data displays in social studies.

V. Curriculum Outcomes

The pages that follow provide details regarding both specific curriculum outcomes and the four topics/units that comprise *Geometry and Applications in Mathematics 111/112*. The specific curriculum outcomes are presented initially, then the details of the units follow in a series of two-page spreads. (See Figure 4 on next page.)

This guide presents the curriculum for *Geometry and Applications in Mathematics 111/112* so that a teacher may readily view the scope of the outcomes which students are expected to meet during the year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how the students' learnings in this course are part of a bigger picture of concept and skill development. (See Appendix B for a complete listing of the SCOs for grades 9 and 10.)

Within each unit, the specific curriculum outcomes are presented on two-page spreads. At the top of each page, the overarching topic is presented, with the appropriate SCO(s) displayed in the left-hand column. The second column of the layout is entitled "Elaboration-Instructional Strategies/Suggestions" and provides a clarification of the specific curriculum outcome(s), as well as some suggestions of possible strategies and/or activities which might be used to achieve the outcome(s). While the strategies and/or suggestions presented are not intended to be rigidly applied, they will help to further clarify the specific curriculum outcome(s) and to illustrate ways to work toward the outcome(s) while maintaining an emphasis on problem solving, communications, reasoning, and connections. To readily distinguish between activities and instructional strategies, activities are introduced in this column of the layout by the symbol ❒. As well, curriculum extensions intended for students in the Level 1 course are indicated with the *** ** *   * ** *** symbol. This symbol not only brackets text discussing differentiation for students in the Level 1 course, but also appears at the top of each page on which such text is located.

The third column of the two-page spread, "Worthwhile Tasks for Instruction and/or Assessment," might be used for assessment purposes or serve to further clarify the specific curriculum outcome(s). As well, those tasks regularly incorporate one or more of the four unifying ideas of the curriculum. These sample tasks are intended as examples only, and teachers will want to tailor them to meet the needs and interests of the students in their classrooms. The final column of each display is entitled "Suggested Resources" and will, over time, become a collection of useful references to resources which are particularly valuable with respect to achieving the outcome(s).

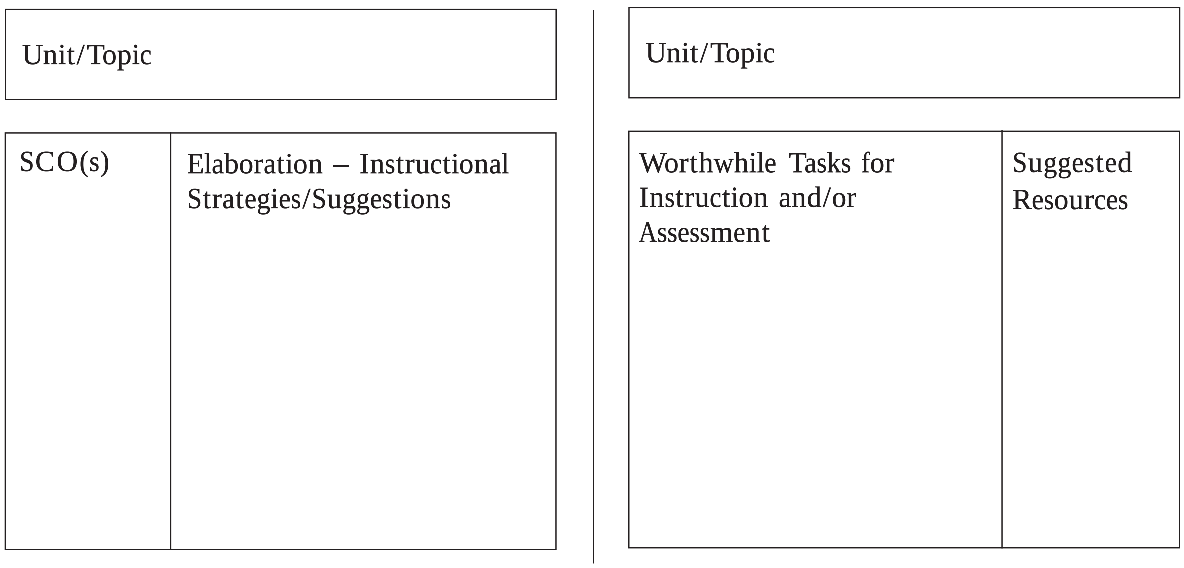


Figure 4: Layout of a 2-Page Spread

Specific Curriculum Outcomes

(by GCO)

GCO A: Students will demonstrate number sense and apply number theory concepts.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

ii) *order real numbers, represent them in multiple ways and apply appropriate representations to solve problems*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

A6 develop an understanding of factorial notation and apply it to calculating permutations and combinations

Elaboration

A6 Students will find it valuable to develop factorial notation in connection with tree diagrams and the fundamental counting principle, and will understand $n!$ to represent the number of possible arrangements of n distinct objects. Factorial notation will be applied when calculating permutations and combinations, in connection with SCOs G7, G8 and B8. Unit 3, pp. 102, 104, 106

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

ii) *derive, analyze and apply computational procedures in situations involving all representations of real numbers*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

B8 determine probabilities using permutations and combinations

Elaboration

B8 Evaluating permutations and combinations is addressed in SCOs G7 and G8. Students apply these counting techniques to the determination of probabilities in B8. Unit 3, p. 106

GCO C: Students will explore, recognize, represent and apply patterns and relationships, both informally and formally.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *iii)* *interpret algebraic equations and inequalities geometrically and geometric relationships algebraically* SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to C20 (111) represent circles using parametric equations	C20(111) This outcome is intended for students in the Level 1 course only. Students will use a system of two parametric equations to represent a circle and, thereby, reach a greater understanding of the means by which technological devices "draw" circles. This outcome will be developed in connection with SCOs C36(111) and C37(111). Unit 4, p. 140
v) *analyze and explain the behaviours, transformations and general properties of types of equations and relations* C36 (111) demonstrate an understanding of the relationship between angle rotation and the coordinates of a rotating point C37 (111) describe and apply parameter changes within parametric equations of circles	C36(111) This outcome is intended for students in the Level 1 course only. Students will need to extend their previous understanding of trigonometric ratios involving acute angles in right triangles to the broader range of values associated with the 360-degree rotation of a point. This understanding will be fundamental to their success with SCOs C20(111) and C37(111). Unit 4, p. 140 C37(111) This outcome is intended for students in the Level 1 course only. Students will do this in connection with, but following, SCOs C36(111) and C20(111). Unit 4, p. 142

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to

iii) *apply measurement formulas and procedures in a wide variety of contexts*

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

D1 develop and apply formulas for distance and midpoint

Elaboration

D1 Within the context of coordinate geometry, students will develop formulas for distance and midpoint. They will apply these formulas in analytical proofs (see SCOs E11 and E4) and when developing and applying the equation of a circle (E15).

Unit 4, pp. 128, 130, 136

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *iii) analyze and apply Euclidean transformations, including representing and applying translations as vectors*	
SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to E3 write the equations of circles and ellipses in transformational form and as mapping rules to visualize and sketch graphs	E3 Writing equations of circles and ellipses in transformational form and as mapping rules facilitates the identification of key characteristics which describe their shapes and simplify sketching. Students will need to master the technique of completing the square to rewrite equations from general form (i.e., $Ax^2 + By^2 + Cx + Dy + E = 0$) into transformational form (or standard form). (Note: Facility with completing the square will serve students well in work with quadratics in later courses.) Students will develop these skills in conjunction with work on SCOs E13, E14, E15 and E16. Unit 4, p. 138
iv) represent problem situations with geometric models and apply properties of figures	
E4 apply properties of circles	E4 Students will not only learn numerous properties of circles, but will also apply them to practical situations and to situations involving both Euclidean and analytical proofs. (See, for example, related SCOs D1, E11 and E15.) Unit 4, pp. 118, 120, 124, 134, 136
v) make and test conjectures about, and deduce properties of and relationships between, 2- and 3-D figures in multiple contexts	
E5 apply inductive reasoning to make conjectures in geometric situations	E5 Mathematical reasoning is one of the unifying ideas in mathematics teaching and learning. Here, students will explore geometric situations (e.g., through paper folding) and make conjectures regarding geometric properties which they observe. This inductive process will precede proving and applying theorems deductively. Unit 4, pp. 118, 120, 128

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

	Elaboration
E7 investigate and make and prove conjectures associated with chord properties of circles	E7 Students will make conjectures regarding the properties of chords in circles, prove them, and apply them in both Euclidean and analytical situations. (See also SCOs E5, E11 and E15.) Also, students will examine the concept of converse (E12) within the context of chord properties. Unit 4, pp. 118, 120, 122, 126, 138
E8 investigate and make and prove conjectures associated with angle relationships in circles	E8 Angles within circles (e.g., central angles, inscribed angles) and their relationships provide a context for students to build inductive and deductive reasoning skills. Unit 4, pp. 118, 122, 132
E9 investigate and make and prove conjectures associated with tangent properties of circles	E9 Tangent properties of circles provide a context for students to build inductive and deductive reasoning skills. Unit 4, pp. 118, 132
vi) demonstrate an understanding of the operation of axiomatic systems and the connections among reasoning, justification and proof	
E11 write proofs using various axiomatic systems and assess the validity of deductive arguments	E11 Mathematical reasoning is one of the unifying ideas in mathematics teaching and learning. Students will be exposed to Euclidean, transformational and analytical systems of proof, and will write and assess both synthetic (e.g., Euclidean) and analytical (i.e., coordinate) proofs. Unit 4, pp. 118, 120, 124, 126, 128, 130, 132, 138
E12 demonstrate an understanding of the concept of converse	E12 Students will explore the concept of converse in the context provided by SCO E7. Students should understand that only some theorems have converses which are true. Also, students should understand the use of "if and only if" terminology in situations in which a theorem and its converse are both true. Unit 4, pp. 118, 120, 122

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties and relationships.

	Elaboration
viii) explore and apply, using multiple representations, circles, ellipses and parabolas and, in 3-D, spheres and ellipsoids	
E13 analyze and translate between symbolic, graphic, and written representations of circles and ellipses	E13 Students will translate between various representations of circles and ellipses as needed to best explore their characteristics and solve problems. Facility with these translations will be developed in connection with SCOs E3, E14, E15 and E16. Unit 4, p. 138
E14 translate between different forms of equations of circles and ellipses	E14 Students will translate among transformational, standard and general forms of circles and ellipses. Developing these skills will be closely associated with SCO E3. Unit 4, p. 138
E15 solve problems involving the equations and characteristics of circles and ellipses	E15 Students will use the knowledge and skills developed around SCOs E3, E13, E14 and E16 to solve problems (both contextual and analytical) involving circles and ellipses. Unit 4, pp. 134, 136, 138
E16 demonstrate an understanding of the transformational relationship between a circle and an ellipse	E16 Students will visualize an ellipse as a circle stretched either horizontally or vertically. Mastery of the algebraic representation of stretches will be built upon students' experiences with stretches in quadratic contexts in Year 10. E16 will be addressed in conjunction with E3, E13, E14 and E15. Unit 4, p. 138

GCO F: Students will solve problems involving the collection, display and analysis of data.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to *i) understand sampling issues and their role with respect to statistical claims*	
SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to F1 draw inferences about a population from a sample	F1 Since it is often impractical to gather information about entire populations, sampling is a common statistical technique. Students will need to understand issues with respect to sampling strategies and sample size in order to properly draw inferences from sample data. These issues are addressed in SCOs F2 and F4. Unit 1, pp. 48, 52
F2 identify bias in data collection, interpretation and presentation	F2 Students should understand that some sampling methods (e.g., convenience, self-selected) may produce samples that are not representative of the population as a whole. They will need to identify the bias that can enter the interpretation of results based on such sampling. Unit 1, p. 48
F4 demonstrate an understanding of how the sample size affects the variation in sample results	F4 By conducting experiments/simulations and examining the data collected, students should understand that larger sample sizes increase the likelihood that the statistical results will approximate expected values or population characteristics. This outcome will be addressed in connection with SCO F15. Unit 1, pp. 52, 54, 68, 70
iv) determine, interpret and apply as appropriate a wide variety of statistical measures and distributions	
F7 draw inferences from graphs, tables and reports	F7 Students will draw inferences based on such information as data distribution (e.g., normal vs. skewed) and data variability (e.g., range). This outcome will be addressed in connection with SCOs F10 and F15. Unit 1, pp. 58, 60, 62, 72, 74
F8 apply characteristics of normal distributions	F8 Students will connect previous learnings regarding normal distributions and standard deviation to 95% confidence intervals as determined by formula. Unit 1, p. 44, 50, 54, 58, 60, 70

GCO F: Students will solve problems involving the collection, display and analysis of data.

	Elaboration
F9 demonstrate an understanding of the difference between sample standard deviation and population standard deviation	F9 (Note: This outcome falls within the optional portion of the statistics unit and will only be addressed if the optional portion is included in students' studies.) Students will extend their previous knowledge of (50%) box plots to the construction, interpretation and application of 90% box plots. This will occur in conjunction with SCOs $G1_2$ and $G3_2$. Unit 1, pp. 38
F10 interpret and apply histograms	F10 In connection with SCOs $G3_2$ and F7, students will construct and interpret histograms and frequency and probability bar graphs, and recognize conditions under which they begin to resemble normal distributions. Unit 1, pp. 34, 36, 40, 42
F11 determine, interpret and apply confidence intervals	F11 Students will determine confidence intervals by formula, and interpret and apply the results. (Note: Optionally, students may also determine confidence intervals using 90% box plots.) In conjunction with SCO F8, confidence intervals will also be connected to normal distributions and standard deviation. Unit 1, pp. 56, 58, 60, 62, 72, 74
v) design and conduct relevant statistical experiments and analyze and communicate the results using a range of statistical arguments	
F14 (111) distinguish between descriptive and inferential statistics	F14(111) This outcome is intended for students in the Level 1 course only. In connection with SCO F18 (111), students will formulate and test hypotheses. The chi-square statistic (see F19(111) and F20(111)) will be introduced as one means of hypothesis testing. Unit 1, p. 46
F15 design and conduct surveys and/or simulate data collection to explore sampling variability	F15 By designing and conducting experiments/simulations and examining the data collected, students will understand the impact of sampling issues on statistical results. This outcome will be addressed in connection with SCO F4. Unit 1, pp. 40, 42, 50, 66
F16 (111) demonstrate an understanding of the difference between situations that involve a binomial experiment and those that do not	F16 Students will understand that the type of questions asked, and the type of sampling conducted, will influence the presentation and interpretation of results. Unit 1, p. 64, 66

GCO F: Students will solve problems involving the collection, display and analysis of data.

	Elaboration
vi) test hypotheses using appropriate statistics	
F18 (adv) identity the characteristics of a binomial experiment	F18(adv) This outcome is intended for students in the Level 1 course only, and will be addressed in connection with SCOs F11 and F19(adv). Unit 1, pp. 64
F19 (adv) demonstrate an understanding of the differences in the quality of sampling methods	F19(adv) This outcome is intended for students in the Level 1 course only. Students will apply the chi-square statistic to test hypotheses. This outcome will be addressed in conjunction with SCOs F20(adv) and $G2_2$(adv). Unit 1, pp. 48
F20 (adv) demonstrate an understanding of how a confidence level affects a confidence interval	F20(adv) This outcome is intended for students in the Level 1 course only. Students will use the chi-square statistic to test a null hypothesis with respect to two populations, in addition to comparing a population to a theoretical model. Unit 1, pp. 56, 58, 60, 72, 74
F21 demonstrate an understanding of how a confidence level affects a confidence interval	Unit 1, pp. 58, 60
F22 demonstrate an understanding of how a confidence level affects a confidence interval	Unit 1, pp. 58, 60, 62

GCO G: Students will represent and solve problems involving uncertainty.

	Elaboration
KSCO: By the end of grade 12, students will have achieved the outcomes for entry-grade 9 and will also be expected to i) *design and conduct experiments a/o simulations to model and solve a wide variety of relevant probability problems, and interpret and judge the probabilistic arguments of others* SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to $G1_3$ develop and apply simulations to solve problems	$G1_3$ In general, students will design simulations to model problem situations and use them to solve the problems. Further, students in the Level 1 course will use simulations to model binomial trials. (See also SCO G11(111).) Unit 3, pp. 98, 110
ii) *build and apply formal concepts and techniques of theoretical probability*	
$G2_3$ demonstrate an understanding that determining probability requires the quantifying of outcomes	$G2_3$ Students will understand that determining probabilities requires counting both the number of possible outcomes and the number of successful outcomes. In this context, tree diagrams, area models, the fundamental counting principle, and permutations and combinations are all means of counting. Unit 3, p. 88
$G3_3$ demonstrate an understanding of the fundamental counting principle and apply it to calculate probabilities of dependent and independent events	$G3_3$ Students will understand that, if an event A can occur in n ways and an unrelated event B in m ways, then A and then B can occur in n x m ways. They will also understand how to apply the principle when events are related to (dependent upon) one another. Unit 3, pp. 88, 90
G4 apply area diagrams and tree diagrams to interpret and determine probabilities of independent and dependent events	G4 Students will use diagrams to assist with quantifying outcomes and determining probabilities. Unit 3, p. 92

GCO G: Students will represent and solve problems involving uncertainty.

	Elaboration
G5 (111) determine conditional probability	G5(111) This outcome is intended for students in the Level 1 course only. Students will determine conditional probabilities both by counting techniques and by formula. Unit 3, pp. 94, 96
G7 distinguish between situations that involve combinations and permutations	G7 Students will distinguish between situations involving unordered collections of objects (i.e., combinations) and those involving an ordering of objects (i.e., permutations). Making these distinctions will be critical with respect to calculating permutations and combinations (SCO G8) and determining probabilities involving permutations and combinations (B8). Unit 3, pp. 100, 104
G8 develop and apply formulae to evaluate permutations and combinations	G8 By exploring situations involving permutations and combinations, students will develop and apply formulae to calculate them and determine probabilities involving them. This outcome will be addressed in conjunction with SCOs A6, G7 and B8. Unit 3, pp. 94, 104, 106

GCO G: Students will represent and solve problems involving uncertainty.

Elaboration

iv) relate probability and statistical situations

G9 demonstrate an understanding of binomial expansion and its connection to combinations

G9 Students will understand and apply the pattern of exponents in the binomial expansion, as well as the connection between the coefficients and combinations. This outcome will be addressed in connection with SCO G10. Unit 3, p. 108

G10 connect Pascal's Triangle with combinatorial coefficients

G10 Students will generate Pascal's triangle and connect its entries with combinatorial coefficients. Unit 3, p. 108

G11 (111) connect binomial expansions, combinations, and the probability of binomial trials

G11(111) This outcome is intended for students in the Level 1 course only. Repeated trials of experiments with only two possible outcomes are called binomial trials. Students will connect combinations and probability with respect to binomial trials. This outcome will be addressed in conjunction with SCOs $G1_3$ and G12(111). Unit 3, pp. 110, 114

G12 (111) demonstrate an understanding of and solve problems using random variables and binomial distributions

G12 Students will understand random variables and solve problems involving binomial distributions. Unit 3, pp. 112, 114

Independent Study

Elaboration

Further to the topics that involve SCOs which contribute to students achieving the general curriculum outcomes, an Independent Study unit is included in *Geometry and Applications in Mathematics 111/112*. This unit is intended to assist in the development of students' independent learning skills while allowing them to explore topics outside the prescribed curriculum. While the following SCOs fall outside the framework of GCOs A through G, they are to be considered integral to the curriculum.

SCO: By the end of *Geometry and Applications in Mathematics 111/112*, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I1 Students will research mathematical topics outside the prescribed curriculum and demonstrate an understanding of their topics by sharing the results of their research with their peers. (See also SCO I2.) Unit 2, pp. 74, 76, 78, 80

I2 communicate the results of the independent research

I2 Students will present the results of their research to their peers in one of a number of possible formats. Note that the presentation of results will be intimately connected to the achievement of SCO I3. Unit 2, pp. 74, 76, 78, 80

I3 demonstrate an understanding of the mathematical topics presented by other students

I3 Students will provide information in one or more of a number of formats to demonstrate that they have learned from the presentations of their peers. This outcome will be achieved in connection with SCO I2. Unit 2, pp. 78, 80

Unit 1
Statistics

(15-20 Hours)

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F10 interpret and apply histograms

Elaboration – Instructional Strategies/Suggestions

F10 Students in this course should begin their study of statistics by reviewing ways of organizing and describing data. They should construct histograms (both with and without technology) and frequency polygons. When raw data is presented, students should be able to calculate the mean and median of the data. When a histogram is presented without the raw data, it is often not possible to calculate the mean and median precisely. When this occurs, students should be able to estimate the mean and the median of the data based on the histogram. Students should use frequency polygons to help them focus on the shape and spread (or dispersion) of the histogram. They will also measure and interpret spread using standard deviation. Students should be able to determine whether or not data appears to be normally distributed based on a histogram and frequency polygon of the data.

Students should take some time to examine various graphs and situations to describe the information presented. For example, they might determine the percentage of data that lies above or below a certain value.

Worthwhile Tasks for Instruction and/or Assessment

F10 *Activity*

F10.1 The Canadian Kennel Association did a study of litter size amongst certain breeds of dogs. Below are the results of the litter sizes last year of one particular breed, in one particular province:

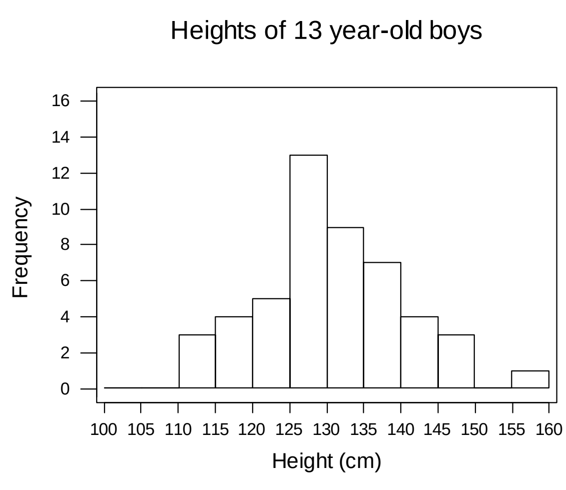
3 6 5 6 5 5 7 5 7 6 6 6 4 6 5 6 4 3 5 6 8 3 6 9 2 5 7 9

4 5 8 2 5 7 6 6 7 7 8 4 5 6 5 9 10 3 4 5 11 2 7 9 5 3 7

6 7 4 5 8 9 3 6 5 7 5 6 4 8 7 3 9 10 4 5 12 5 8 9 5 6 8

a) Determine the mean and the median of the set of data.
b) Construct a histogram with bin-width 1 and a frequency polygon for this data.
c) Describe the shape of the frequency polygon.
d) What proportion of the litters had more than 6 babies? Between 3 and 8 babies?
e) What proportion of observations are at most 8 babies?
f) What proportion of litters contain between 5 and 10 (inclusive) offspring?

F10.2 The histogram below shows the heights of 13 year-old boys in a large urban school.



a) Draw the frequency polygon and describe its shape.
b) How many 13 year-old boys are in this school?
c) What do you think is the mean height of 13 year-old boys in this school?
d) What do you think is the median height of 13 year-old boys in this school?
e) What percentage of boys is between 130cm and 150cm?
f) What percentage of boys is less than 140cm?
g) Give some examples of heights that are not typical of this group, and explain why you chose the heights you did.

(continued on the next 2 page spread....)

Suggested Resources

Statistics (revised)

Outcomes

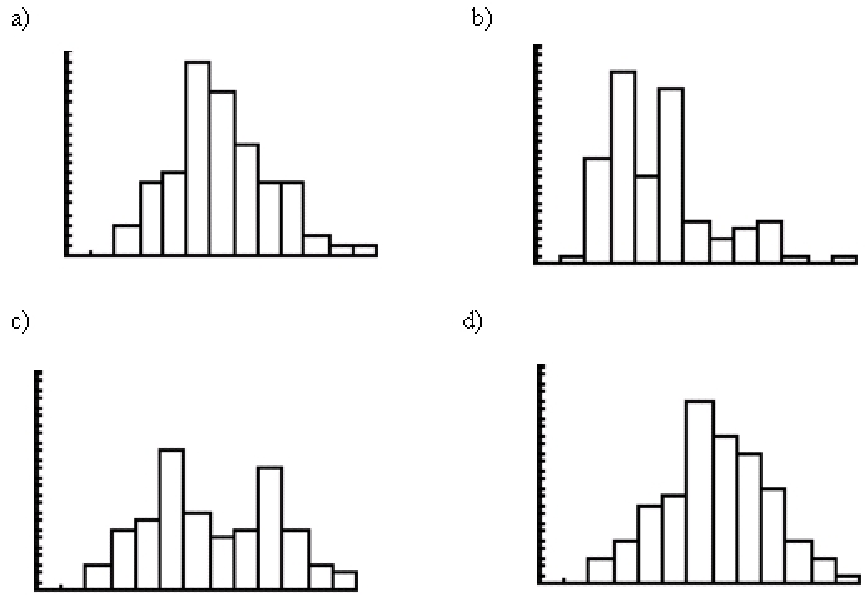
SCO: In this course, students will be expected to

F10 interpret and apply histograms

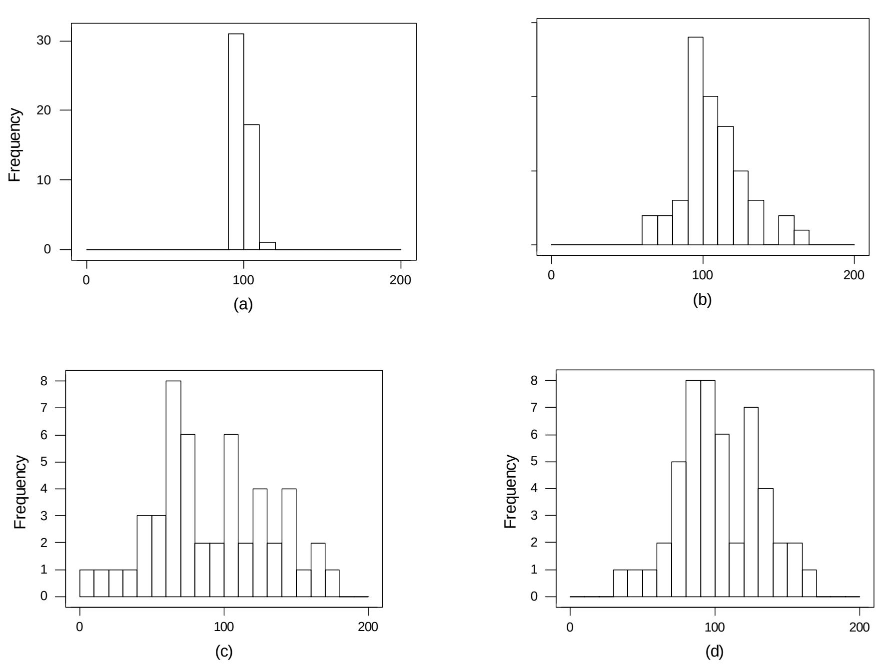
Worthwhile Tasks for Instruction and/or Assessment

(...continued from the previous 2 page spread.)

F10.3 Which histograms appear to represent data that is approximately normally distributed? Explain your answer for each:



F10.4 Which of the following histograms has the largest standard deviation? the smallest? Explain your choices.



Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F9 demonstrate an understanding of the difference between sample standard deviation and population standard deviation

Elaboration – Instructional Strategies/Suggestions

F9 Before students can understand the difference between population standard deviation (σ_x) and sample standard deviation (S_x), they must first understand the difference between a population and a sample, as well as the difference between the sample mean and population mean. A population is the set of all the possible outcomes of interest to the investigator. A sample is a subset of the population. For example, considering all the Grade 11 students in a particular school would represent the population of Grade 11 students in that school. If we put all the names of the Grade 11 students in a hat and selected a subset of students by selecting names at random, we would have a sample. Further, if we are interested, for example, in the mean shoe size of Grade 11 students in this school, calculating the mean based on the entire population data would be called the population mean, μ. Calculating the mean based on the sample data would be called the sample mean, $\overline{x}$. One important note is that we often find entire populations too large to work with for a number of reasons such as cost and time; therefore we use random samples which will hopefully be reflective of the entire population. If these samples are large enough (see Elaboration for F8/G3/F4 for discussion of sample sizes), we expect the value of the sample mean to be very close to the value of the population mean.

The calculation of standard deviation also depends on whether we are dealing with a population or a sample. If a set of data represents a population, then:

$$\sigma_x = \sqrt{\frac{(x_1-\mu)^2+(x_2-\mu)^2+\ldots+(x_n-\mu)^2}{n}}$$, where $x_1, x_2, \ldots, x_n$ represent the data in the population.

However, when we are given a sample, we cannot find the value of the population standard deviation because we do not have the data set for the entire population. Therefore, we must use a value computed from the sample that is a good estimator of σ_x. We use a divisor of $n-1$ rather than n because, on average, the resulting value tends to be a bit closer to the true value of σ_x. That is:

$$S_x = \sqrt{\frac{\left(x_1-\overline{x}\right)^2+\left(x_2-\overline{x}\right)^2+\ldots+\left(x_n-\overline{x}\right)^2}{n-1}}$$, where $x_1, x_2, \ldots, x_n$ represent the data in the sample.

The mathematical verification of this fact is beyond the scope of this course.

Students should be able to differentiate between when to use σ_x and S_x. While students should know how to calculate standard deviation both with and without technology, they should use technology for large data sets as the calculation of standard deviation can be quite lengthy. Note that the TI-83 Plus will calculate both standard deviations.

As well, students should be able to use the mathematical symbols involved in describing the measurements in data:

- sample mean ($\overline{x}$)
- population mean (μ)
- sample standard deviation (S_x)
- population standard deviation (σ_x).

(Note: There is no difference in the calculation of $\overline{x}$ and μ for *finite* populations.)

Statistics (revised)

Worthwhile Tasks for Instruction and/or Assessment

F9 *Activity*

F9.1. The senate at a university with 12 000 students is interested in knowing how many hours per week students spend studying outside of class. Two hundred fifty students were selected at random and asked, "How many hours per week do you spend studying?"

a) What is the population of interest?
b) What group constitutes the sample?

F9.2. A biologist is researching bats and their ability to detect insects within close proximity of them. The following data represents the distances (cm) at which 11 different bats first detected a nearby insect:

62 23 27 56 52 34 42 40 68 45 83

a) Is this a sample or a population? Explain.
b) Without using the statistical capabilities of your calculator, determine the standard deviation.
c) Check your answer to (b) using technology.

F9.3. Mr. Sweeney has a small class of 15 math students. He is interested in knowing the mean and standard deviation of the class on their quiz. Their marks are given below.

82 76 65 78 81 90 52 93 50 89 70 85 59 60 52

a) Explain why this group represents a population and not a sample.
b) Calculate the mean and standard deviation of this data.
c) Create a set of data that has a smaller standard deviation and explain how you created it.

F9.4. The mathematics department at a small university expects all students enrolling in Mathematics 1000 to take an entry test. The exam is marked out of 20. The following shows the results of all 40 students who wrote the exam.

9 12 9 10 15 12 11 12 10 17 7 12 12 12 14 10 11 13 5 12
12 12 11 9 12 8 14 14 11 9 16 12 11 13 14 15 12 11 11 14

a) Construct a histogram and a frequency polygon.
b) Does the distribution appear roughly "bell-shaped"? Explain.
c) Does this data set represent a sample or a population? Explain
d) Using technology, find the mean and standard deviation.
e) How many scores fall within one standard deviation of the mean? Express this as a percentage of all the scores.

Suggested Resources

See the following website for more discussion on the use of n versus $n-1$ in the standard deviation formulas:

http://herkimershideaway.org /writings/unbiae.htm

See the following website for a sample experiment using the standard deviation formulas:

http://mathcentral.uregina.ca/QQ /database/QQ.09.99/freeman2.html

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

A3 demonstrate an understanding of the application of random numbers to statistical sampling

F15 design and conduct surveys and/or simulate data collection to explore variability

F10 interpret and apply histograms

Elaboration – Instructional Strategies/Suggestions

A3/F15/F10 Students should be introduced to creating data through simulations. A simulation is the imitation of chance behaviour, based on a model that accurately reflects the experiment under consideration. The data generated in a simulation is always collected in a random fashion. Flipping a coin, drawing names from a hat, and utilizing random number generators are common methods used to generate data for simulations. For example, if we wanted to predict the average number of female children in a family with 4 children, we could use a coin. We could let "flipping a head" represent a male child, and "flipping a tail" represent a female child. Then, we could flip the coin 4 times and count the number of tails. This would represent the number of female children in that family. If we repeated this process 100 times and found the average number of females per family, we would have conducted a valid simulation.

Likewise, we could use a random number generator. We could randomly generate an integer from 0 to 4 (inclusive) from an appropriate probability distribution to represent the number of female children in a family with four children. Generating 100 such numbers should provide us with a large enough data set to accurately predict the average number of female children in a family with 4 children.

Students should design and conduct simulations to model real world phenomena. As well, students should realize that, because their data sets are randomly generated, they will usually generate data sets that are different from those of other students. They should organize their data into graphs and look for patterns that emerge.

It is also important that students distinguish between a simulation and drawing a sample from an actual population. In the example of female children above, the random number generator can be used to *imitate* surveying various families at random and recording the number of female children in each family. On the other hand, if we wished to take a sample of actual families and record the number of female children in each family, we might assign a number to each family in the population of interest and use a random number generator to select the families for the sample.

F15 When designing a simulation or survey, students should keep in mind that data must be created (in the case of a simulation) or collected (in the case of a survey) in a random fashion. Assumptions must also be stated, a trial must be defined, and the survey or simulation described. When randomly generating an integer from 0 to 4 (inclusive) to represent the number of female children in a family with four children, this would represent one trial of the experiment. This procedure would have to be repeated a large enough number of times to obtain a suitably reliable *simulated* sample. If each outcome from 0 to 4 in a trial is equally probable, then the assumption has been made that in reality each possible number of female children is equally probable and that there have been no outside factors that might influence the number of female children in a particular family.

Worthwhile Tasks for Instruction and/or Assessment

A3/F15/F10 *Activity*

A3/F15.1. Mytown High School produced 150 graduates last year. As part of a curriculum review, the government would like to ask 20 graduates of this school their perception of the value of the mathematics curriculum. Describe a method that the government might use to randomly select 20 students.

A3/F15.2. In a psychological study, subjects are shown 10 different images and asked to select one at random. There are a total of 100 subjects. Design a simulation that models this situation.

A3/F15/F10.1. A certain game of chance is based on randomly selecting three numbers from 0 to 99 and adding the numbers. A person wins if the resulting sum is a multiple of 5.

a) Using technology, design and conduct a simulation that models this situation for 30 games. Record the number of wins.
b) Repeat part (a) 5 times. You should now have six numbers, with each number representing the number of wins in 30 games.
c) Collect results from other students until you have a data set of at least 50 numbers. Construct a histogram.
d) Does the histogram appear bell-shaped? Explain.

A3/F15/F10.2. Ask students to work in pairs using measuring tapes to measure and record their heights. Then ask students to measure the distance between their finger tips (which are pointing in opposite directions) when they hold their arms outstretched parallel to the floor.

a) Have students compute the ratio of their arm span to their height.
b) Ask students to create a distribution of the class results.
c) Ask students if the distribution is bell-shaped. Have them explain why they think their answer is correct. If students collected more ratios, what would be the effect on the distribution?
d) Calculate the mean and standard deviation.
e) Using their results, ask students to determine what percentage of all students in the school would have a ratio below one. Ask them why their sample may not be indicative of the population.

(Continued on the next two-page spread...)

Suggested Resources

There is a free application (APP) for the TI-83 Plus called *ProbSim* that performs simulations with dice, spinners, coins, cards, and random numbers. It is available at www.education.ti.com.

There is a free program for Windows-based computers called *Winstats*. This program:

- performs simulation
- creates histograms, scatter plots, and boxplots
- calculates confidence intervals and many other advanced statistical computations

It is available at http://math.exeter.edu/rparris

Some Winstats tutorials may be found at http://www.gov.nf.ca/edu/sp/mathres/softwaretutorials/softwaretutorials.htm

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

A3 demonstrate an understanding of the application of random numbers to statistical sampling

F15 design and conduct surveys and/or simulate data collection to explore variability

F10 interpret and apply histograms

Elaboration – Instructional Strategies/Suggestions

Worthwhile Tasks for Instruction and/or Assessment

(...Continued from previous two-page spread)

A3/F15/F10.3. In this activity, students will explore random sampling using letters from a Scrabble® game. Each group will need one complete set of letters.

a) Make a histogram of the point values for the population of the tiles. (Blank tiles represent zero points.)
b) Is the histogram bell-shaped? Explain.
c) Find μ .
d) Randomly select 30 tiles, without replacement, from the Scrabble® bag and find the mean score of the sample. After you have your sample of 30 collected, put the tiles back in the bag.
f) Repeat part (d) until you have at least 10 means.
g) Collect other means from your classmates until you have 100 means. Create a histogram of your data collected in part (f). Is the histogram bell-shaped? Is this surprising? Why or why not? How does it compare to the histogram in part a)?
h) Find the mean of your data collected in part (f). How does it compare with the population mean?

Suggested Resources

Statistics (revised)

Outcomes
SCO: In this course, students will be expected to

F8 apply characteristics of normal distributions

Elaboration – Instructional Strategies/Suggestions

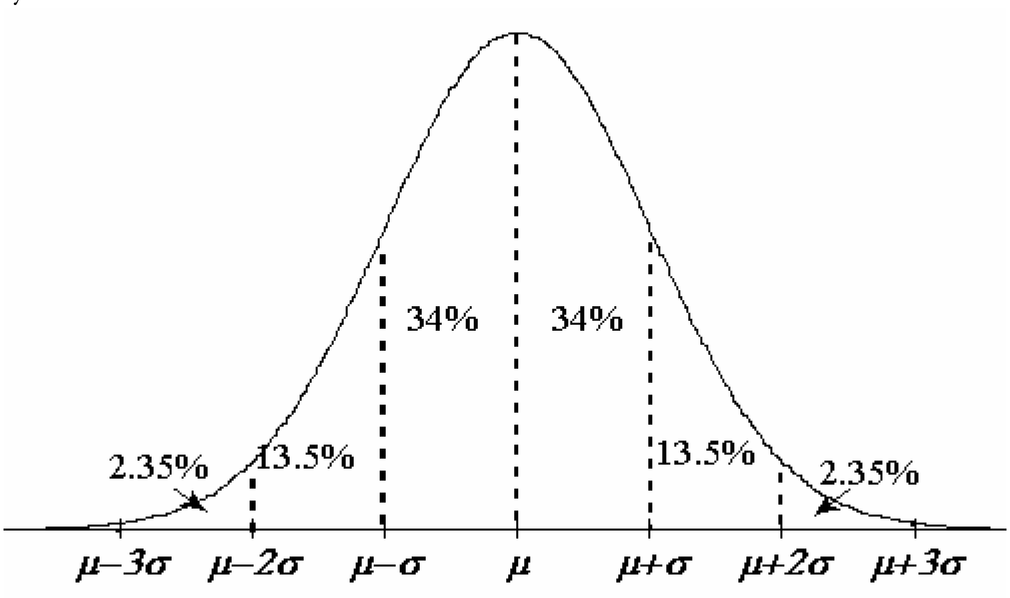
Students have studied properties of the normal distribution in a previous course. These properties should be reviewed. Students should know that a normal distribution:

- is "bell-shaped";
- has a mean that is equal to the median;
- is symmetric and centred about the mean and median;
- has most of its values clustered about the mean

Students should also know and understand the 68%-95%-99.7% rule:

- approximately 68% of the measurements are within 1 standard deviation of the mean
- approximately 95% of the measurements are within 2 standard deviations of the mean (for later work with confidence intervals, students will use the more precise value of 1.96 standard deviations from the mean)
- approximately 99.7% of the measurements are within 3 standard deviations of the mean

Students should be able to recognize a data set as being 'approximately' normal when given a histogram. They should explore visual representations of different normal distributions and apply their knowledge of the normal distribution to real-world scenarios. They should explore the symmetric nature of the normal distribution:



These percentages represent the approximate percentages of the members of the population falling within each interval, and should be related back to work on frequency polygons and calculating percentages of population or sample members lying within certain classes on a histogram e.g. refer to F10.1 and F10.2.

Note: In the diagram, μ and σ may be replaced with $\bar{x}$ and S_x if dealing with a sample that is normally distributed.

Worthwhile Tasks for Instruction and/or Assessment

F8 *Activity*

F8.1. A company is manufacturing hair dryers. The lifespan of a hairdryer is normally distributed with a mean of 6.5 years and a standard deviation of 1.5 years.

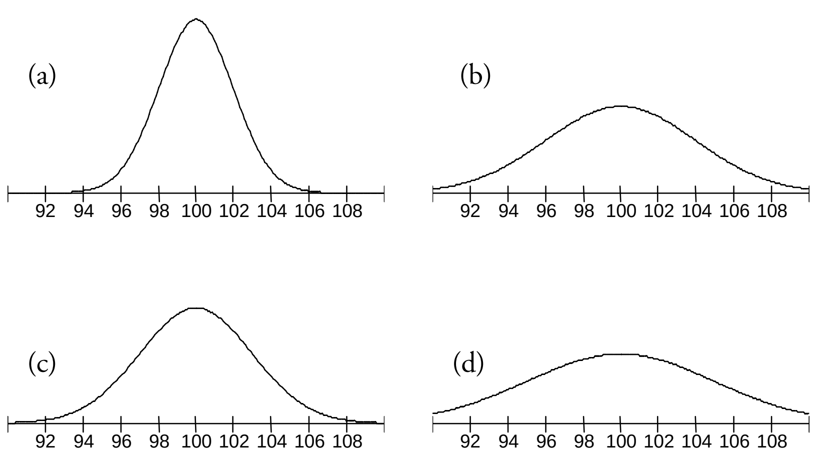
a) Draw a visual representation of this situation.
b) What percentage of hair dryers is expected to last between 5 and 8 years?
c) What percentage of hair dryers is expected to last between 3.5 and 8 years?
d) What percentage of hair dryers is expected to last more than 8 years?
e) The company decides to offer a one year warranty. Does this seem like a good idea? Explain.

F8.2. A popular restaurant in Halifax does not take reservations, so there is usually a waiting time before customers are seated. The following data shows the waiting time (in minutes) of 100 randomly selected customers:

18	18	21	18	10	14	19	14	18	20	17	25	20	23	12	17	21	19	22	18
9	11	14	19	9	17	19	10	20	15	15	14	19	15	14	17	14	25	20	17
15	21	14	26	14	6	17	17	20	19	27	11	13	19	18	16	20	24	19	10
9	17	24	17	16	12	14	18	18	18	13	15	18	17	15	25	17	20	12	22
23	14	26	20	23	22	24	15	16	18	21	22	12	20	16	21	27	19	18	25

a) Is this a sample or a population? Explain.
b) Construct a histogram of the data. Does the data appear normally distributed? Explain.
c) Using technology, calculate the mean and standard deviation of the data.
d) What percentage of data lies within 1 standard deviation of the mean? 2 standard deviations? Do your percentages suggest the data is normally distributed? Explain.

F8.3. Which normal distribution curve has the largest standard deviation? the smallest? Explain.



Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F14 (111) distinguish between descriptive and inferential statistics

Elaboration – Instructional Strategies/Suggestions

Data, or numerical statistics derived from data, are essential for making many kinds of decisions in our lives. Statistical displays like bar graphs, stem-and-leaf plots, box plots, and histograms, are ways we organize and present information. Descriptive statistics deals with organizing and summarizing data for effective presentation and for increased understanding of trends in the data.

Often the individuals or objects that are being studied are a sample that comes from a much larger population. The investigator or researcher may be interested in more information than just data summarization. Inferential statistics involves generalizing from a sample to the population from which it was selected. This often involves some risk since some conclusions about the population might be reached on the basis of available, but incomplete information. The sample may possibly be unrepresentative of the population from which it came. So, an important part of the development of inference techniques would involve quantification of associated risks.

For example, at most universities students are able to register for classes using a telephone. To assess the effectiveness of this system, one university has developed a set of questions to ask a sample of 150 randomly selected students who have used the system. This will result in a database of information. To make sense of the data and to describe student responses, it is desirable to summarize the data using various graphical displays. This would also make the results more accessible to others. In addition, inferential methods could be employed to draw various conclusions about the experiences of all students who used the system.

Some work on histograms from outcome F10 may be used as examples of descriptive statistics.

Worthwhile Tasks for Instruction and/or Assessment

F14 *Activity*

F14.1. The following data shows the number of shots on net that a minor hockey league team has taken in its last 40 games:

27 23 23 24 30 13 22 26 22 27 22 26 28 22 29 21 23 20 23 24
26 27 17 24 25 26 18 23 23 22 21 31 20 16 35 21 33 19 22 28

a) Using technology, calculate the mean, median, and standard deviation of the data.
b) Construct a histogram of the above data.
c) If you had to write a report to the team's general manager describing these data, what would you say?
d) If you were asked to predict how many shots you thought the team would take in the next game, what would be your response? Explain.
e) A friend tells you he predicts the team will take at least 30 shots on net in their next game. What would be your response? Why?
f) In which of the above questions were you using descriptive statistics? inferential statistics? Explain.

Suggested Resources

Statistics (revised)

Outcomes
SCO: In this course, students will be expected to

A3 demonstrate an understanding of the application of random numbers to statistical sampling

F2 identify bias in data collection, interpretation, and presentation

F19 demonstrate an understanding of the differences in the quality of sampling methods

Elaboration – Instructional Strategies/Suggestions

A3/F2/F19 To sample people in a mall is fast, cheap, and convenient. However, think about which people in the mall might be invited to participate. Often it is those who are well dressed, respectable looking and friendly, because they look easier to approach. The sample from malls may over-represent the middle class and retired, and under-represent the poor. Sampling techniques such as this one often produce data that is unrepresentative of the population. A sample is said to be biased if it systematically favours certain outcomes. Inferences about a population based on a sample that is biased are often inaccurate. It is preferable to collect unbiased samples. Even though unbiased samples will result in inferences that are at least slightly different from the population, unbiased samples are, on average, better representative of the population, and therefore inferences based on unbiased samples tend to be more accurate.

Students should understand the difference between a biased and an unbiased sample. A unbiased sample is usually generated, at least in part, randomly. They should be familiar with and be able to identify samples that tend to be unbiased:

- simple random sample – a sample that is selected from a population in a way that ensures that every different possible sample of the desired size has the same chance of being selected.
- stratified random sample – selecting a simple random sample from each of the given number of subpopulations, or strata.
- cluster sample – the population is divided into groups (or clusters), a cluster is randomly selected, and every member of that cluster is included in the sample.
- systematic sample – a sample that is chosen according to a formula or rule

Students should also be familiar with and be able to identify samples that tend to be biased:

- convenience sample – a sample that uses results or data that are conveniently and readily obtained.
- voluntary response sample – a sample where results are collected on a volunteer basis, such as a call-in poll or an online voting survey.

Worthwhile Tasks for Instruction and/or Assessment

A3/F2/F19 *Activity*

A3/F2/F19.1. During the previous calendar year, a city's small claims court processed 1562 cases. A legal researcher would like to select a random sample of 50 cases to obtain information regarding the average award in such cases. Assuming that the researcher wants his sample to be as unbiased as possible, describe two different methods the researcher could use to select his sample.

A3/F2/F19.2. A sociologist wants to know the opinions of employed adult women about government funding for day care. She obtains a list of the 520 members of a local business and professional women's club and mails a questionnaire to the first 100 people on the list. 63 surveys are returned.

a) What type of sample is this?
b) What are some sources of bias in her survey?
c) How would you improve her sampling method to reduce bias?

F2.1. Identify the sampling method used and identify possible sources of bias in each:

a) Shona stood by the mall entrance and questioned every tenth person.
b) Avril mailed questionnaires to all of the people who had rented videos at her store in the past three months.
c) Dominic randomly selected one of his hometown's eight skating arenas, and then surveyed all of the people who attended the next hockey game.
d) Ron visited several day-care centres and asked a few questions of children whom he judged to be typical four-year-olds.

A3/F2/F19.3. Marla stood by the front door of the local theatre immediately at the end of the movie, and asked every 5th person who left how much money they had spent at the popcorn counter. Of the 90 people asked, only 52 agreed to respond. Marla told the theatre management that the typical moviegoer spends $5.00 on treats before or during the movie.

a) What sampling method did she use?
b) Do you think these results are biased? Justify your reasoning.
c) Explain in a paragraph to Marla how she might have improved her results to be more representative of all moviegoers.

Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

A3 demonstrate an understanding of the application of random numbers to statistical sampling

F8 apply characteristics of the normal distribution

F15 design and conduct surveys and/or simulate data collection to explore sampling variability

Elaboration – Instructional Strategies/Suggestions

A3/F8/F15 Students have already been exposed to simulations and how simulations can be used to generate data. Students should now simulate sampling procedures from normal populations where the population mean (μ) and population standard deviation (σ) are known. Students can then compare sample means ($\bar{x}$) to μ .

For example, consider an IQ test where the scores are normally distributed with $\mu = 100$ and $\sigma = 15$. A random sample of 200 people is to be selected from this population. In order to simulate taking a random sample from this population, we need a method of generating numbers from a normal distribution with specified values of μ and σ for a specified number of trials. The randNorm(function on the TI-83 Plus is one method of generating such data. This function can be accessed by using the following keystrokes:

□~~~ PRB – 6: randNorm(

The syntax for this command is:

randNorm(μ, σ ,number of trials)

Therefore, if a student wanted to simulate taking a random sample of 200 people from the IQ test population given above, they should execute the following command:

randNorm(100,15,200)$\rightarrow L_1$ (Keystrokes □~~~ PRB - 6: randNorm(100,15,200) ↵ψℵ)

The "$\rightarrow L_1$" at the end of the command stores the list of 200 numbers in the first list, L_1 .

Students should then calculate the mean of their sample*, and compare this to the population mean (100). This process should be repeated a number of times to explore sampling variability. Students should realize that μ remains constant while $\bar{x}$ varies. In later sections, students will learn how to use $\bar{x}$ to estimate the value of μ when μ is unknown.

*Note: On the TI-83 Plus, it is possible to find the mean of a set of data without storing the data in a list. We can use the mean(command, which can be accessed using the following keystrokes:

ψ□~~3: mean(

For example, if we execute the command mean(randNorm(100,15,200)), the TI-83 Plus will generate a sample of size 200 from a normal distribution with $\mu = 100$ and $\sigma = 15$, and then return only the mean of this sample; it will not display the data in the simulated sample.

Note: While students are expected to be able to use graphing calculators and/or other technology for performing simulations, it is NOT expected that they would memorize keystrokes and commands for the purposes of summative evaluation.

If available, computer software such as MINITAB can be used to carry out such simulations.

Worthwhile Tasks for Instruction and/or Assessment

A3/F8/F15 *Activity*

A3/F8/F15.1. The amount of soft-drink in a particular brand of cola is normally distributed with $\mu = 355$ ml and $\sigma = 5.3$ ml.

a) Design a simulation to generate a sample of 100 soft drinks from this population.
b) Conduct your experiment and find the sample mean.
c) Compare the sample mean to the population mean. Do you think that ⌡ is a good indicator of μ? Why or why not?
d) Repeat your simulation 3 more times and find the mean of each sample. Are your means the same? Explain.
e) If you did not know that $\mu = 355$ ml, would you be able to tell which value of ⌡ was the closest to μ? Explain.

A3/F8/F15.2. The lifespan of a certain brand of car is normally distributed with $\mu = 10$ years and $\sigma = 2$ years.

a) Design a simulation to generate the lifespan of 30 cars from this population.
b) Create a histogram of your data collected. Describe the shape of the histogram.
c) Conduct your simulation again, and generate the lifespan of 100 cars.
d) Create a histogram of your 100 lifespans. How does this histogram compare to the one you created in part (d)?
e) Based on your results to the above, how could you make your histogram appear more bell-shaped? Explain

A3/F8/F15.3. Paper Activity Part 1: Obtain 7 different colors of 8” by 11” paper e.g.1 sheet each of yellow and white, 3 sheets each of pink and brown, 6 each of orange and blue, and 9 sheets of green. Cut paper into one inch squares. Place each set of coloured paper into transparent containers such as plastic water bottles with the tops cut off, arranged to form an approximation of a “bell” curve (note this is NOT truly a normal population). Let each colour represent a number from 1 to 7, with green = 4. Discuss attributes such as the mean and standard deviation (dispersion) of the distribution.

Suggested Resources

Statistics (revised)

Outcomes
SCO: In this course, students will be expected to

G3 graph and interpret sampling distributions of the sample mean

F4 demonstrate an understanding of how sample size affects the variation in sample results

Elaboration – Instructional Strategies/Suggestions

G3/F4 Students should be introduced to sampling distributions of the sample mean. They should explore how the means of samples of size n are distributed if samples are repeatedly collected.

To create a good approximation of the sampling distribution of the sample mean, we often need the means from a large number of samples. Therefore, it may be beneficial to explore the sampling distribution of the sample mean as a whole group activity. Consider the following example. The time taken for a certain reading test for 12 year-old children is normally distributed with a mean of 60 minutes and a standard deviation of 15 minutes. Figure 1 represents this normal distribution. Students could then simulate collecting a number of random samples of size 30 from this population and find the mean time for each sample. The mean times could then be collected as a class and a histogram constructed. (Note: the class should generate at least 100 samples, and the same scale for the horizontal axis that was used for the population distribution should be used). A histogram based on 100 random samples is provided in Figure 2.

Population distribution of times

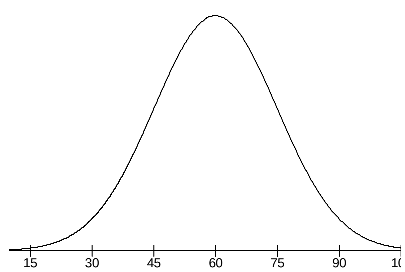


Figure 1

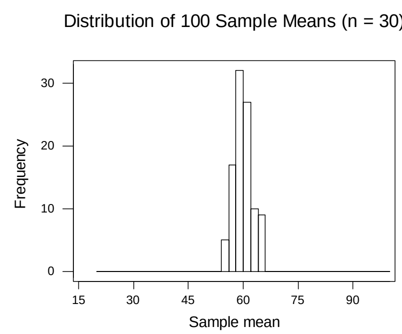


Figure 2

The histogram in Figure 2 is an approximation of the sampling distribution of the sample mean. Students should note that:

- it appears approximately normal.
- the sampling distribution is centred at approximately 60 minutes, which is the population mean. (Calculating the mean of the data in the histogram should confirm this fact.)
- the histogram of the sample means has a smaller standard deviation. That is, there is less variability in the sampling distribution of the sample mean than in the population distribution.

Students should then explore the relationship between σ and the standard deviation of the data in their histogram. Students may need guidance to discover that the standard deviation of their simulated data should be approximately $\frac{\sigma}{\sqrt{n}}$, which in the above example is $\frac{15}{\sqrt{30}} \approx 2.74$.

Students should then repeat this exercise with a larger sample size (such as 60) to see the effect on the variability of the sampling distribution. A larger sample size will produce a more clustered histogram, which results in less variability.

Worthwhile Tasks for Instruction and/or Assessment

G3/F4 *Activity*

G3.1. An airline reports that the average weight of passenger baggage is 25kg with a standard deviation of 8.2kg.

a) Draw a visual representation of this population.
b) Design a simulation to collect a random sample of 40 weights from this population.
c) Conduct your simulation and find the mean of your sample.
d) Repeat part (c) until you have 10 sample means.
e) Collect results from other students until you have at least 100 means.
f) Construct a histogram of your data.
g) Compare your histogram in part (f) to your visual representation in part (a). What do they have in common? What is different?
h) Calculate the mean and standard deviation of your data. How do these values compare with μ and σ ?

G3/F4.1. The age of university students in a certain province is normally distributed with an age of 24.1 years and a standard deviation of 2.3 years.

a) Describe how to create an approximate sampling distribution of the sample mean for a sample size of 50.
b) At what value would you expect your sampling distribution to be centred?
c) What would you predict the standard deviation of your sampling distribution to be?
d) If you were creating an approximate sampling distribution of the sample mean based on a sample size of 100, how would it be similar to the sampling distribution you created in part (c)? How would it be different?

G3.2. <u>Paper Activity Part 2 (continued from the previous Worthwhile Tasks):</u> Place the paper squares into a bag and shake vigorously. Have each student draw several samples of size 10 (see below – a larger sample size may be desired), ***replacing each sample before drawing the next,*** and record each sample mean, then combine class results until there are at least 100 sample means and construct a histogram. The class should make visual comparisons of the original population with the distribution of sample means, particularly the means and standard deviations, then calculate the mean of the sample means and compare to the population mean. Repeat with larger sample sizes. If the sample size is 30 or greater, students can also compare the value of $\frac{\sigma}{\sqrt{n}}$ to the standard deviation of the sampling distribution of the sample means.

Discuss the hypothetical situation of obtaining an infinite number of sample means and the shape of the resulting distribution.

Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F8 apply characteristics of normal distributions

G3 graph and interpret sampling distributions of the sample mean

F4 demonstrate an understanding of how sample size affects the variation in sample results

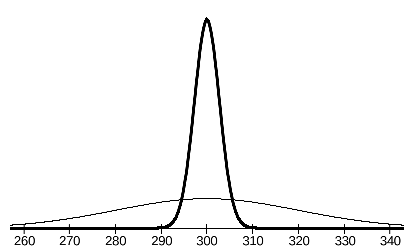
Elaboration – Instructional Strategies/Suggestions

F8/G3/F4 Students have explored approximate sampling distributions of the sample mean through simulations. However, because their histograms were based on a finite number of sample means, they could not create the actual sampling distribution of the sample mean. To describe the actual sampling distribution of the sample mean, we use the Central Limit Theorem:

- If samples of size *n* are drawn at random from any population with a finite mean and standard deviation, then the sampling distribution of the sample mean $\bar{x}$ is approximately normal when *n* is large (when $n \geq 30$).
- the mean of the sampling distribution is equal to the population mean ($\mu_{\bar{x}} = \mu$).
- the standard deviation of the sampling distribution is equivalent to the following: $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$.

Students should understand and apply the Central Limit Theorem. Given a population where μ and σ are known, students should be able to determine the mean and standard deviation of the sampling distribution. Students should also understand that as the sample size increases, the variability in the sampling distribution decreases, and that the sampling distribution becomes more normal. This result is true *regardless* of the shape of the population distribution (for more on this, the online tutorial recommended under **Suggested Resources** on the next page can be accessed). Activity F8/G3/F4.3 on the next page can also be used to illustrate this point for students.

Students could be presented with examples such as the following. A company claims that one of its brands of cakes has an average fat content of $\mu = 300$ grams with a standard deviation of 10 grams. Students could be asked to describe the sampling distribution of the sample mean if samples of size 50 were randomly and repeatedly taken from this population. Students could use the Central Limit Theorem to determine the mean of the sampling distribution ($\mu_{\bar{x}} = \mu = 300$ grams) and the standard deviation of the sampling distribution ($\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{50}} \approx 2.828$). Both the population distribution and the sampling distribution are graphed below (the sampling distribution is bolded):



Students should notice that the sampling distribution has less variability than the original population. Students could then be asked to describe the sampling distribution of the sample mean for samples of size 100. Again, using the Central Limit Theorem, students should determine that $\mu_{\bar{x}} = \mu = 300$ grams and $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{20}{\sqrt{100}} = 2$. This sampling distribution has less variability than the original population and that of the sampling distribution based on a sample size of 50.

Worthwhile Tasks for Instruction and/or Assessment

F8/G3/F4 *Activity*

F8/G3/F4.1. The base of live aspen trees in a national park is normally distributed with a mean of 34.5cm and a standard deviation of 10.3cm.

a) If samples of size 30 were repeatedly collected, what would be the mean of the sample means?
b) If samples of size 30 were repeatedly collected, what would be the standard deviation of the sample means?
c) How would your sampling distribution compare to (a) and (b) if samples of size 400 were repeatedly collected?

F8/G3/F4.2. The age of professional football players in a certain league is normally distributed with an average age of $\mu = 27.8$ and $\sigma = 3.5$ years. The following data shows the age of 40 randomly selected players from this league:

32.1	23.8	25.8	33.6	27.9	28.3	26.2	29.2	35.5	25.5
31.4	29.8	33.4	31.3	32.4	26.3	30.0	28.2	24.0	27.2
30.4	31.7	35.2	30.3	22.2	28.0	27.9	29.2	23.7	29.9
28.1	31.2	30.4	26.9	30.2	27.9	34.7	30.4	28.9	27.2

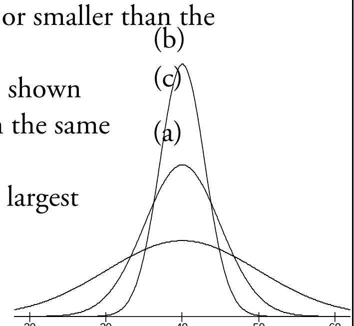
a) What is the sample mean? Is it close to the expected value? Explain.
b) If samples of size 40 were to be repeatedly collected, accurately predict the values of the mean and standard deviation of the sample means.

F8/G3/F4.3. Place 100 identical pieces of paper with the number 1 written on them in a clear plastic container such as a soft drink bottle with the top cut off. Repeat this for the numbers 2 through 5, using pieces of paper identical to those used in the first container. This will be the population, with $\mu = 3$ and $\sigma \approx 1.41$.

a) Construct a histogram for the distribution.
b) Place all the numbered pieces of paper in a bag and mix thoroughly. Randomly select a sample of size 30, without replacement, from the bag, and then calculate the sample mean and standard deviation. Compare these values to the corresponding population values.
c) Repeat the procedure in b) until 100 such samples have been selected.
d) Construct a histogram for the sample means you obtained. How does its shape compare with that of the population? Is the standard deviation of the sampling distribution of sample means larger or smaller than the population standard deviation?

F8/G3/F4.4. Three sampling distributions of sample means are shown below. All three distributions were obtained from the same population. However, the sample sizes differ.

a) Which sampling distribution results from the largest sample size? the smallest? Explain.
b) What is the mean of the original population? How do you know?



Suggested Resources

For an online tutorial on the Central Limit Theorem see the following:

http://www.wadsworth.com/psychology_d/templates/student_resources/workshops/stat_workshp/cnt_lim_therm/cnt_lim_therm_01.html

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F1 draw inferences about a population based on a sample

F11 determine, interpret, and apply confidence intervals

F20 demonstrate an understanding of how a confidence level affects a confidence interval

Elaboration – Instructional Strategies/Suggestions

F1/F11 Students should understand what confidence intervals are and how they can be used to make inferences about the population. Students should be able to identify and understand what is meant by a point estimate, an interval estimator, a confidence level, and a confidence interval. When initially introduced, students should focus on interpreting and understanding confidence intervals, not calculating them.

While the population mean is fixed, it is often not known. A confidence interval is a method used to infer the population mean based on information collected from a sample. Students should understand that a confidence interval for the population mean is an interval of plausible values for the population mean. It is constructed so that the value of the population mean will likely be captured inside the interval with a chosen degree of confidence. In this course, students will work with three different confidence levels: 90%, 95%, and 99%. Students should also compare 90%, 95%, and 99% confidence interval for the same data set to explore effect of the confidence level on the confidence interval. They should understand that a higher confidence level will result in a larger interval for a given sample size. However, if we increase the sample size we can decrease the interval while maintaining the same level of confidence.

There should also be some class discussion with reference to cases that may require different levels of confidence. For example, most polls regarding preferences for political parties are reported as accurate "19 times out of 20" (or with 95% confidence). On the other hand, the medical profession would require 99% or greater confidence, and likely a very large sample size, when examining the side effects of a new drug.

To help clarify this, students could be presented with the following example. A random sample of 100 people was selected and their sleeping habits were studied. It was discovered that the mean time taken to fall asleep was $\bar{x} = 23.2$ minutes with a standard deviation of $S_x = 6.3$ minutes. From this information, the researchers conclude that they are 95% confidence that the actual time it takes a person to fall asleep at night is between 22.0 minutes and 24.4 minutes. Students should understand that the researchers do not know the population mean or standard deviation because they only studied a sample of 100 people. Instead, the researchers will use the information to create a plausible range of the population mean. Students should identify that the point estimate is 23.3 minutes because this single number is used as a plausible value of the population mean. The interval estimator (or confidence interval) is $(22.0, 24.4)$ because this is the interval that the researchers believe contains the true population mean. However, because a sample will have different characteristics than a population, the researchers cannot be 100% confident that their interval actually contains the true population mean. Instead, the mathematical procedure used to create this confidence interval will entrap the population mean 95% of the time (for more elaboration on this subtle but important point, see the last paragraph below). Therefore, the confidence level is 95%. Students should not be initially concerned about how the confidence interval was derived. Instead, they should understand what a confidence interval is and what information about the population it gives.

Students could then be given the 90% and 99% confidence intervals for this set of data to explore the relationship between the confidence level and the confidence interval. The 90% and 99% confidence intervals are provided below:

- 90% -- between 22.2 and 24.2 minutes
- 99% -- between 21.6 and 24.8 minutes

It is tempting to say there is a 95% chance that the population mean is between 22.0 minutes and 24.4 minutes. However, this is incorrect. The 95% refers to the percentage of all possible samples resulting in an interval that contains μ. That is, if we take sample after sample from the population and use each one individually to calculate a confidence interval, in the long run 95% of them will capture μ.

Worthwhile Tasks for Instruction and/or Assessment

F1/F11/F20 *Activity*

F1/F11/F20.1. A botanist collects a sample of 50 iris petals and measures the length of each. It is found that $\bar{x} = 5.55$ cm and $S_x = 0.57$ cm. He then reports that he is 95% confident that average petal length is between 5.39cm and 5.71cm .

a) Identify the point estimate, the confidence interval, and the confidence level.
b) Explain what information the confidence interval gives about the population of iris petal length.
c) How would the length of a 99% confidence interval be different from that of a 95% confidence interval?
d) If you did not know the point estimate but were still given that the confidence interval is between 5.39cm and 5.71cm, how could you determine the point estimate?

F11/F20.1. Explain what is meant by a 90%, 95% and a 99% confidence interval. How are these intervals similar? How are they different?

Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F1 draw inferences about a population based on a sample

F7 draw inferences from graphs, tables, and reports

F8 apply characteristics of normal distributions

F11 determine, interpret, and apply confidence intervals

F20 demonstrate an understanding of how a confidence level affects a confidence interval

F21 demonstrate an understanding of the role of the Central Limit Theorem in the development of confidence intervals

F22 distinguish between the calculation of confidence intervals for a known population mean versus an unknown population mean

G3 graph and interpret sample distributions of the sample mean

Elaboration – Instructional Strategies/Suggestions

F1/F7/F8/F11/F20/F22/G3 Once students have an understanding of what a confidence interval is, they should determine and interpret confidence intervals from samples where the population mean and standard deviation is known. While this type of question is unrealistic (there is no need to calculate a plausible interval of μ if we already know the value of μ), it should help students further understand the meaning of a confidence interval.

The construction of a confidence interval is based on the Central Limit Theorem. If we construct a sampling distribution of the sample mean based on a sample of size *n*, we know that the distribution will be approximately normal with $\mu_{\bar{x}} = \mu$ and $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}}$. To create a 95% confidence interval, we use the fact that approximately 95% of the data in a normal distribution lies within 1.96 standard deviations of the mean (see Figure 1). If we take a sample of size *n* and calculate $\bar{x}$, it has a 95% chance of lying within $1.96\sigma_{\bar{x}}$ of $\mu_{\bar{x}}$. Therefore, if we create the interval $\bar{x} \pm 1.96\sigma_{\bar{x}}$, we have a 95% chance of creating an interval that contains $\mu_{\bar{x}}$, which equals our original population mean (see Figure 2). Since there is a 5% chance of obtaining a sample that is beyond 1.96 standard deviations of the mean, there is a 5% chance that our confidence interval will not contain the population mean.

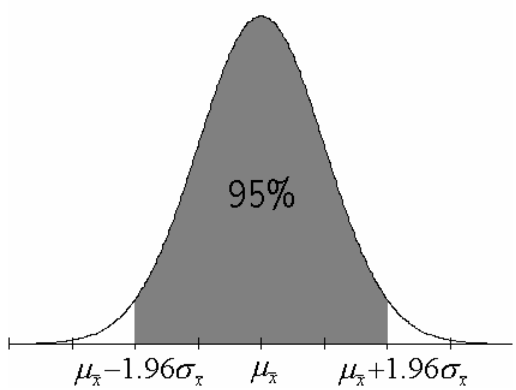


Figure 1

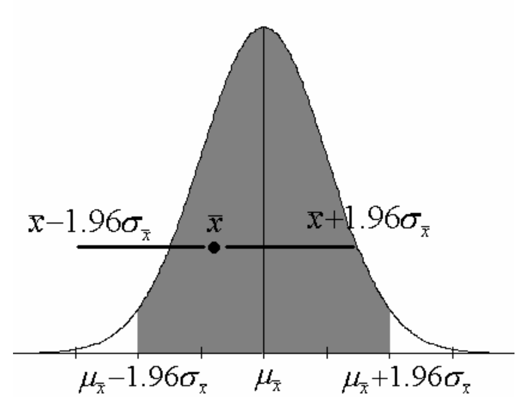


Figure 2

Students should determine and interpret 90%, 95%, and 99% confidence intervals from sample data where μ and σ are known.

(continued on the next 2 page spread....)

Worthwhile Tasks for Instruction and/or Assessment

F1/F7/F8/F11/F20/F22/G3 *Activity*

F1/F8/F11/F20/F22.1. The development time for a particular type of photographic printing paper is normally distributed with $\mu = 30$ minutes and $\sigma = 7.5$ minutes. A random sample of 50 developing times is collected and the sample mean is determined to be $\bar{x} = 31.9$ minutes.

a) What is the point estimate?
b) If you were to determine the 90%, 95%, and 99% confidence interval, which would be the smallest? the largest? Explain how you know.
c) Determine the 90%, 95%, and 99% confidence intervals for the mean developing time.
d) Does each confidence interval contain the population mean? Explain.
e) Another random sample, this time of 200 developing times, is collected and the sample mean is $\bar{x} = 31.8$ minutes. Determine the 90%, 95%, and 99% confidence intervals based on this sample, then compare to the corresponding intervals constructed in part (c) for the sample of size 50. What do you notice?
f) Compare your 95% interval for sample size 200, with the 90% interval for sample size 50. Which is shorter? Explain why this happens.

F1/F7/F8/F11/F22.1. The length of Digby neck clams in a particular bay is normally distributed with $\mu = 445$ cm and $\sigma = 100$ cm. The following represents the lengths of 40 randomly selected Digby neck clams:

446	407	447	380	400	511	467	257	613	530
599	359	489	451	352	710	322	410	633	520
624	519	427	503	547	275	519	505	303	396
408	308	358	623	426	402	347	443	460	588

a) What is the sample mean?
b) Determine the 90%, 95%, and 99% confidence intervals for the mean length of Digby neck clams in this particular bay.
c) Another random sample of 100 Digby neck clams was taken from the same bay and has the same sample mean as part (a). If a 95% confidence interval was calculated using this new sample, how would it differ from the 95% confidence interval calculated in part (b)?

F11/F20.1. A sample is taken from a normal population. Based on this sample, Lenna claims she is 90% confident that the population mean is between 45.6 and 50.2. Based on the same sample, Phillip claims he is 95% confident that the population mean is between 46.6 and 49.2. How do you know that someone has made a calculation error?

(continued on the next 2 page spread....)

Suggested Resources

Winstats is a free program that has an excellent confidence interval demonstration. It is available at http://math.exeter.edu/rparris

Tutorials for Winstats may be found at http://www.gov.nl.ca/edu/sp/mathres/mathresources.htm

Applets demonstrating confidence intervals can be found at:

http://www.ruf.rice.edu/~lane/stat_sim/conf_interval

http://www.stat.sc.edu/~west/applets/ci.html

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F1 draw inferences about a population based on a sample

F7 draw inferences from graphs, tables, and reports

F8 apply characteristics of normal distributions

F11 determine, interpret, and apply confidence intervals

F20 demonstrate an understanding of how a confidence level affects a confidence interval

F21 demonstrate an understanding of the role of the Central Limit Theorem in the development of confidence intervals

F22 distinguish between the calculation of confidence intervals for a known population mean versus an unknown population mean

G3 graph and interpret sampling distributions of the sample mean

Elaboration – Instructional Strategies/Suggestions

(...continued from last 2 page spread.)

Consider the following example. The automatic opening device of a military cargo parachute has been designed to open when the parachute is a certain distance above the ground. The opening altitude is normally distributed with $\mu = 250$ m and $\sigma = 15$ m. A random sample of 40 opening altitudes is taken, and it is determined that $\bar{x} = 246.83$ m. Students should be able to calculate the 95% confidence interval as follows:

$$\bar{x} \pm 1.96\sigma_{\bar{x}}$$
$$246.83 \pm 1.96\frac{\sigma}{\sqrt{n}}$$
$$246.83 \pm 1.96\frac{15}{\sqrt{40}}$$
$$\text{from } 242.18 \text{ to } 251.48$$

That is, we are 95% confidence that the mean opening altitude is between 242.18m and 251.48m. Students should note that $\frac{\sigma}{\sqrt{n}}$ was substituted for $\sigma_{\bar{x}}$. This is a direct result of the Central Limit Theorem. Students should note that, in this instance, their confidence interval contains the population mean. This will happen for approximately 95% of our samples since this is a 95% confidence interval. Students could also calculate the 90% and 99% confidence intervals by understanding that:

- 90% of the data in a normal distribution will lie within 1.645 standard deviations of the mean
- 99% of the data in a normal distribution will lie within 2.56 standard deviations of the mean

To calculate the 90% confidence interval, for example, students follow the same calculation as above, except they replace 1.96 with 1.645:

$$\bar{x} \pm 1.645\sigma_{\bar{x}}$$
$$246.83 \pm 1.645\frac{\sigma}{\sqrt{n}}$$
$$246.83 \pm 1.645\frac{15}{\sqrt{40}}$$
$$\text{from } 242.93 \text{ to } 250.73$$

Again, students should realize that this interval contains the population mean, and that this will happen for 90% of the samples since this is a 90% confidence interval. Students should also note that the 90% confidence interval is smaller than the 95% confidence interval, reaffirming the fact that higher confidence levels will result in larger confidence intervals for samples of the same size.

It is very important that students also consider the effect of sample size on the length of a confidence interval. While higher levels of confidence will produce longer confidence intervals when constructed from samples of the same size, using larger sample sizes will produce shorter confidence intervals. It is possible, for example, that a 90% confidence interval based on a sample of size 40 will be longer than a 95% confidence interval based on a sample of size 100 from the same population. Students could repeat the above example, but this time assuming a sample size of 100 with the same sample mean of $\bar{x} = 246.83$ m. They should compare the lengths of the 90%, 95%, and 99% confidence intervals with the corresponding intervals based on the smaller sample size

Worthwhile Tasks for Instruction and/or Assessment

F1/F7/F8/F11/F20/F22/G3 *Activity*

(...continued from the last 2 page spread.)

F1/F7/F8/F11/F22/G3.1. In the assembly line of the Canyonaro four-wheel-drive sport vehicle, the amount of time taken to install the exhaust system is normally distributed with $\mu = 7.5$ minutes and $\sigma = 1.7$ minutes.

a) Using technology, design and conduct a simulation that randomly selects a sample of 50 installation times.
b) Find the mean of your sample.
c) Using your sample, create a 90% confidence interval.
d) Repeat parts (a), (b), and (c) until you have 10 confidence intervals.
e) Collect confidence intervals from other students until you have at least 100 confidence intervals.
f) What percentage of your confidence intervals contains the population mean? Is this close to what you expected? Explain.
g) Repeat the simulation one more time, but this time for a sample of 100 installation times. Create a 90% confidence interval and compare its length to the ones you created in parts (a), (b), and (c). What do you notice? Why does this happen?

Suggested Resources

Statistics (revised)

Outcomes

SCO: By the end of Course 2YZ, students will be expected to

F1 draw inferences about a population based on a sample

F2 identify bias in data collection, interpretation, and presentation

F7 draw inferences from graphs, tables, and reports

F11 determine, interpret, and apply confidence intervals

F22 distinguish between the calculation of confidence intervals for a known population mean versus an unknown population mean

Elaboration – Instructional Strategies/Suggestions

F1/F2/F7/F11/F22 Students should determine and interpret confidence intervals from large samples ($n \geq 30$) without knowing the values of μ and σ . They will need to approximate σ from their sample data. When a simple random sample of size $n \geq 30$ is taken from a finite population without replacement, and the sample size is no more than 5% of the population size, we can approximate σ by S_x. We can also approximate σ using S_x when dealing with infinite populations or with finite populations where sampling with replacement takes place.

Students should explore examples similar to the following. A sociologist is studying the length of courtship before marriage in Kyoto, Japan. A random sample of 56 couples was interviewed. It was found that the mean length of courtship was 4.3 years with a sample standard deviation of 1.0 years. Students could be asked to create a 99% confidence interval for the mean length of courtship in Kyoto. Students should realize that neither μ nor σ are known. In this case, we know that $\bar{x} = 4.3$ years and $S_x = 1.0$ years. However, since the sample size is greater than 30, and the sample is small compared to the number of couples in Kyoto, Japan, students should realize that they can approximate σ with S_x. They could then perform the following calculation to construct their confidence interval:

$$\bar{x} \pm 2.56 \frac{S_x}{\sqrt{n}}$$

$$4.3 \pm 2.56 \frac{1.0}{\sqrt{56}}$$

from 3.96 to 4.64

Students could then explain that they are 99% confident that the mean length of courtship in Kyoto, Japan, is between 3.96 years and 4.64 years. Students should also discuss the limitations of the data in this example. They should understand that their confidence interval is only applicable to Kyoto, Japan. They could not apply their confidence interval to a different part of the world. For example, they could not claim that this is true for all of Japan, as the length of courtship may be different in rural areas of the country. (Note: The shape of the distribution of courtship lengths is not important for the construction of the confidence interval since the Central Limit Theorem holds regardless of the shape of the parent distribution).

Students should also understand what is meant by the term 'margin of error.' The margin of error indicates the accuracy of the point estimate. For example, if an article reports that the mean waiting time in a bank line is 9.3 minutes with a margin of error of ± 2.1 minutes, and that these results are accurate 19 times out of 20, students should recognize that we are dealing with a 95% confidence interval (as stated by the "19 times out of 20"), and that the confidence interval is given by:

point estimate ± margin of error

$$9.3 \pm 2.1$$

from 7.2 to 11.4

Students should determine the margin of error for different confidence intervals. The margin of error is calculated by the expression $z \frac{S_x}{\sqrt{n}}$ where *z* is:

- 1.645 for a 90% confidence interval;
- 1.96 for a 95% confidence interval; and
- 2.56 for a 99% confidence interval.

Worthwhile Tasks for Instruction and/or Assessment

F1/F2/F7/F11/F22 *Activity*

F1/F2/F11/F22.1. A random sample of 35 red pine trees was selected from a large forest containing 100000 trees. The mean diameter was determined to be $\bar{x} = 25.3$ cm with $S_x = 3.6$ cm.

a) If we were to create a 95% confidence interval for the mean diameter of red pine trees in this forest, would we be allowed to approximate σ with S_x? Explain.
b) Create a 95% confidence interval and explain its meaning.
c) Can you make any conclusions about the mean diameter for all red pine trees in Canada? Why or why not?

F1/F2/F7/F11/F22.1. The following data shows a random sample of the ages of football players in a large European soccer league consisting of 1000 players:

26	24	25	36	26	32	31	34
32	27	32	23	24	29	30	30
29	33	25	32	24	25	28	22
22	24	23	33	32	31	26	28
32	25	22	28	25	29	25	27

a) What is the mean and standard deviation of the sample?
b) Determine a 99% confidence level for the mean age of soccer players in this league, and explain the meaning of this confidence interval.
c) What is the point estimate?
d) What is the margin of error of your confidence interval? Explain the meaning of the margin of error.
e) Based on this data, what, if anything, can you conclude about the mean age of soccer players in a Canadian soccer league? Explain.

F1/F7/F11/F22.1. A report claims that the average family income in a large city is \$32000. It states the results are accurate 19 times out of 20 and have a margin of error of $\pm\$2500$.

a) What is the confidence level in this situation? Explain what it means.
b) What is the confidence interval?
c) Explain the meaning of the confidence interval.

Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F16adv demonstrate an understanding of the difference between situations that involve a binomial experiment and those that do not

F18adv identify the characteristics of a binomial experiment

Elaboration – Instructional Strategies/Suggestions

The following material applies only to students in the advanced course.

F16adv/F18adv Students should also understand the difference between a population proportion (*p*) and a sample proportion $\left(\hat{p}\right)$. For example, if 60% of people in a certain community believe that smoking should be banned in public areas, then $p = .60$. If, in a sample of size 40 from this community, a researcher finds that 22 people believe in a public smoking ban, then $\hat{p} = \frac{\text{number of successes}}{\text{number of trials}} = \frac{22}{40} = 0.55$. Students should also recognize that probability is always expressed as a number between 0 and 1.

Students will explore binomial experiments. Students should be able to identify the characteristics of a binomial experiment:

- it consists of a fixed number of identical trials *n*.
- each trial can result in one of two outcomes labelled success (*S*) and failure (*F*).
- outcomes of different trials are independent; that is, the outcome of one trial does not affect the outcome of any other trial.
- the probability that a trial results in *S* is equal to the population proportion (*p*) and is the same for each trial.

Students should be able to distinguish between situations that are binomial experiments and those that are not, and justify their reasoning. For example, students could be given the following situation: a fair coin is flipped 200 times and the number of heads is recorded. Students should realize that this is a binomial experiment because:

- there are 200 trials.
- each trial has two possible outcomes: a head (success) and a tail (failure).
- the probability of obtaining a head for one trial does not influence the probability of obtaining a head for another trial.
- the probability of obtaining a head in each trial is the same: 0.50.

Students could also be given situations that are not binomial and explain why they are not binomial. For example, a biology student is taking a biology quiz consisting of 20 true/false questions. He knows the answers to 15 of the questions, but has to guess the other 5 answers. In this example, students should realize that this is not a binomial experiment because the probability of success is not the same for each trial. In some trials, the probability of success is 0.50 (when the student is guessing), and in other trials, the probability of success is 1 (when the student knows the answer).

Students should also understand that many surveys conducted in real-world situations do not represent true binomial experiments because the sample is collected without replacement from a finite population, which does not result in independent trials. Consider the following example. In a community with a population of 1000, 60% of the people believe that smoking should be banned in public areas. A researcher randomly selects 40 people from this community and asks them if they believe that smoking should be banned in public areas. The probability of success for the first trial is $\frac{600}{1000} = 0.60$. Students should realize that the first person interviewed will not be interviewed again. If on the first trial, the researcher selected a person who believes in banning smoking in public areas, then the probability of success on the second trial decreases to $\frac{599}{999} = .5996$ since the first person cannot be selected again. Even though there is little difference in the probability of success between the first and second trials, the fact that a difference exists means that this is not a true binomial experiment. However, as long as the sample is not too large in comparison with the population (as long as the population is at least 20 times bigger than the sample), students should recognize that the situation could be approximated by a binomial experiment.

Worthwhile Tasks for Instruction and/or Assessment

F16adv/F18adv *Activity*

F16/F18.1. Which of the following are binomial experiments? Which are not? Explain your reasoning.

a) A dice is rolled 600 times and the number of 3s is recorded.
b) In a town of 500 people, 50 people are asked, "Do you own two or more cars?"
c) A student is taking a true-false quiz consisting of 100 questions. The student decides not to read the questions and selects an answer (true or false) at random. The number of correct responses is recorded.
d) A student taking the same exam in part (c) has studied and knows the answers to 80 of the questions. He guesses at the remaining 20 questions in a random fashion. The number of correct responses is recorded.
e) 50 people have entered a contest. Each person is asked to randomly select one of three boxes. The number of people who select each box is recorded.

F16/F18.2. Which of the following can be approximated by a binomial experiment?

a) In a city of 300 000 people, 100 people are randomly selected and asked, "Are you 18 years of age or older?"
b) In a town of 1000 people, 200 people are randomly selected and asked, "Do you own at least one cat?"
c) In a city of 40 000 people, 500 people are randomly selected for a taste test. People must decided between Cola A, Cola B, and Cola C.

F16/F18.3. In a large city of 100 000 people, it is known that 70% of the people use the public transit system at least 3 times a week. In a survey of 200 people, 136 people claim to use the public transit system at least 3 times a week.

a) Explain why this situation is not a binomial experiment.
b) Explain why we can approximate this situation with a binomial experiment.
c) What is p? p ? Explain how you determined each.

Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F16adv demonstrate an understanding of the difference between situations that involve a binomial experiment and those that do not

A3 demonstrate an understanding of the application of random numbers to statistical sampling

F15 design and conduct surveys and simulate data collection to explore sampling variability

Elaboration – Instructional Strategies/Suggestions

The following material applies only to students in the advanced course.

F16adv/A3/F15 Students should imitate collecting data from binomial situations using simulations. They should recognize how to use resources such as dice, coins, spinners, random number tables, and technology to design and create such simulations.

For example, students could be told that 70% of a student population in a certain province take a high school French course. Students could then be asked to design a simulation that would allow them to take a random sample of 50 students from this population and calculate the sample proportion of students who take French. Some possibilities include:

- Create a spinner divided into 2 regions. One region contains 70% of the area and is labelled 'Takes a French course.' The remaining 30% is labelled 'Does not take a French course.' Student could use the spinner 50 times, record the number of times it lands in the first region, and calculate the sample proportion.
- Use a random number table or the **randInt(** function on the TI-83 Plus. Let 0, 1, 2, 3, 4, 5, 6 represent a student who takes a French course, and 7, 8, 9 represent a student who does not take a French course. Students could generate 50 numbers, record the number of successes, and calculate the sample proportion.

Students should conduct simulations for given binomial experiments a number of times to understand variability in sampling. They should recognize that sampling from a binomial experiment will not always give the same sample proportion.

Note: The randBin(function on the TI-83 Plus or TI-84 Plus can be used to generate random data for a binomial experiment. This function can be accessed through the following keystrokes:

□~~~ 7:randBin(

The syntax for this command is:

randBin(number of trials, probability of success, number of samples)

In the above population of high school students in a particular province, we could generate a sample proportion by using the command randBin(50,0.70). If we wanted to collect 100 sample proportions, we could use the command randBin(50,0.70,100).

Worthwhile Tasks for Instruction and/or Assessment

F16adv/A3/F15 *Activity*

F16/A3/F15.1. For each of the following, design a simulation that would allow you to imitate collecting a random sample from the population.

a) Sixty-five percent of students at a large university take a Calculus course. A random sample of 75 students is taken from this population and asked, "Have you taken a Calculus course?"
b) Five out of six people in a large city believe that dogs should be on leashes in public areas. A random sample of 30 people from the city is selected and asked, "Do you believe that dogs should be on leashes in public areas?"
c) Sixty-eight percent of households in a certain community each have a DVD player. A random sample of 250 households is selected and an occupant is asked, "Do you have a DVD player in your house?"
d) Fifty percent of students in a particular high school are involved in extra-curricular activities. A random sample of 92 students is selected and asked, "Are you involved in any extra-curricular activity?"

F16/A3/F15.2. A team of eye surgeons has developed a technique for a risky eye operation to restore the sight of people blinded from a certain disease. In a population of 20000 people who have had the surgery, 75% of the patients recovered their eyesight.

a) Explain why this situation can be approximated by a binomial experiment.
b) Design and conduct a simulation that mimics collection of a random sample of 60 people from this population. Calculate p from your sample.
c) Repeat part (a) until you have 3 sample proportions.
d) Are all of your sample proportions the same? Explain why you think this happens.

Project

F16/A3/F15.3. The principal of the school wishes to determine whether students prefer the current lunch period of 40 minutes or a new proposed lunch period of 30 minutes that would also allow the school day to close 10 minutes earlier in the afternoon. Design and conduct a survey which would allow you to collect a random sample of student preferences. Compare your value for the sample proportion with others in your class.

Suggested Resources

There is a free application (APP) for the TI-83 Plus called *ProbSim* that performs simulations with dice, spinners, coins, cards, and random numbers. It is available at www.education.ti.com along with a tutorial document.

Winstats is a free program that can also perform simulations. It is available at http://math.exeter.edu/rparris

Tutorials for Winstats may be found at http://www.gov.nl.ca/edu/sp/mathres/mathresources.htm

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F4 demonstrate an understanding of how sample size affects the variation in sample results

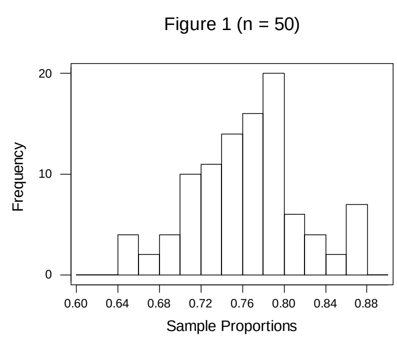
G3 graph and interpret sample distributions of the sample proportion

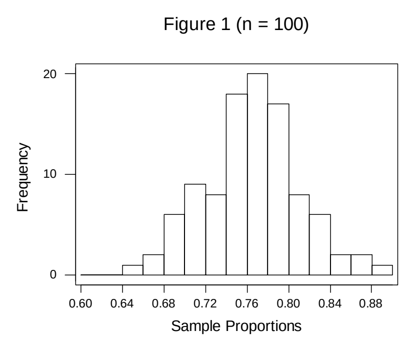
Elaboration – Instructional Strategies/Suggestions

The following material applies only to students in the advanced course.

G3/F4 Students should be introduced to sampling distributions of the sample proportion. They should explore how the sample proportions of samples of size *n* are distributed if samples are repeatedly collected.

To create a good approximation of the sampling distribution of the sample proportion, we often need the sample proportions from a large number of samples. Therefore, it may be beneficial to explore the sampling distribution of the sample proportion as a whole group activity. For example, in a particular province, 75% of all doctors have a private practice. Students could simulate collecting a number of random samples of size 50 from this population and determine p (the proportion of doctors in each sample who have a private practice). The sample proportions could then be collected as a class and a histogram constructed. (Note: the class should generate at least 100 samples.) A histogram based on 100 random samples for a sample size of 50 is provided in Figure 1. Students should then repeat the process for a larger sample size (such as 100). A





histogram based on 100 random samples for a sample size of 100 is provided in Figure 2. (Note: the same horizontal scale should be used for both histograms.)

Both histograms are approximate sampling distributions of the sample proportion. Students should note that:

- the histograms appear approximately normal , and the one based on samples of size 100 is more normal than the one for samples of size 50
- the sampling distributions are centred at approximately 0.75, which is the population proportion. (Calculating the mean of the data in the histograms should confirm this fact.)
- the histogram based on the sample size of 100 appears more clustered than the histogram based on 50 samples. That is, a larger sample size will produce a more clustered histogram, which results in less variability. (Calculating the standard deviation of both sets of data will confirm this fact.)

Students should then explore the relationship between the population proportion *p* and the standard deviation of the data in their histograms (s_p). Students may need guidance to discover that the standard deviation of their data can be approximated by $\sqrt{\frac{p(1-p)}{n}}$.

As a follow-up and for further investigation of these results, students can use technology to each generate 100 sample proportions for the above activity and create their own histograms to make comparisons with those of classmates. For the TI-83 Plus, the command would be randBin(50,.75,100) for sample size 50, and ranBin(100,.75,100) for sample size 100.

Worthwhile Tasks for Instruction and/or Assessment

F4/G3 *Activity*

F4/G3.1. Suppose that 77% of all households use coupons at grocery stores.

a) Design a simulation to imitate collection of a random sample of 60 households from this population.
b) Conduct your simulation and find the sample proportion of households which use coupons.
c) Repeat part (c) until you have 10 sample proportions.
d) Collect results from other students until you have at least 100 means.
e) Construct a histogram of your data. Describe the shape of your histogram.
f) Calculate the mean and standard deviation of your sample proportions. How could you have predicted these values using the population proportion?

F4/G3.2. Sixty percent of all school children watch television in the morning before attending school.

a) Describe how to create an approximate sampling distribution of the sample proportion of school children who watch television before school for a sample size of 50.
b) At what value would you expect your sampling distribution to be centred?
c) What would you predict the standard deviation of your sampling distribution to be?
d) If you were creating an approximate sampling distribution of the sample proportion based on a sample size of 200, how would it be similar to the sampling distribution you created in part (c)? How would it be different?

Suggested Resources

Statistics (revised)

Outcomes
SCO: In this course, students will be expected to

F8 apply characteristics of normal distributions

G3 graph and interpret sampling distributions of the sample mean

F4 demonstrate an understanding of how sample size affects the variation in sample results

Elaboration – Instructional Strategies/Suggestions

The following material applies only to students in the advanced course.

F8/G3/F4 Students have explored approximate sampling distributions of the sample proportion through simulations. However, because their histograms were based on a finite number of sample means, they could not create the actual sampling distribution of the sample proportion. The following results summarize the properties of the sampling distribution of the sample proportion where p is the population proportion:

- For random samples of size n where $np > 5$ and $n(1-p) > 5$, the sampling distribution of the sample proportion will be approximately normally distributed. (For the purposes of this course, these restrictions do not need to be mentioned to students.)
- the mean of the sampling distribution is equal to the population proportion $\left(\mu_p = p\right)$.
- the standard deviation of the sampling distribution is equivalent to the following: $\sigma_p = \sqrt{\frac{p(1-p)}{n}}$.

Students should understand and apply these results. Given a binomial situation where p is known, students should be able to determine the mean and standard deviation of the sampling distribution. They should understand that as the sample size increases, the variability in the sampling distribution decreases.

Students could be presented with examples such as the following. 60% of employees for a large company exercise during their lunch break. Students could be asked to describe the sampling distribution of the sample proportion if samples of size 30 were randomly and repeatedly taken from this population. Students could use the Central Limit Theorem to determine the mean of the sampling distribution ($\mu_p = p = 0.30$ grams) and the standard deviation of the sampling distribution ($\sigma_p = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.30(0.70)}{30}} = 0.084$). Students could then describe the sampling distribution if samples of size 50 were randomly and repeated taken. Students could then compare the standard deviation of both sampling distributions to recognize that a larger sample size results in a lower standard deviation and less variability.

Worthwhile Tasks for Instruction and/or Assessment

F8/G3/F4 *Activity*

F8/G3/F4.1. In a large city, it is known that 15% of people have two or more jobs.

a) If random samples of size 150 were repeatedly taken from this population and a sampling distribution of the sample proportion of people with two or more jobs was created, what would be the shape of the distribution?

b) Where would the sampling distribution of p be centred?

c) What would be the standard deviation of the sampling distribution of p?

F8/G3/F4.1. In a particular province, 50% of the voting public voted in the last election.

a) If random samples of 40 were repeatedly taken from this population and p was calculated for each sample, what would be the mean and the standard deviation of the sampling distribution of p?

b) Design and conduct a simulation for this situation. Repeat your simulation until you have 10 sample proportions.

c) Collect other sample proportions from your classmates until you have at least 100 sample proportions.

d) Construct a histogram of your data in part (c). Describe its shape.

e) Calculate the mean and standard deviation of the data you collected in part (c). Are your results close to the expected values? Explain.

f) If random samples of size 100 were randomly taken and the histogram of the sample proportions was constructed, how would it compare to the histogram you created in part (d)?

Suggested Resources

Statistics (revised)

Outcomes

SCO: In this course, students will be expected to

F11 determine, interpret, and apply confidence intervals

F1 draw inferences about a population based on a sample

F20 demonstrate an understanding of how a confidence level affects a confidence interval

F2 identify bias in data collection, interpretation, and presentation

F7 draw inferences from graphs, tables, and reports

Elaboration – Instructional Strategies/Suggestions

The following material applies only to students in the advanced course.

F11/F1/F20/F2/F7 Students have already explored confidence intervals to find plausible ranges for population means. Now, they will use confidence intervals to find plausible ranges for population proportions.

Students have been introduced to the terminology of confidence intervals in previous outcomes, so they should be familiar with the terms point estimate, interval estimator, confidence interval, margin of error, and confidence level. They can be exposed to examples similar to the following both as a review and to understand how these terms apply to confidence intervals for population proportions.

Example An airline randomly selects 100 passengers and asks them if they were satisfied with their in-flight meal. In a report based on this survey, the airline claims that 34% of passengers are satisfied with the in-flight meals. Their results are accurate 19 times out of 20 with a margin of error of $\pm 9\%$. Students should be able to identify the point estimate (0.34), the interval estimator / confidence interval (from 0.25 to 0.43), the margin of error (± 0.09), and the confidence level (95%). Students should also realize that while they do not know the true value of *p*, they are 95% confident that it lies between 0.25 and 0.43 (or between 25% and 43%).

Students should also construct 90%, 95%, and 99% confidence intervals for population proportions. The construction of confidence intervals for the population proportion is similar to the construction of confidence intervals for the population mean. Since the sampling distribution for p is normally distributed with $\mu_p = p$ and $\sigma_p = \sqrt{\frac{p(1-p)}{n}}$, students should realize that a 95% confidence interval is constructed by:

$$p \pm 1.96\sigma_p$$

$$= \quad p \pm 1.96\sqrt{\frac{p(1-p)}{n}}$$

However, since we usually do not know the value of *p* (and therefore cannot calculate σ_p), we approximate it with p when *n* is sufficiently large ($n \geq 30$). Therefore, when constructing 95% confidence intervals for sample sizes greater than or equal to 30, students will use:

$$p \pm 1.96\sqrt{\frac{p(1-p)}{n}}$$

(Constructing confidence intervals for sample sizes less than 30 are beyond the scope of this course.)

Students should also realize that to create a 90% confidence interval, we use $p \pm 1.645\sqrt{\frac{p(1-p)}{n}}$, and for a 99% confidence interval, we use $p \pm 2.56\sqrt{\frac{p(1-p)}{n}}$.

(continued on the next 2 page spread....)

Worthwhile Tasks for Instruction and/or Assessment

F11/F1/F20/F2/F7 *Activity*

F11/F1/F20/F2/F7.1. In a national survey of 400 Canadians from the ages of 20-35, 37.5% of those interviewed claimed they exercise for at least 4 hours a week. The results were considered accurate within 4%, 9 times out of 10.

a) Are you dealing with a 90%, 95%, or 99% confidence interval? How do you know?
b) How many people in the survey claimed to exercise at least 4 hours a week?
c) What is the margin of error?
d) What is the confidence interval? Explain its meaning.
e) What are some limitations of this survey?
f) If the writers of the article created a 99% confidence interval based on this data, how would it be different? How would it be the same?
g) How would the confidence interval change if the sample size was increased to 500 but the sample proportion happed to remain the same?

F11/F1.1. A random sample of 500 first-year university students is obtained and 355 of respondents claim to be attending the university of their first choice.

a) What is the sample proportion of students who claim to be attending the university of their first choice?
b) Construct a 90%, 95%, and 99% confidence interval for the population proportion of first-year students attending the university of their first choice.
c) If the sample size was decreased but the sample proportion happened to remain the same, how would your confidence intervals change?

F11/F1.2. Create a situation that would result in a 99% confidence interval from 73.5% to 86.5% based on a sample size of 200. Create at least three questions that could be asked about the situation and share them with a classmate. You may wish to ask questions about the sample proportion and population proportion, for example.

F11.1. A survey shows that 76% of high school students regularly attend dances. The survey was reported to be accurate to within 6.2%, 19 times out of 20. How many people were surveyed?

F11/F1.3. A fishing lodge in northern Alberta claims that 75% of its guests catch northern pike over 9.0kg. A random sample of 83 guests indicated that 61 of them catch northern pike over 9.0kg.

a) Create a 95% confidence interval for the population proportion of guests who caught northern pike over 9.0kg.
b) Do you think the lodge's claim is correct? Explain

(continued on the next 2 page spread....)

Suggested Resources

Statistics (revised)

Outcomes
SCO: In this course, students will be expected to

F11 determine, interpret, and apply confidence intervals

F1 draw inferences about a population based on a sample

F20 demonstrate an understanding of how a confidence level affects a confidence interval

F2 identify bias in data collection, interpretation, and presentation

F7 draw inferences from graphs, tables, and reports

Elaboration – Instructional Strategies/Suggestions

(...continued from the last 2 page spread.)

The following material applies only to students in the advanced course.

Students could be presented with examples similar to the following. A car rental company randomly selects 250 customers from Nova Scotia and asks them if they were satisfied with their last rental. 195 people claimed that they were satisfied. Students could be asked to calculate the sample proportion ($p = 0.78$) and the 90%, 95% and 99% confidence intervals; e.g. the 90% confidence interval is:

$$p \pm 1.645\sqrt{\frac{p(1-p)}{n}}$$

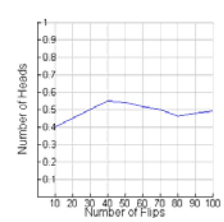
$$= 0.78 \pm 1.645\sqrt{\frac{0.78(1-0.78)}{250}}$$

$$= 0.78 \pm 0.043$$

from 0.737 to 0.823

Students should claim that they believe that the actual percentage of customers in Nova Scotia who were satisfied with their last car rental is between 73.7% and 82.3%. They should also claim they are 90% confident of this. When calculating the 95% and 99% confidence intervals based on the same sample, students should recognize that a higher confidence level results in a larger interval. Students should also be aware of the limitations of such a survey. The car rental company only randomly selected customers from Nova Scotia. Since they did not include customers from other provinces, they cannot apply this confidence interval to customers from other parts of Canada.

It is very important that students also consider the effect of sample size on the length of a confidence interval. While higher levels of confidence will produce longer confidence intervals when constructed from samples of the same size, using larger sample sizes will produce shorter confidence intervals. It is possible, for example, that a 90% confidence interval based on a sample of size 40 will be longer than a 95% confidence interval based on a sample of size 100 from the same population. Students could repeat the above example, but this time assuming a sample size of 500 with 390 claiming they were satisfied. They should compare the lengths of the 90%, 95%, and 99% confidence intervals with the corresponding intervals based on the smaller sample size. If one were to flip a fair coin only once, then there is no chance that the sample proportion of heads would match the population proportion of $p = 0.5$. If it were flipped twice, while it is possible that one head would be obtained (giving $p = 0.5 = p$), it is also very possible to obtain no heads ($p = 0$) or two heads ($p = 1$). However, as the number of repetitions is increased, it is generally the case that the sample proportion more consistently approximates the population proportion (at ten flips, for example, one would usually get anywhere from 3 to 7 heads, meaning p would range from 0.3 to 0.7, all of which provide better estimates of p than for two of the possible outcomes in the two flip situation. For 50 flips, one would usually obtain somewhere between 20 and 30 heads, all outcomes which give values for p that are even more reasonable estimates of p). An interesting activity here might be to have students construct a broken line graph, similar to the one shown, based on an experiment with flipping coins. One should see that the sample proportions more consistently approximate the population proportion as the number of flips is increased. This would also intuitively lead one to conclude that the larger sample sizes would usually allow for better estimates of the population proportion and therefore shorter confidence intervals than similar intervals based on smaller sample sizes.



Worthwhile Tasks for Instruction and/or Assessment

(...continued from the last 2 page spread.)
F11/F1/F20/F2/F7 *Activity*

F11/F1.4. A random sample of 267 Canadian doctors showed that 215 provided at least some charity care (i.e. treated people at no cost).

a) Let p represent the proportion of all Canadian doctors who provide some charity care. Find a point estimate for p.
b) Find a 99% confidence interval for p. Give a brief explanation of the meaning of your answer in the context of this problem.
c) Another random sample of 100 Canadian doctors showed that 79 of them provided at least some charity care. Find a 99% confidence for p and compare it to the one you found in (b). Which interval is of more use in drawing conclusions about the entire population of Canadian doctors and the proportion that provide at least some charity care? Explain, giving at least two reasons for your answer.

Suggested Resources

Unit 2
Independent Study
(10-15 Hours)

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I2 communicate the results of the independent research

Elaboration – Instructional Strategies/Suggestions

I1/I2

The purpose of this independent study is 1) to prepare students for learning independently, and 2) to provide students with the opportunity to explore

- in more depth, mathematical content that they have been exposed to but would like to know more about
- new mathematical content areas not yet explored
- mathematical topics of interest
- historical studies and connections to the math we study
- mathematics in our lives, and related to careers
- mathematics through the Internet
- how people learn mathematics

Approximately 10–15 hours of class time should be devoted to this research project. Teachers should allow time for

- students to present the results of their research and learnings to other students (presentation time of about 10 minutes per student should be allowed for). If students are working collaboratively on this project, it is expected that each would be responsible for gathering certain information and thus could be held responsible for the oral presentation that deals with that part of the project.
- initial discussion and discovering of ways to get information, what it means to learn mathematics independently, and why that is important (the resources should supply activities to stimulate this).
- an introduction to topics not yet studied to whet students' appetites.
- discussing the expectations and assessment rubrics for the student presentations at the end of the unit and how they will be assessed during the unit.
- brainstorming, topic-webbing, developing action plans and time lines, conferencing ... .

Managing the Project

The managing of the project should be closely teacher-directed:

- Various topics will determine the appropriate group size.
- Some students may wish to work independently.
- Students will choose appropriate topics (perhaps from a teacher-prepared list) that are appropriate and of interest to them. Teachers should ensure the availability of reference resources (material and human) in or around the school, or the community and teachers should give final approval for each topic.

. . . continued

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

New Topics for Secondary School Mathematics, Matrices, NCTM, 1988

de Lange, Jan, *Meaningful Math, Matrices*, WINGS for Learning, 1992

de Lange, Jan, *Flying Through Math*, T*rig, Vectors, and Flying*, WINGS for Learning, 1991

Froelich, Gary et al., *Discrete Mathematics Through Applications*, W. H. Freeman and Company, New York, 1994

Jacobs, Harold R., *Mathematics, A Human Endeavor*, Third Edition, W. H. Freeman and Company, New York, 1994

Serra, Michael, *Discovering Geometry*, Second Edition, Key Curriculum Press, 1997

Charles, Randall et al., *How to Evaluate Progress in Problem Solving*, NCTM, 1992

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I2 communicate the results of the independent research

Elaboration – Instructional Strategies/Suggestions

. . . continued

I1/I2

Brainstorming or topic-webbing should take place.

Action plans should detail the tasks that have to be completed (it is assumed that each student will have responsibility for independent work within the structure of the project), for example:

- write letters to gather data or request materials
- make phone calls for information, read texts, newspapers, flyers, journals, and reports
- complete library search
- interview resource people
- reflect on and share ideas with group members
- prepare oral presentation
- prepare written submission

Each student or group should detail their own time lines to match the teacher's.

- Deadlines should be determined for the work that has to be completed, and for bringing completed work to class. Regular conferences regarding progress are crucial. This includes conferencing with group members and with the teacher.

Suggested Topic and Content Areas

New Content

vectors with respect to navigation and forces graph theory (4-colour problem and travelling salesman problem), or other topics found under the heading 'Discrete Mathematics' applications of matrices, such as Markov Chains and Leontief Input—Output models

Topics for More Depth

proof in mathematics
algebraic manipulation
functions, compositions of functions—connection to art
regression analysis
the story on infinity and zero
parametric equations
conics from a geometric perspective

Mathematics in our Lives

Fibonacci numbers—connecting to the world
geometry in our lives—patterns, design, architecture
mathematics in jobs—interview a person about how he/she uses math in his/her job
do statistics lie?
consumer mathematics
career options
leisure mathematics—non-routine, recreational problems, logic, math games, puzzles, games of chance
the Internet as a source of mathematics information

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I2 communicate the results of the independent research

I3 demonstrate an understanding of the mathematical topics presented by other students

I1/I2/I3 Teachers might facilitate this unit by

Elaboration – Instructional Strategies/Suggestions

focussing 10–15 hours on the project all at one time-teachers should be aware of the time needed to gather and compile information

spreading the project over a term, with some class periods being designated to the project introduction, and as checkpoints, each with a particular expectation, and for finalization, preparing and performing presentations

integrating it with a topic going on at the same time in the classroom—statistics, algebra, indirect measurement

Expectations for assessment must be made clear to students:

All students must be involved with the presentation on what mathematics they have researched:

- oral presentation to class, or
- oral presentation on video and played to the class, or
- conversation between students or among group members in front of class or on video, or
- teacher/student interview (private or in front of class) or,
- some other variation.

Executive summaries must be distributed to the class at the time of presentation. This means that students should summarize the new mathematics learned so that other students can read over the summary, see a couple of examples, and have a pretty good feeling for the new topic.

Assessment

Criteria for written submission should be made clear.

Criteria for presentations should be made clear (students should not simply read their written submission).

Criteria should be prepared by the teacher and discussed with the students prior to the assigning of the project.

A rubric (students could help design it) should be included that allows for assessment on the written work as well as the presentation.

Peer evaluation should occur during presentations and from each group member on the group effort.

continued ...

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

Assessment Alternatives in Mathematics, Equals Publishing/Lawrence Hall of Science, University of California, 1989

Mathematics Assessment: Myths, Models, Good Questions, and Practical Suggestions, NCTM, 1992

Charles, Randall et al., *How to Evaluate Progress in Problem Solving*, NCTM, 1992

Independent Study

Outcomes

SCO: In this course, students will be expected to

I1 demonstrate an understanding of a mathematical topic through independent research

I2 communicate the results of the independent research

I3 demonstrate an understanding of the mathematical topics presented by other students

Elaboration – Instructional Strategies/Suggestions

. . . continued

I1/I2/I3 A possible rubric for the written component might look something like the following:

Top Level

- contains a complete report with clear, coherent, unambiguous, and elegant explanations
- includes clear and simple diagrams, charts, graphs, etc.
- communicates effectively to an identified audience
- shows understanding of the mathematical ideas and processes
- identifies all the important elements of the topic
- includes examples and counter-examples
- gives strong supporting arguments

Second Level

- contains good solid report with some of the characteristics above
- explains less elegantly, less completely
- does not go beyond the requirements of the project (or topic)

Third Level

- contains a complete report but the explanation may be muddled
- presents arguments but incomplete
- includes diagrams but inappropriate or unclear
- indicates understanding of mathematical ideas, but not expressed clearly

Fourth Level

- omits significant parts
- has major errors
- uses inappropriate strategies

See the books in the Suggested Resources column for more examples of rubrics for evaluating projects and open-ended activities.

I3 Students should collect executive summaries from students who are currently presenting. Students should ask questions for clarification at the end of presentations. Students might demonstrate their learnings from presentations of others by completing a questionnaire that focusses on the highlights of a presentation. On a test, teachers might ask students to discuss, showing examples, what was learned from any presentation. Another strategy for assessment might be through a conversation between the teacher and student about someone else's project.

Independent Study

Tasks for Instruction and/or Assessment

Resources/Notes

Unit 3
Probability
(20 Hours)

Probability

Outcomes

SCO: In this course, students will be expected to

$G2_3$ demonstrate an understanding that determining probability requires the quantifying of outcomes

$G3_3$ demonstrate an understanding of the fundamental counting principle and apply it to calculate probabilities of dependent and independent events

Elaboration—Instructional Strategies/Suggestions

$G2_3$ Every day students experience a variety of situations. Some involve making decisions based on their previous knowledge of similar situations.

- Should they do their math homework tonight or during their spare period before math class tomorrow?
- Should they challenge a friend to a game of racquetball or blockers?
- Should they buy a ticket on a car raffle?
- Should they take their umbrella today?

Before making the decision, what they must ask themselves is What is the chance of this decision working out in my favour?

In probability, the goal is to assign numbers between 0 and 1 inclusive to events that interest us, but for which we do not know the outcome.

In their previous studies (grades 7–9) students have created and solved problems using probabilities, including the use of tree and area diagrams and simulations. They have compared theoretical and experimental probabilities of both single and complementary events and dependent and independent events. They have examined how to calculate complementary events as well as two independent events, A and B, the probability of A and B is equal to $P(A) \times P(B)$.

Sometimes the task of listing and counting all the outcomes in a given situation is unrealistic, since the sample space may contain hundreds or thousands of outcomes.

$G2_3/G3_3$ The fundamental counting principle enables students to find the number of outcomes without listing and counting each one. For independent events, if the number of ways of choosing event A is $n(A)$ and the number of ways of choosing event B is $n(B)$, then

$$n(A \text{ and } B) = n(A) \times n(B), \text{ and}$$
$$n(A \text{ or } B) = n(A) + n(B).$$

The first is the multiplication principle, the second, the addition principle.

Sometimes events are not independent. For example, suppose a box contains three red marbles and two blue marbles, all the same size. A marble is drawn at random. The probability that it is red is $\frac{3}{5}$. If the marble is then replaced, the probability of picking a red marble again is $\frac{3}{5}$. However, if it is not replaced, then when another marble is picked the probability of its being red is now $\frac{2}{4}$. The second selection of a marble is dependent on the first selection not being returned to the box.

continued ...

Probability

Worthwhile Tasks for Instruction and/or Assessment

$G2_3/G3_3$

Activity

1) Two students are playing “grab” with a deck of special “grab” cards. One student has a triangular-shaped deck with 16 ones, 12 twos, 8 threes, and 4 fours. The other has a rectangular shaped deck with 10 each of ones, twos, threes, and fours. The decks are well shuffled and each student’s plays the top card simultaneously. A “grab” is made when two cards match (a double).
   a) There are 40 cards in each deck. What is the total number of pairs of cards that could be played?
   b) How many of these are “double ones,” that is, a one from the triangular deck and a one from the rectangular deck?
   c) How many are i) double twos?ii) double threes? iii) double fours?
   d) For equally likely outcomes, the probability of an event is “the number of outcomes that correspond to the event” divided by what?
   e) So, the probability of a double one is “what” divided by “the total number of pairs”?
   f) Use this principle and your answers to (c) to find the probability of i) a double one ii) a double two iii) a double.
   g) A circular deck has 10 ones, 20 twos, 10 threes, and no fours. Calculate the probability of a grab if a triangular deck is played against a circular deck.

Performance

2) Telephone numbers are often used as random number generators. Assume that a computer randomly generates the last digit of a telephone number. What is the probability that the number is
   a) an 8 or 9?
   b) odd or under 4?
   c) odd or greater than 2?

3) A airplane holds 176 passengers: 35 seats are reserved for business class, including 15 aisle seats; 40 of the remaining seats are aisle seats. If a passenger arrives late and is randomly assigned a seat, find the probability of that person getting an aisle seat or one in the business section.

4) Use the given table, which represents the number of people who died from accidents by age group to find the following. In each case assume that one person is selected at random from this group:
   a) the probability of selecting someone under 5 or over 74
   b) the probability of selecting someone between 16 and 64
   c) the probability of selecting someone under 45 or between 25 and 74

Age	Number
0-4	3,843
5-14	4,226
15-24	19,975
25-44	27,201
45-64	14,733
65-774	8,499
75 and over	16,800

Suggested Resources

Flewelling, Gary et at., *Mathematics 10 A Search for Meaning*. Toronto: Gage 1987.

Probability

Outcomes

SCO: In this course, students will be expected to

$G3_3$ demonstrate an understanding of the fundamental counting principle and apply it to calculate probabilities of dependent and independent events

Elaboration—Instructional Strategies/Suggestions

continued ...

$G3_3$ How is the fundamental counting principle related to probability? Consider the marble situation described at the bottom page 84. The probability of selecting red is $\frac{3}{5}$, while the probability of selecting blue is $\frac{2}{4}$. The probability of selecting a red and a blue without replacement would be $P(r \text{ and } b)\frac{3}{5}\times\frac{2}{4}=\frac{6}{20}$.

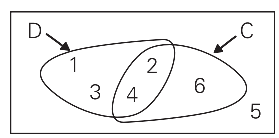
Now, let us consider another situation:

Consider the experiment of a single toss of a standard die. There are six equally likely outcomes: 1, 2, 3, 4, 5, and 6. Define certain events as follows:

A: observe a 2
B: observe a 6
C: observe an even number
D: observe a number less than 5.

$P(A)=\frac{1}{6}(\text{observe a } 2)$, $P(B)=\frac{1}{6}(\text{observe a } 6)$. What about *P*(*A* or *B*) (observe a 2 or 6)? This can be shown two ways: $\frac{n(A)+n(B)}{\text{total number of ways}} = \frac{1+1}{6} = \frac{2}{6}$

or *P* (*A* or *B*) = *P* (*A*) + *P* (*B*)= $\frac{1}{6}=\frac{1}{6}=\frac{2}{6}$. Will this be true for any two events? The events "observe a 2", and "observe a 6" are called mutually exclusive events, or disjoint, because one can observe only a 2 or a 6, not both at the same time. On the other hand, events like C and D above have at least one element in common and therefore are not mutually exclusive.



Consider the events C and D. The event (C or D) includes all the outcomes in C or D or both.

That is, P(C or D) = P(observe an even number or a number less than five)

= *P* (observe 2, 4, 6, or observe 1, 2, 3, 4)

Every outcome except 5 is included in (C or D). Thus there are exactly five favourable outcomes. Thus *P* (*C* or *D*) = $\frac{5}{6}$

But $P(C)+P(D)=\frac{3}{6}+\frac{4}{6}=\frac{7}{6}$, which cannot be possible since it exceeds 1.

The outcomes 2 and 4 are contained in both C and D and must be removed. There is an overlap.

$$P(C \text{ or } D)=P(C)+P(D)-P(C \text{ and } D)=\frac{3}{6}+\frac{4}{6}-\frac{2}{6}=\frac{5}{6}.$$

Probability

Worthwhile Tasks for Instruction and/or Assessment

$G3_3$

Performance

1) Discuss whether the following pairs of events are mutually exclusive and whether they are independent.
   a) The weather is fine; I walk to work.
   b) I cut a deck of cards and have a Queen; you cut a 5.
   c) I cut the deck and have a red card; you cut a card with an odd number.
   d) I select a voter who registered Liberal; you select a voter who is registered Tory.
   e) I found a value for x to be greater than –2; you found x to have a value greater than 3.
   f) I selected two cards from the deck, the first was a face-card, the second was red.
2) If 366 different possible birthdays are each written on a different slip of paper and put in a hat and mixed,
   a) find the probability of making one selection and getting a birthday in April or October
   b) find the probability of making one selection that is the first day of a month or a July date
3) A store owner has three student part-time employees who work independently of each other. The store cannot open if all three are absent at the same time.
   a) If each of them averages an absenteeism rate of 5%, find the probability that the store cannot open on a particular day.
   b) If the absenteeism rates are 2.5%, 3%, and 6% respectively for three different employees, find the probability that the store cannot open on a particular day.
   c) Should the owner be concerned about opening in either situation a) or b)? Explain.
4) There are 6 defective bolts in a bin of 80 bolts. The entire bin is approved for shipping if no defects show up when 3 are randomly selected.
   a) Find the probability of approval if the selected bolts are replaced, are not replaced.
   b) Compare the results. Which procedure is more likely to reveal a defective bolt? Which procedure do you think is better? Explain.
5) Mary randomly selects a card from an ordinary deck of 52 playing cards. What is the probability that Mary will select either an ace or a diamond? Below is Fred's solution. Explain what Fred is thinking. Will his attempt lead to a correct answer? Explain.

$$P\left(\text{ace or diamond}\right)=\tfrac{4+13}{52}=\tfrac{17}{52}$$

Journal

6) Consider the table of experimental results. Comment on the following solution attempts.
   a) If one of the 2072 subjects is randomly selected, the probability of getting someone who took Seldane or a placebo is

	Seldane	Placebo	Control	Total
Drowsiness	70	54	113	237
No drowsiness	711	611	513	1835
	781	665	626	2072

$$\tfrac{781}{2072}+\tfrac{665}{2072}=\tfrac{1446}{4144}=0.3489$$

   b) If one of the 2072 subjects is randomly selected, the probability of getting someone who took Seldane or experienced drowsiness can be found by:

$$\tfrac{781}{2072}+\tfrac{237}{2072}=\tfrac{1018}{4144}=0.491$$

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

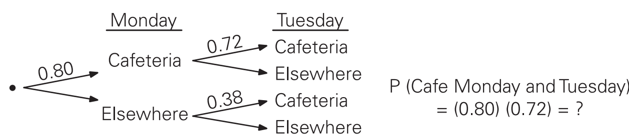
G4 apply area and tree diagrams to interpret and determine probabilities

Elaboration—Instructional Strategies/Suggestions

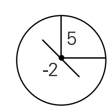
G4 Students have studied area and tree diagrams since grade 7 and have applied them to help establish the sample space, or the total number of possible outcomes in a situation. In this course their experiences with these diagrams will be extended to probability tree diagrams and area diagrams that will help students visualize and calculate the probabilities given certain situations.

Consider the following situation. Students at Yore High School have two choices for where to eat lunch, in the cafeteria or elsewhere outside the school. Mildred, the manager of the cafeteria, needs to be able to predict how many students can be expected to eat in the cafeteria over the long run. Mildred asks the math class to conduct a survey. The results show that if a student eats in the cafeteria on a given day, the probability that he or she will eat there the next day is 72%. If a student does not eat in the cafeteria on a given day, the probability that he or she will eat in the cafeteria the next day is 38%. On Monday, 80% of the students ate in the cafeteria. What can Mildred expect for Tuesday?

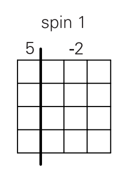
A good way to organize all these statistics is with a probability tree diagram:



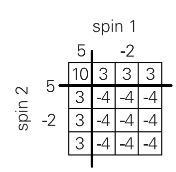
Geometric or area models will be useful to some students as these models provide a pictorial representation of the analysis which provides the students with a visual insight into the concept of probability. Consider the following situation. One of the events at your school's spring fair is a game of chance involving points. For each turn, a player spins and gets the points indicated in the area in that the spinner lands. Each player should add the numbers obtained by spinning twice. What are all the possible sums? What are the probabilities for obtaining each of these sums?



Students will notice that the spinner suggests that getting –2 will happen three-quarters of the time, while getting 5 will occur one quarter of the time. Using a grid of 16 squares to represent



the probability of 1, they would draw a vertical line (as in fig. 1) to represent the probabilities for the first spin (1/4 and 3/4). They would then separate the grid horizontally (as in fig. 2) to represent the probabilities of getting a –2 or 5 on the second spin. They would then analyse the grid to find the probabilities of obtaining the sums –4, 3, and 10.



P(–4) = 9/16, P(3) = (3/16)×2, and

P(10)= 1/16.

Now, using these results the students can be asked to create a situation where a player must accomplish something in order to win the game.

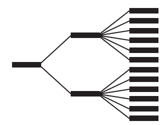
Probability

Worthwhile Tasks for Instruction and/or Assessment

G4

Pencil and Paper

1) This incomplete tree diagram lists all the outcomes of tossing a coin and then rolling a die.



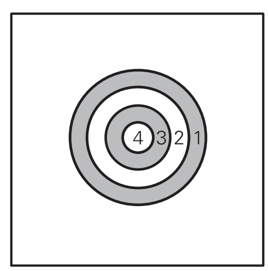
   a) Copy and complete the diagram.
   b) How many pairs of outcomes are there in this multiple event?
   c) What is the probability of tossing a head on the coin and then rolling a six on the die?
   d) What is the probability of tossing a head on the coin and then rolling an even number?
   e) What is the probability of *not* tossing a head on the coin and then rolling an even number of the die?

2) In a restaurant there are four kinds of soup, 12 entrees, six desserts, and three drinks. How many different four-course meals can a patron choose from? If 4 of the 12 entrees are chicken and two of the desserts involve cherries, what is the probability that someone will order wonton soup, a chicken dinner, a cherry dessert, and milk?

3) Licence plates for cars often have three letters of the alphabet, then three digits from 0 to 9. How many possible different licence plates can be produced? What is the probability of having the plate "CAR 000"?

Performance

4) The dart board at the right consists of four concentric circles whose centre is the centre of the square board. The side length of the square is 36 cm. The circles have radii 2 cm, 4 cm, 6 cm, and 8 cm respectively. A dart hitting the bull's eye or one of the shaded rings scores the indicated number of points. A hit anywhere else on the board scores 0 points. Assume that a dart thrown at random hits the board. Determine the probability of scoring:
   i) 4 points ii) 3 points iii) 2 points iv) 1 point v) 0 points



5) The following problem illustrates the usefulness of geometric probability. A tape recording is made of a meeting between a senator and her aide. Their conversation starts at the 21st minute on a 60-minute tape and lasts 8 minutes. While playing back the tape the aide accidentally erases 15 minutes of the tape.
   a) What is the probability that the entire conversation was erased?
   b) What is the probability that some part of the conversation was erased?
   c) Suppose the exact portion of the conversation on the tape is not known, except that it began sometime after the 21st minute. What is the probability that the entire conversation was erased?

6) Consider finding the area of the region bounded by the ellipse $4x^2 + y^2 = 4$. Enclose the ellipse in a rectangle whose sides pass through the x- and y-intercepts, and then consider the rectangular region to be a dart board. Suppose several darts thrown at random hit the rectangular region.
   a) Explain how probability can be used to approximate the area of the region bounded by the ellipse.
   b) Explain how probability can be used to approximate the area of the region bounded by the equation $y = -x^2 + 4$.

Suggested Resources

Probability

Outcomes

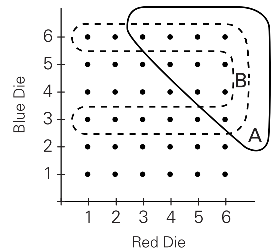
SCO: In this course, students will be expected to

G5 (111) determine conditional probabilities

Elaboration—Instructional Strategies/Suggestions

G5(111) Ask the students a question such as "What is the probability that event A occurs if it is known that event B has occurred?" You should, through specific examples and some discussion, be able to get the class to arrive at a definition for conditional probability. For example:

If two dice, one red and one blue, are thrown and it is known that the blue die shows a number divisible by three, ask students what the probability is that the total on both dice is greater than 8? The condition that the number on the first die be divisible by three changes the sample space under consideration.



In particular, the new sample space contains only the 12 points shown inside the dashed closed curve at the right. For how many of these points is the total greater than 8? In light of the fact that all 36 points in the original sample space were assumed to be equally likely, students should agree that it seems reasonable to say that all 12 points in this sample space are equally likely. Then, given the condition that the number on the blue die is divisible by three, students should calculate the probability of having a total greater than 8 is equal to $\frac{5}{12}$.

For any two events A and B, the symbol "$P(A|B)$" is used to designate the probability that event A occurs given that event B has occurred. This is called a conditional probability because the condition is given that event B has occurred.

To evaluate $P(A|B)$ reconsider the above problem. Let the original sample space be the set of 36 possible outcomes shown in the diagram, let A be the set of points for which the total number of spots showing is greater than 8, and let B be the set of points for which the number of spots showing on the first die is divisible by three. Then $A \cap B$, pronounced 'A intersect B' consists of the 5 points indicated in the diagram by the triangular shape. In this case, to determine the conditional probability $P(A|B)$, divide the number of points in $A \cap B$ by the number of points in B. Of course, if the points of the original sample space were not equally likely, the result could not be obtained by simply counting points. Therefore, the probability of event A given that event B has occurred is defined as the probability of $A \cap B$ divided by the probability of B.

The probability that event A occurs if it is known that event B has already occurred is known as "conditional probability." It is symbolized as $P(A|B)$, and calculated using

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \leftarrow \text{ new sample space}.$$

If the first of three tosses of a fair coin is heads, find the probability of getting exactly two heads in three tosses.

Solution: Let A be the event "getting exactly two heads."
Let E be the event "getting a head on the first throw."

$$(A \cap E) = \{HHT, HTH\}$$

$$\text{so,} \quad P(A \cap E) = \tfrac{2}{8} = \tfrac{1}{4}, P(E) = \tfrac{1}{2}$$

$$\therefore P(A|E) = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

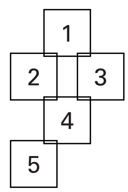
Probability

Worthwhile Tasks for Instruction and/or Assessment

G5(111)

Performance

1) What is the probability of getting two fives when two dice are thrown and it is known that at least one landed with a five up?
2) Assuming the probability of being born male is 0.5. In a family of three children it is known that at least one child is male. What is the probability that all three children are male?
3) A weather report indicates an 80% probability of rain on Monday, 60% on Tuesday, and 20% on Wednesday. What is the probability that it will rain on at least one of the three days?
4) In the MAKE-A-NUMBER game, you draw a Condition Card. Then you draw two Number Cards from a stack of only five cards and place them side-by-side to make a two-digit number. If the two-digit number fits your Condition Card, you score one point.



1. Condition

The number is divisible by 3.

Probability: ________

2. Condition

The sum of the digits of the number is 5.

Probability: ________

3. Condition

The number is greater thatn 40.

Probability: ________

4. Condition

The number is a prime number.

Probability: ________

5. Condition

The tens digit of the number is greater than the ones.

Probability: ________

6. Condition

The units digit of the number is divisible by the tens digit

Probability: ________

Determine the probability of scoring with these Condition Cards.

Suggested Resources

Shulte, Albert P., ed. *Teaching Statistics and Probability*. *1981 Yearbook*. Reston, VA: NCTM, 1981.

Probability

Outcomes

SCO: In this course, students will be expected to

G5 (111) determine conditional probabilities

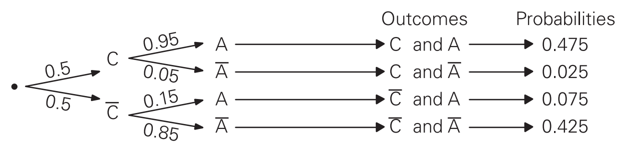
Elaboration—Instructional Strategies/Suggestions

G5(111) Tree diagrams are often used to organize all the possible combined outcomes of a multiple event. For example: A fast food outlet requires prospective employees to take an employability exam as a basis for hiring. They estimate that 50% of the applicants would complete a full year's work. They also estimate that 15% of the unqualified applicants pass the exam and 5% who are qualified fail the exam. All who pass the exam are hired and none of the other are. What fraction of applicants are hired? If an applicant passes the exam, what is the probability that he or she will complete a year's work?

Let C represent the event "applicant completes a year"

Let A represent the event "applicant passes exam"

Thus $\overline{C}$ and $\overline{A}$ are the complementary events.



The first and third paths represent hired students, so $P(A) = .475 + .075 = .55$. So 55% of the applicants are hired. The first path shows 47.5% of the applicants are both hired and work a year, and so the conditional probability (see previous two-page spread) of completing a year, given the passing the exam and being hired, is

$$P(C|A) = P(C \cap A)/P(A) = .475/.55 = 0.86.$$

So, 86% of those hired complete a year's work. An area diagram example: Suppose that Tom is a 60% free throw shooter in basketball. At the end of a game he was fouled and his team is losing by two points. He will shoot "one-and-one." What is the probability that he misses the second shot? To solve this problem, students could use an area model like that on the right. The probability of making the first shot is shown in fig. 1, then if he makes the first shot, he gets the second shot. Fig. 2 shows the probability of missing the second shot.

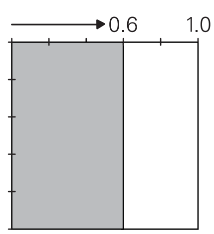


fig. 1

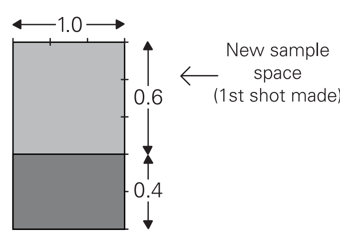


fig. 2

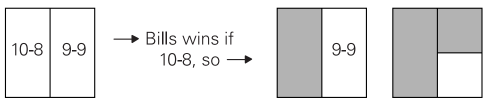
Probability

Worthwhile Tasks for Instruction and/or Assessment

G5(111)

Performance

1) Two gamblers play a game for a stake that goes to the first player to gain 10 points. If the game is stopped when the score is 9 to 8, in favour of Bill, what is the probability that Bill will win when the game is resumed? Use an area model to help ... (It is assumed that both players have equal chances of winning each point.)
   If the score is 9–8 then the next score will be ...



   If the game goes to 9–9, either one might win.
   a) What can you conclude from this?
   b) What would be the solution to the problem if Bill was winning 9–7 when the game is stopped?

2) As archers, Rita hits the target $\frac{2}{5}$ of the time and David $\frac{1}{3}$, of the time. They are going to have a contest with David shooting first. They alternate shots until one wins by hitting the target. Who is favoured? What is each contestant's probability of winning?

3) A certain restaurant offers select-your-own desserts. That is, a person may select one item from each of the categories listed:

Ice Cream	Sauce	Extras
vanilla	chocolate	cherries
strawberry	caramel	peanuts
chocolate mint		

   a) Using a tree diagram, list all possible desserts that can be ordered.
   b) Would you expect the choices of a dessert to be equally likely for most customers?
   c) If the probability of selecting chocolate ice cream is 40%, and vanilla is 10%, chocolate sauce is 70%, and cherries 20%, describe the dessert with the highest probability of being selected.

4) A certain model of automobile can be ordered with one of three engine sizes, with or without air conditioning, and with automatic or manual transmission.
   a) Show, by means of a tree diagram, all the possible ways this model car can be ordered.
   b) Suppose you want the car with the smallest engine, air conditioning, and manual transmission. A General American agency tells you there is only one of the cars on hand. What is the probability that it has the features you want, if you assume the outcomes to be equally likely?

5) Jennifer dresses in a skirt and a blouse by choosing one item from each category.

Skirts	Blouses
tan, plaid, gray, stripe 1, stripe 2, stripe 3	white, pink 1, pink 2, red

   a) Show, by means of a tree diagram, all the outfits she can make if one has three striped skirts and two pink blouses and only one of everything else.
   b) What's the probability of her wearing something striped and white knowing that she already has a striped skirt on?

Suggested Resources

Newan, Claire et al. *Exploring Probability. Quantitative Literacy Series.* White Plains, NY: Dale Seymour Publications, 1987.

Probability

Outcomes

SCO: In this course, students will be expected to

$G1_3$ develop and apply simulations to solve problems

Elaboration—Instructional Strategies/Suggestions

$G1_3$ Simulation is a procedure developed for answering questions about real problems by running experiments that closely resemble the real situation.

Suppose the students want to find the probability that a three-child family contains exactly one girl. If students cannot compute the theoretical answer and do not have the time to locate three-child families for observation, the best plan might be to simulate the outcomes for three-child families. One way to accomplish this is to toss coins to represent the three births. A head could represent the birth of a girl. Then, observing exactly one head in a toss of three coins would be similar, in terms of probability, to observing exactly one girl in a three-child family. Students could easily toss the three coins many times to estimate the probability of seeing exactly one head. The result gives them an estimate of the probability of seeing exactly one girl in a three-child family. This is a simple problem to simulate, but the idea is very useful in complex problems for which theoretical probabilities may be nearly impossible to obtain.

Students need experience thinking through complete simulation processes. When choosing a simple device to model the key components in the problem they have to be careful to choose a model that generates outcomes with probabilities to match those of the real situation. Students could use devices such as coins, dice, spinners, objects in a bag, and random numbers.

Students need to understand that the experimental probability approaches the theoretical probability as the number of trials increases. They should also realize that knowing the probability of an event gives them no predicting power as to what the outcome of the next trial will be. However, after enough trials, they should be able to predict with some confidence what the overall results will be.

When conducting simulations students should follow a certain process such as the one outlined: (see next page for an actual class activity).

Step 1: State the problem clearly.

Step 2: Define the key components.

Step 3: State the underlying assumptions.

Step 4: Select a model to generate the outcomes for a key component.

Step 5: Define and conduct a trial.

Step 6: Record the observation of interest.

Step 7: Repeat steps 5 and 6 until 50 trials are reached.

Step 8: Summarize the information and draw conclusions.

Probability

Worthwhile Tasks for Instruction and/or Assessment

$G1_3$

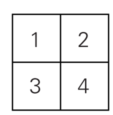
Pencil and Paper

1) Consider the following problem:
   Marie has not studied for her history exam. She knows none of the answers on the seven-question true-and-false section of the test. She decides to guess at all seven. Estimate the probability that Marie will guess the correct answers to four or more of the seven questions.
   Ask students to complete the following:
   a) What are you being asked to do?
   b) To perform a simulation, what assumptions should you make?
   c) Describe the model you would choose to perform the simulation.
   d) Pretend that you are watching the simulation. Describe what you observe for the entire simulation.
   e) What conclusion do you think would be made?

2) Suppose a stick or a piece of raw spaghetti has been broken at two random points. What is the probability that the three pieces will form a triangle? (The pieces must touch end to end.)
   a) Describe the process that might be used to estimate the answer using experimental probability.
   b) Instead, Robert is going to use a simulation. He assumes the spaghetti is 100 units long, and he is going to generate two random numbers between 0 and 100 using each as a side of a triangle. How would Robert find the third side? How would Robert check to see if the numbers represent the lengths of the side of a triangle?
   c) Perform this simulation to find the answer.

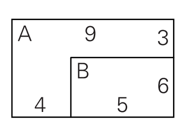
Performance

3) Dale, a parachutist, jumps from an airplane and lands in a field. What are the chances that Dale will land in a particular numbered plot? Make a field grid using a normal sheet of graph paper divided into four equal areas.
   a) Model the situation by tossing a thumbtack onto the grid from a metre or more away. (If the tack bounces off the sheet—don't count it as a toss.)



   In your response consider several questions:
   Is there an equal chance to land in each plot?
   How many times did Dale land in plot 1?
   Discuss the experimental probability results versus the theoretical probability results for the given field.
   b) Conduct the experiment again, but use a field divided into plots A and B to find the probability that Dale will land in Plot A.
   c) Perform a simulation to answer the same problem as in (b). Compare the results of the simulation with that of the experiment. Comment.

4) Perform simulations to solve the following problems:
   a) What is the probability that all five children in a family will be girls?
   b) A couple leaves for work anytime between 7:00 and 8:00 am. Their newspaper arrives any time between 6:30 and 7:30 am. What is the probability that they get the paper before they leave for work?



Suggested Resources

Zawojewski, Judith. *Dealing With Data and Chance. Curriculum and Evaluation Standards for School Mathematics Addenda Series. Grades 5–8.* Reston, VA: NCTM, 1992.

Probability

Outcomes

SCO: In this course, students will be expected to

G7 distinguish between situations that involve permutations and combinations

Elaboration—Instructional Strategies/Suggestions

G7 Before describing different situations in terms of permutations and combinations, students need to have an opportunity to solve simple counting problems (see elaboration for $G2_3$, p. 84). They may wish to organize their work into systematic lists and/or tree diagrams. As the number of choices increases, they will see the need for a way to count more efficiently. For example:

a) How many different routes can you take from Sydney to Halifax through Antigonish?
b) How many routes are there from Antigonish to either Halifax or Sydney?

Following this, the class might be split into two groups—one will do Problem A, the other Problem B. Students should present their solution to the class.

Problem A: Suppose there were three people, Adam, Marie, and Brian, standing in line at a banking machine. In how many different ways could they order themselves?

Problem B: The executive of the student council has five members. In how many ways can a committee of three people be formed?

Solutions might look like:

Problem A: using a systematic list:A M B, A B M, M B A, M A B, B A M, B M A

Problem B: using a systematic list : if Adam, Marie, and Brian along with Dennis and Elaine were on the executive, then to select committees of three, starting with Adam, Marie and Brian, the five permutations in the answer to A above would result in the same five people being the committee, so they represent one combination.

The essential difference between these two situations needs to be discussed and emphasized. Eventually, Problem A should be described as a permutation (order is important), Problem B as a combination (order not important).

Probability

Worthwhile Tasks for Instruction and/or Assessment

G7

Pencil and Paper

1) For each of the following, decide whether permutations or combinations are involved.
   a) The number of committees of two that can be formed from a group of 12 people.
   b) The number of possible lineups for a baseball team that can be formed from 12 people without regard to position (a baseball team consists of nine players, as follows: pitcher; catcher; first, second, and third basemen; shortstop; right, centre, and left fielders).
   c) The number of five-letter licence plates that can be formed from 12 different letters.
   d) The number of six subsets that can be formed from 12 different letters.
   e) The number of five-man basketball teams that can be formed from 10 players.
   f) The number of ordered triples that can be formed from 10 different numbers.
   g) The number of ordered triples that can be formed from the numbers 1, 1, 1, 3, 3, 5, 5, 5, 5, and 4.

2) The manager of a baseball team needs to decide the batting order for the season opener. In how many ways can the first four batters be arranged on the batting roster? Is this a permutation or combination question? Explain.

3) As a promotion, a record store placed 12 tapes in one basket and 10 compact discs in another. Pierre was the one millionth customer and was allowed to select 4 tapes and 4 compact discs. To find how many selections that can Pierre make, does one use permutations or combinations? Explain.

4) Three identical red balls (R) and two identical white balls (W) are placed in a box. How many ways are there of selecting the balls in the following order?

   RWRRW

5) a) Find the total number of arrangements of the letters of the word "SILK."
   b) Find the total number of arrangements of the letters of the word "SILL."
   c) How are your answers in a) and b) alike? How are they different?

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

A6 develop an understanding of factorial notation and apply it to calculating permutations and combinations

G8 develop and apply formulas to evaluate permutations and combinations

Elaboration—Instructional Strategies/Suggestions

A6 As students refine their methods of counting, moving from tree and area diagrams and listing through the fundamental counting principles, they should learn to recognize and use *n*! (*n* factorial) to represent the number of ways to arrange *n* distinct objects. For example, the product rule can be used to find the number of possible arrangements for three people standing in a line. There are three people to choose from for the front of the line. For each of these choices, there are two people to choose from for the second position in the line. For each of these choices, there is one person to choose from the end of the line. Therefore, there are $3\times2\times1$ or six possible arrangements.

In another example, at a music festival, eight trumpet players competed in the Baroque class. After the judging, they were awarded first, second, third... down to eigth place. In how many ways could their placements be awarded?

If all the trumpet players were given a position first, second, third, ... , eighth, then the total number of possible standings could be calculated by using reasoning like: There are eight people eligible for first, which leaves seven eligible for second, six people for third ... leading to a calculation $8\times7\times6\times5\times4\times3\times2\times1$. This product can be written in a compact form as 8! and is read “eight factorial.”

In general, $n!=n(n-1)(n-2)...(3)(2)(1)$, where $n\in N$ and $0!=1$.

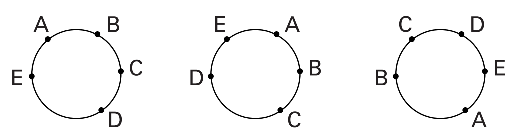
A6/G8 If there are only three prizes to be given to the 8 trumpeters, how many ways could placement be awarded?

Students should reason that eight people could come first, only seven could come second, and six could come third $\rightarrow 8\times7\times6\rightarrow336$. This could be worded “How many permutations are there of eight distinct objects taken three at a time?”

The symbol commonly used to represent this is ${}_8P_3$, or ${}_nP_r$ for the number of “*n*” objects taken “*r*” at a time. Students should notice that

$$
\begin{aligned}
&{}_8P_3=8\times7\times6\\
&\text{also, } {}_8P_3=\frac{8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}\\
&\text{so, } {}_8P_3=\frac{8!}{5!}\\
&{}_8P_3=\frac{8!}{(8-3)!}\\
&\text{so, } {}_nP_r=\frac{n!}{(n-r)!}
\end{aligned}
$$

Students should note that when five people are to be arranged in a straight line there would be 5! or 120 ways to do this. However, if the same five people were to be arranged around a table in the order, say A, B, C, D, and E, their relative position to each other would not be distinguishable.



Thus, the total number of arrangements would be:

$$\frac{{}_5P_5}{5}=\frac{5!}{5!}=4!=24$$

Probability

Worthwhile Tasks for Instruction and/or Assessment

A6

Pencil and Paper

1) The town of Karsville, which has 32 505 automobiles, is designing its own licence plates for residents to place on the front of their automobiles.
   a) Ask students to use counting principles to determine the best of the following three options and explain their choice:
      i) a licence made from using four single-digit numerals from 1 to 9
      ii) a licence made of three single-digit numerals from 1 to 9 and one letter from the alphabet
      iii) a licence made from three single-digit numerals from 1 to 9 and two letters from the alphabet.
   b) Ask students to select the best combination of single-digits from 1 to 9 and letters from the alphabet to suit the purposes of this town and to defend their selection.

2) The figure shows three black marbles and two white marbles.
Suppose they are in a box. Without looking in the box, choose two of the five marbles. How many ways are there to select two marbles that are the same colour? Each a different colour?

A6/G8

Pencil and Paper

a) Indicate which of the following are true (T) and which are false (F).
   i) $\frac{5!}{4!} = 5 \times 4$
   ii) $10 \times 9 \times 8 = \frac{10!}{7!}$
   iii) ${}_8P_2 = 56$
   iv) ${}_{100}P_4 = 100 \times 100 \times 99 \times 98 \times 97$

b) Create a story where each expression above would be used in the solution.

7) There are five points, no three of which are collinear, on a plane.
   a) How many segments can be formed using these five points as endpoints?
   b) If consecutive points are joined, a convex polygon is formed. How many diagonals does this polygon have?

8) A local pizza restaurant has a special on its four-ingredient 20 cm pizza. If there are 15 ingredients from which to choose, how many different “specials” are possible?

9) Explain why the following theorem would be true:

   A circular arrangement of ‘n’ items can be calculated using: $\frac{{}_nP_n}{n} = (n-1)!$

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

A6 develop an understanding of factorial notation and apply it to calculating permutations and combinations

G8 develop and apply formulas to evaluate permutations and combinations

G7 distinguish between situations that involve permutations and combinations

Elaboration—Instructional Strategies/Suggestions

A6/G8 Refer back to the problem where there are five members on the executive of the student council. If these five were elected from a list of 10 candidates for executive position, such as president, vice president, secretary, the number of ways 10 people can be slotted into five positions would be found using permutations ${}_{10}P_5 = \frac{10!}{(10-5)!} = 30240$.

A6/G8/G7 From these five people a committee of three is struck. If the five people are represented by *A*, B, C, D, and E, then clearly a committee with A, B, and C is the same as a committee with C, A, and B. So, the order of the selection is not important and the arrangement is called a combination. Therefore, since ABC, ACB, BAC, BCA, CAB, and CBA are all considered the same committee, they represent only one committee of three selected from the five people. The number of permutations of A, B, and C is 3!. Thus, the number of committees from the original list of 10 candidates

$$= \frac{\text{number of ways the executive was chosen}}{3!}$$

$$= \frac{30240}{3!}$$

$$= 5040$$

That is $${}_{10}C_3 = \frac{{}_{10}P_5}{3!} = 5040$$

and the number of committees from the five member executive selected would be

${}_5C_3 = \frac{{}_5P_3}{3!} = 10$.

A combination of "*n*" objects taken "*r*" at a time is any subset of size "*r*" taken from the "*n*" objects The number is denoted by $\binom{n}{r}$ (read "*n*" choose "*r*"), or ${}_nC_r$.
The number $\binom{n}{r}$ can be evaluated by investigating the connection between permutations and combinations.

Since $${}_nP_r = \frac{n!}{(n-r)}. \text{ Thus, in general, } \binom{n}{r} = \frac{{}_nP_r}{r!}. \quad \therefore {}_nC_r = \frac{n!}{r!(n-r)!}.$$

For example: A committee of size 4 and a committee of size 3 are to be assigned from a group of 10 people. How many ways can this be done if no person is assigned to both committees? Solution: First committee ${}_{10}C_4 = \binom{10}{4} = 210$ ways, and there are 6 people left for the second committee. Second committee ${}_6C_3 = \binom{6}{3} = 20$ ways. Therefore the two committees can be assigned $210 \quad 20 = 4200$ ways. Note: If the smaller committee was selected first then $\binom{10}{3}\binom{7}{4} = 120 \bullet 35 = 4200$ ways.

Probability

Worthwhile Tasks for Instruction and/or Assessment

A6/G8/G7

Pencil and Paper

1) a) Which of the following will produce the number of greatest magnitude? (Use estimation first.) Which will produce the smallest ?

   i) 6! iv) 3!¸ 4 vii) $\frac{9!}{7!}$

   ii) 11! v) $\frac{9!}{2!}$ viii) $\frac{100!}{2!}$

   iii) $\frac{15!}{12!}$ vi) $\frac{9!}{2!}$ ix) 4! - 3! x) $\frac{7!}{6!}$

   b) Pick three of the above expressions and create a problem in which these symbols would be used in the solution.

A6

Pencil and Paper

2) Write each as a ratio of factorials.

   a) 7´ 6´ 5 d) 30´ 29´ 29´ 12´ 11´ 10´ 9

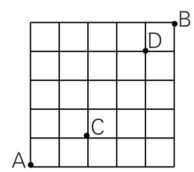
   b) 19´ 9´ 8´ 7´ 5´ 19 e) 20´ 19´ 18´ 17

   c) 10´ 9´ 8´ 7´ 6 f) $\frac{50\times49\times48\times47\times46}{5!}$

G8/G7

Performance

3) A government committee of size 9 is to be selected from five liberals, four reformers, and four new democrats. How many ways can this be done it each of the three parties must be equally represented?
4) Explain in words why you think a combination lock is called a combination lock instead of a permutation lock.
5) A fly goes from A to B in the grid by travelling only to the right or upwards. How many possible routes are there? How many routes are there that go through C, but not through D?



6) Linda, Gino, and Sam each draw 3 cards from a deck of 52 playing cards and do not replace them.
   a) If Linda goes first, in how many ways can she pick 3 cards?
   b) In how many ways can Gino draw his cards after Linda has drawn hers?
   c) Finally, in how many ways can Sam draw her cards?
7) A quarterback on a football team has seven different plays to use in a game. In order to confuse the defence of the other team, the quarterback does not want to repeat the same sequence of plays too often. How many different sequences of three plays has she to choose from if no play is repeated?
8) Mr. Burble teaches 182 students mathematics at Harry High. He tells his students that they must do these six problems, but that they can do them in any order. Is it possible for each of his students to do them in a different order? Explain.

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

B8 determine probabilities using permutations and combinations

G8 develop and apply formulas to evaluate permutations and combinations

Elaboration—Instructional Strategies/Suggestions

B8/G8 Students should now apply ${}_nP_r$ and ${}_nC_r$ to probability problems.

One practical use of permutations and combinations is in the field of probability. For example, a deck of 52 cards is shuffled well. What is the probability that A, K, Q of spades will be dealt to you as the first three cards?

The students would reason that since they want to see a particular three cards from 52 possible cards, they would use ${}_nP_r$ or ${}_{52}P_3$.

$${}_{52}P_3 = \frac{52!}{(51-3)!} \Rightarrow \frac{52!}{49!} \Rightarrow 132600$$

and only one of those outcomes is favourable, so

$$P(A,K,Q) = \frac{1}{132600}$$

Combinations are sometimes used along with other counting techniques. For example, suppose that a 17-member student council at the high school consists of 9 girls and 8 boys. A committee of 4 council members must have 2 girls and 2 boys. There are ${}_9C_2$ or 9!/(7!2!)=36 ways of selecting the 2 girls, and ${}_8C_2$ or 8!/(6!2!)=28 ways of selecting 2 boys. Because the committee must include 2 girls and 2 boys, there are $36 \times 28 = 1008$ ways of forming the committee. If the 4 committee members are selected at random, the probability that the committee will consist of 2 boys and 2 girls is 1008/2380, or about 0.424.

Probability

Worthwhile Tasks for Instruction and/or Assessment

B8/G8

Performance

1. Nine people try out for nine positions on a baseball team. Each position is filled by selecting players at random. Assume all players are equally qualified for every position.

   (a) In how many ways could the positions be filled?

   (b) What is the probability that Duffy will be the pitcher?

   (c) What is the probability that David, George or Duffy will be first baseman?

   (d) What is the probability that David, George or Duffy will be first baseman and Eleanor or Georgina will be pitcher?

2. The numbers on a raffle ticket contain three digits. The first digit cannot be zero.

   (a) What is the probability of ticket number 917 winning the grand prize? What assumption did you make?

   (b) What is the probability that a ticket with three as a second digit wins the grand prize?

3. If a coin is flipped five times, what is the probability of flipping a sequence that contains four heads and one tail?

4. (a) Give a detailed mathematical reason why Nova Scotia changed their automobile licence plates to ones that contain three letters of the alphabet and three while numbers less than ten.

   (b) What would be the probability of having licence plate that reads MTH 101.

5. Fran ran out and bought 20 6/49 tickets when she heard that the jackpot was $10 000 000. Anna told her she was nuts. Fran says, "If I only buy one ticket I only have one chance of winning, but if I buy 20 tickets I have 20 chances." How should Anna respond to help Fran understand both her chances of winning, and how wise is her investment?

Suggested Resources

Hannah, Frank L., "Probability," Venture Publishing, 1993.

Forrster, Paul A. "Algebra and Trigonometry," Addison-Wesley, 1994.

Probability

Outcomes

SCO: In this course, students will be expected to

G10 connect Pascal's Triangle with combinatorial coefficients

G9 demonstrate an understanding of binomial expansion and its connection to combinations

Elaboration—Instructional Strategies/Suggestions

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
. . . . . .

Pascal's Triangle

G10 Students should be asked to take binomials like $(x + y)$ and find simplified expressions for $(x + y)^2$, $(x + y)^3$, $(x + y)^4$, etc., and look for patterns in their coefficients. They should be able to find a connection between the expansion power and that same row in the Pascal's Triangle with respect to the coefficient values. $\left((x+y)^0\right)=1$ is the top row (row 0).

G9 The counting techniques, discussed in G3, p. 142, can be useful in the multiplying of polynomials. Looking at the product of $(a + b)(c + d) = ac + bc + ad + bd$, students should notice that each term in the expansion has one factor from $(a + b)$ and one factor from $(c + d)$. e.g., ac has two factors a and c. The a is from $(a + b)$ and the c is from $(c + d)$. Thus the number of terms in the expansion is four since there are two choices from $(a + b)$ and two choices from $(c + d)$. Students should also notice that since there are two factors $(a + b)$, and $(c + d)$ there are two factors in each term of the expansion.

$$(a+b)(c+d) \rightarrow ac+bc+ad+bd \rightarrow 4 \text{ (each term has two factors)}$$

$$\uparrow$$

two factors in each term

The product of one binomial and itself follows the same pattern.
$(x+y)^2 \rightarrow (x+y)(x+y) \rightarrow xx+xy+yx+yy$, but the multiplication would be completed by collecting the like terms and using exponents: $x2 + 2xy + y2$. Students should consider $(x+y)^5=(x+y)(x+y)\ldots(x+y) \rightarrow xxxxx+xxxyy+\ldots+yyyyy$. Each term is made up of five factors and using exponents will look like $x^a y^b$ where

$$\text{a + b = 5, e.g.,} \quad \begin{array}{l} xxxxx \rightarrow x^5 \rightarrow x^5y^0 \rightarrow 5+0=5 \\ xxxy \rightarrow x^4y \rightarrow x^4y^1 \rightarrow 4+1=5 \\ xxxyy \rightarrow x^3y^2 \rightarrow 3+2=5 \end{array}$$

G10/G9 In collecting the like terms, how many terms will be made up of the two factors x^2y^3? To answer this students should count the number of ways to make x^2y^3, e.g., the two factors of x must come fro wo of the five factors in each term of $(x + y)^5$. This can be done $\binom{5}{2}$ or ${}_5C_2$ = 10 ways. The three factors of y must come from the remaining three factors in each term of $(x + y)^5$ and this can be done in only one way. So the coefficient of x^2y^3 (${}_5C_3$) will be 10. Students should note that these coefficients are values in the fifth row of Pascals' Triangle.

Students should examine the pattern changes in the signs between terms when $(x – y)^5$ is expanded. Because the second term in the expression $(x – y)^5$ could be considered negative $(–y)$, then the terms in the expansion that have odd numbers of y-factors will be negative. When exponents or coefficients are included in the binomial to be expanded $(x^2 + 3y)^3$ students should be aware that for every x-factor, there is now an x^2-factor, and for every y-factor there is now a $3y$-factor, e.g., when x is replaced with x^2 and y with $3y$ the expansion becomes:

$$\left(x^2-3y\right)^3=\binom{3}{3}\left(x^2\right)^3-\binom{3}{2}\left(x^2\right)^2 3y+\binom{3}{1}\left(x^2\right)(3y)^2-\binom{3}{0}(3y)^3$$

Probability

Worthwhile Tasks for Instruction and/or Assessment

Paper and Pencil (G10/G9)

1) What is the coefficient of *the* x^4y^2 term in each of the following?
   a) $(x + y)^6$
   b) $(x - 2y)^6$
   c) $(2x + y)^6$
   d) $(3x - 2y)^6$

2) When examining the terms from left to right, find the specified term in each expansion.
   a) 10 th in $(x - y)^{12}$
   b) 20 th in $(2x - 1)^{19}$
   c) 8 th in $(a + b)^{10}$
   d) 2 nd in $(x^3 - 5)^7$
   e) 3 rd in $(1 - 2x)^9$
   f) 15 th in $(1 + a^2)^{24}$

3) a) Find the sum of the elements in each row, for the first six rows of Pascal's Triangle.
   b) Find the number of subsets in a 0–, 1–, 2–, 3–, 4–, and 5– element set.
   c) How are parts (a) and (b) related?
   d) How many elements are there in an n-element set?

4) Find a decimal approximation for 1.02^{10} by writing it as $(1 + 0.02)^{10}$ and calculating the first five terms of the resulting binomial series.

5) Henrietta is expanding $(3a - 2b^2)^3$. In her work below, explain what she is doing when going from step 2 to step 3. Is her work correct? Explain. What should she do to complete her work?

step 1:	a	$a \; b$	$a \; b$	b
step 2:	a^3	$a^2 \, b$	$a^1 \, b^2$	b^3
step 3:	$(3a)^3$	$(3a)^2 \, (2b^2)$	$(3a)^3 \, (2b^2)^2$	$(2b^2)^3$

G10/G9

Journal

6) Betty Lou missed math class today. Helen phoned her at night to tell her about how combinations are helpful when expanding binomials. Write a paragraph or two about what Helen would have told her.

7) When expanding $(a^2 - 2b)^5$, Wally gets confused about the exponents in his answer. Write a paragraph to Wally to help him remember how to record the exponents on this expansion.

Suggested Resources

Probability

Outcomes

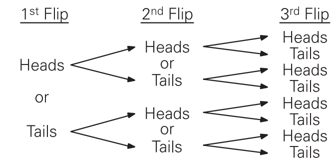
SCO: In this course, students will be expected to

G11 (111) connect binomial expansions, combinations, and the probability of binomial trials

$G1_3$ develop and apply simulations to solve problems

Elaboration—Instructional Strategies/Suggestions

G11(111) Many experiments consist of more than two parts, and if these parts are independent of one another, students can use the concept of a product model or tree diagram to help them with their counting and probability calculations. For example, when flipping a fair coin three times, a tree diagram determines that the sample space has eight outcomes. The probability of any one of them being selected is 1/8.



P (of any one of eight outcomes) $= \left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

$G1_3$/G11(111) The tossing of three coins discussed above could be used as a simulation model. Say the students want to solve the problem "in a family with three children what is the probability that the first two children are girls and the third is a boy."

Experiments that consist of repeated trials of a simple experiment (e.g., tossing a coin) using a model with only two possible outcomes (heads or tails) are called binomial trials. Suppose students needed to find the probability of getting exactly three heads in 10 tosses of a fair coin. Since each trial has two choices there would have to be $2^{10} = 1024$ branches on a tree diagram. The answer would be the sum of all the probabilities of the branches that contain three heads and seven tails. The number of ways 3 heads and 7 tails could be arranged is the "ten choose three ($_{10}C_3$)" or $\binom{10}{3}$, and so the probability of this happening would be $\dfrac{\text{\# of success}}{\text{total number of outcomes}} = \dfrac{_{10}C_3}{2^{10}}$.

Another way to consider this is that the probability of any one of the 2^{10} branches being selected would be the product of $\frac{1}{2}$ for every H and $\frac{1}{2}$ for every T or $\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$. Since every branch with three heads and seven tails has the same probability the answer is the number of these branches times the probability for each branch. The number of branches will be the number of ways of choosing the three heads out of ten tosses $\binom{10}{3}$.

Hence, $\binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$. Students should compare $\dfrac{\binom{10}{3}}{2^{10}}$ and $\binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7$ to see how one is the same as the other.

In general, in binomial trials there are two outcomes for each of n trials. One of the two outcomes is a "success," the other a "failure." These are labelled p and q respectfully and $q = 1 - p$. The number of successes in n trials is labelled s. Thus the probability of getting s successes and $n - s$ failures in n binomial trials is $p^s q^{n-s}$. For example, suppose students conduct an experiment of flipping a coin. The coin is bent, so the probability of heads (success) is 0.3. If they flip the coin five times, what is the probability of three tails and two heads? Students should now be able to answer this with .

Students should recognize this as a term in the binomial series that comes from expanding $(0.7 + 0.3)^5$. Students might want to use the 'randBin' feature on their calculators, or other software technology to conduct experiments or simulations where random samples are needed from populations that include only two possible outcomes (e.g., yeses, and nos).

Probability

Worthwhile Tasks for Instruction and/or Assessment

G11(111)/$G1_3$

Performance

1) Find the probability of getting 10 heads in 15 throws of a bent coin if the probability of heads on the bent coin is .
2) Find the probability of getting exactly two ones in six rolls of a fair die.
3) If n = 4 and p = , for what value of s will $p^s q^{n-s}$ be largest? Answer the same question for n = 4 and p = and for n = 5 and p = .
4) If Jamie is serving he wins a tennis game against Sam with probability , but if he is receiving he wins with probability . Jamie and Sam agree to play five games, and Jamie bets that he can win two in a row. If Jamie wins the toss, should he elect to serve or receive? Draw two tree diagrams and verify your answer.
5) A teacher made up a fair 10-item true and false test. Kira missed a few days just before the test and thought if she answered the questions randomly selecting *T*s and *F*s, she might do allright. When she was done she had 4 *T*s and 6 *F*s. What is the probability that Kira's 4 *T*s and 6 *F*s are correct. Show how to find the answer two ways.

Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

G12 (111) demonstrate an understanding of and solve problems using random variables and binomial distributions

Elaboration—Instructional Strategies/Suggestions

G12(111) Once students have developed the pattern described on the previous two-page spread $\binom{n}{s} p^1 q^{n-3}$, they can use it to calculate the probabilities of other related events: For example:

Let x = number of times the bent coin is "heads" in five flips.
Let $P(x)$ = probability that it is "heads" x times.
Therefore, with the probability of heads being 0.30 ...

no head: $P(0) = \binom{5}{0}(0.3^0)(0.7^5) = 0.16807)$

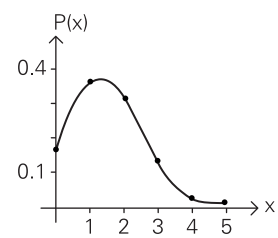
one head: $P(1) = \binom{5}{1}(0.3^1)(0.7^4) = 0.36015$

$\vdots \quad \vdots \quad \vdots$

five heads: $P(5) = \binom{5}{5}(0.3^5)(0.7^0) = 0.00243$

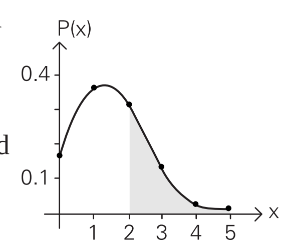
As a check on the answers, students should realize that x is certain to take on one of the values 0 through 5. So P (0 or 1 or 2 or 3 or 4 or 5) must equal 1 or 100%.

The independent variable x is called a random variable since you cannot be sure what value x will have on any one run of the random experiments. The dependent variable *P(x)* is the probability that the value is x. So P is a function of a random variable. The graph of *P(x)* for the above situation is shown.



The function P shows how the total probability, 1.00000, is "distributed" among the possible values of x. This function of a random variable is often called a probability distribution. Since this particular distribution has probabilities that are terms of a binomial series, it is called binomial distribution. It is skewed left since the probability of heads is only 0.3.

Binomial distributions occur when students perform a random experiment repeatedly, and each time there are only two possible outcomes (e.g., heads or tails, boy or girl, win or lose, yes or no). Students have already learned that a normal distribution is the result of recorded measurement of the same phenomena repeated over and over and over again. Since the binomial distribution is the result of a very similar action or fact repeated over and over and over again, it would be expected that it too would approach a normal distribution if the given probability is 0.5. This can be simulated quickly using the 'randBin' feature of the graphing calculator or other software technology.



For example: randBin $(10, 0.5, 10) \rightarrow L_1$. . Once they have found the probability distribution, they can use the properties of probability to calculate the probabilities of related events. For example, if the bent coin is flipped five times, as above, then the probability of getting at least two heads is $P(x \geq 2) = P(2) + P(3) + P(4) + P(5)$.

$$= 0.3087 + 0.1323 + 0.02835 + 0.00243$$

Probability ***
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* *
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Worthwhile Tasks for Instruction and/or Assessment

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* G12(111)

Performance

1) Heredity Problem: If a dark-haired mother and a dark-haired father have a recessive gene for light hair, there is a probability of then having a light-haired baby. For this to happen, each must have a large *x* (dark hair) and a small *x* (light hair) gene. In order for the baby to be light-haired, it must have two small *x* genes
   a) What is their probability of having a dark-haired baby?
   b) If they have three babies, calculate *P* (0), *P* (1), *P* (2), and *P* (3), the probabilities of having exactly 0, 1, 2, and 3 dark-haired babies, respectively.
   c) Show that your answers to part b are reasonable by finding their sum.
   d) Plot the graph of the probability distribution, *P*.

2) Multiple Choice Test Problem: A short multiple choice test has four questions. Each question has five choices, exactly one of which is right. Willie Makitt has not studied for the test, so he guesses at random.
   a) What is his probability of guessing any one answer right? Wrong?
   b) Calculate his probabilities of guessing 0, 1, 2, 3, and 4 answers right.
   c) Perform a calculation that shows your answer to part b is reasonable.
   d) Plot the graph of the probability distribution in part b.
   e) Willie passes the test if he gets at least three answers right. What is his probability of passing?

4) What is the probability of getting exactly 50 heads when 100 coins are tossed?

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Suggested Resources

Probability

Outcomes

SCO: In this course, students will be expected to

G11 (111) connect binomial expansions, combinations, and the probability of binomial trials

G12 (111) demonstrate an understanding of and solve problems using random variables and binomial distributions

Elaboration—Instructional Strategies/Suggestions

G11(111)/G12(111) Using the ideas developed over the last two two-page spreads, students can investigate some of the claims typically made in television and newspaper advertising. For example, a television commercial states that 8 out of 10 cats prefer Purrfect Chow. The claim is based upon a particular test in which 8 out of 10 cats chose Purrfect when given a choice between it and another cat food. A complaint is made by a rival cat food manufacturer. They say that 8 out of 10 would not be unusual, if it is assumed that cats have no particular preference for *Purrfect*. Assuming that cats will choose equally between one food or another randomly, what is the probability of them choosing Purrfect, and what does this mean with respect to the claim made by the other manufacturer?

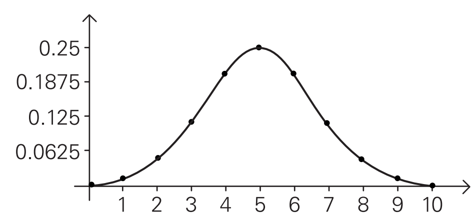
In their solution attempts, students could use the binomial model to calculate the probability that exactly 8 of 10 chose Purrfect:

$$\binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 = 45\left(\frac{1}{256}\right)\left(\frac{1}{4}\right) = \frac{45}{1024} \doteq 0.0439$$

If R is the number choosing Purrfect, then the full probability distribution would be:

r	0	1	2	3	4	5	6	7	8	9	10
$P(R=r)$	$\frac{1}{1024}$	$\frac{10}{1024}$	$\frac{45}{1024}$	$\frac{120}{1024}$	$\frac{210}{1024}$	$\frac{252}{1024}$	$\frac{210}{1024}$	$\frac{120}{1024}$	$\frac{45}{1024}$	$\frac{10}{1024}$	$\frac{1}{1024}$

The graph of this distribution looks quite normal, since the probability is 0.5 that cats will choose Purrfect over the other food choice.



From the graph the result “8 or more out of 10” is likely to occur in only about 5% of all samples of 10 cats. Based on previous study 5% is not very likely and suggests that the assumption made by the rival manufacturers is probably wrong. It appears likely that more than 50% of cats would indeed choose Purrfect.

The probability of 8 or more of the cats choosing Purrfect can be calculated using:

$$\text{P (8 or more)} = \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024} = \frac{56}{1024} \doteq 0.055.$$

Probability

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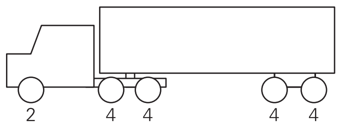
Worthwhile Tasks for Instruction and/or Assessment

*** ** * G11(111)/G12(111)

Performance



1) Eighteen-Wheeler Problem: Large tractor-trailer trucks usually have 18 tires. Suppose that the probability of any one tire blowing out on a cross-country trip is 0.03.
   Ask students the following:
   a) What is the probability that any one tire does not blow out?
   b) What is the probability that
      i) none of the 18 tires blows out?
      ii) exactly one tire blows out?
      iii) exactly two tires blow out?
      iv) more than two tires blow out?
   c) If the trucker wants to have a 95% probability of making the trip without a blowout, what must the reliability of each tire be? That is, what is the probability that any one tire will blow out?

2) Sally claims that she can predict which way a coin will land, either heads or tails. Tommy throws the coin eight times and Sally gets it right six times. Ask students to calculate, on the basis of a binomial model, the probability of
   a) getting six coin tosses correct out of eight
   b) getting six or more coin tosses correct out of eight
   c) Ask students if they think the result supports her claim? Explain your answer.

6) A blind taste test is organized to see if people can tell the difference between two different brands of orange juice. They have 10 "tastes". After each taste they have to say whether it is juice A or juice B. Ask students how often they would expect the participants to get it right before they were reasonably convinced that they could actually tell the difference.

7) A list of people eligible for jury duty contains about 40% women. A judge is responsible for selecting six jurors from this list.
   a) If the judge's selection is made at random, what is the probability that three of the six jurors will be women?
   b) Prepare a probability distribution table and graph for the number of women among the six jurors.
   c) The judge's selection includes only one woman. Ask students if they think this is sufficient reason to suspect the judge of discrimination? Explain.

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Suggested Resources

Unit 4
Circle Geometry
(35 Hours)

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E4 apply properties of circles

E5 apply inductive reasoning to make conjectures in geometric situations

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

E7 investigate and make and prove conjectures associated with chord properties of circles

E8 investigate and make and prove conjectures associated with angle relationships in circles

E9 investigate and make and prove conjectures associated with tangent properties of circles

E12 demonstrate an understanding of the concept of converse

Elaboration—Instructional Strategies/Suggestions

E4 Geometry is a rich field of mathematical study. The world around us is inherently geometric, and hummankind's creations most often reflect geometric principles. The concrete and visual nature of geometry resonates with certain learning styles, and geometry's pervasiveness in our environment facilitates connecting the study of geometry to meaningful, real-world situations. This is as true for circle geometry, the focus of this unit, as for geometry in general. Whether determining the correct location for handles on a bucket, finding the centre of a circle in an irrigation project, or determining the length of a tangent to the earth from an orbiting satellite, properties of circles (and lines, line segments, and/or angles associated with them) come into play.

E11/E5 Geometric figures such as segments, lines, angles, polygons, circles, and planes are each sets of points that are subsets of the universal set called space. In synthetic (Euclidean) geometry, these geometric figures can be drawn anywhere on a plane in space; in analytical (coordinate) geometry, a reference system is added, and important points on the figures are assigned coordinates. Using transformations, these figures—with or without coordinates—can be moved in space by following specific rules. In all perspectives, students seek to discover patterns among figures or within a fixed figure.

Students need many opportunities to explore geometric situations, look for common elements (or patterns) in them, and make appropriate conjectures. They also need to reach an understanding that, while this inductive process of observing multiple cases and conjecturing seems to imply the truth of a relationship, deductive reasoning is required to establish the truth of any conjecture in general. As part of this process, students should also realize that measurements with tools i) are not accurate and ii) deal only with specific cases and are, therefore, not adequate as proofs.

Students should be exposed to a variety of modes of proof, with the understanding that a logical argument can take many different forms. This implies that students should experience Euclidean, coordinate, and transformational approaches and develop an appreciation that each can be advantageous in certain situations. This unit provides the opportunity for students to develop some proficiency with all three modes of proof. If time should be a factor, however, it is sufficient for students to focus on analytical arguments and either Euclidean or transformational.

E7/E8/E9/E5/E11/E12 In particular, contexts will be explored, and theorems conjectured, proven and applied, with respect to chord properties in circles, inscribed and central angle relationships, and tangents to circles. The treatment of these circle topics is not intended to be exhaustive, but is determined to a significant extent by the contexts examined. It should also be noted that the concept of the converse of a theorem will also be explored in relation to some of the theorems developed.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E4/E5/E7

Performance

1) Activity:
   a) Begin with a circle of any size (given to the student).
      – Fold the circle in half—make a crease.
      – Open up the circle and fold in half differently.
      – Open up the circle and investigate the intersection point—compare with your classmates—make a conjecture.
   b) Begin with a circle of any size (given to the student).
      – Make a fold anywhere on the circle, make a crease.
      – Repeat the first step with a second fold and crease.
      – Mark the first fold AB at its end point, the second CD.
      – Fold A onto B, make a crease.
      – Open the circle, fold C onto D, make a crease.
      – Investigate the intersection of the last two creases—compare with your classmates—make a conjecture.
   c) Begin with a circle (make your own) and mark the centre point.
      – By folding create five chords (creases) of different lengths.
      – Fold one end of each chord onto itself, make a crease.
      – Investigate the lengths of these creases from the centre of the circle to the chord—compare with your classmates—make a conjecture.
   d) Begin with any circle (make your own) and mark the centre point.
      – Make fivefolds creating 5 chords all of equal length (fold into the centre).
      – Investigate the distance each is from the centre—compare with your classmates—make a conjecture.

E4/E5/E9

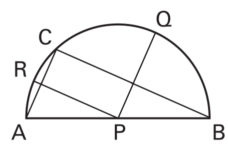
Performance

   e) Begin with a circle on a rectangular sheet of paper and mark the centre.
      – Make a fold at 3 different points on the circumference to produce creases that touch the circle at only those three points.
      – Join the points to the centre and investigate the angles formed between the radii and the tangents—make a conjecture.
      – Place a coordinate system over this situation, or transfer it to a coordinate system, find the slopes of the radius and the tangent to it, compare with your classmates, and make a conjecture.

E4/E11/E8

Performance

2) If P is the centre of semicircle $\overset{\frown}{AB}$, PR bisects $\overset{\frown}{AC}$ and PQ bisects $\overset{\frown}{BC}$, prove PR is perpendicular to PQ.



E12/E11

Performance

3) State and prove the converse of:
   If a point on the hypotenuse of a right triangle is equidistant to all three vertices, then it is the midpoint of the hypotenuse.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E5 apply inductive reasoning to make conjectures in geometric situations

E7 investigate and make and prove conjectures associated with chord properties of circles

E12 demonstrate an understanding of the concept of converse

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

E4 apply properties of circles

Elaboration—Instructional Strategies/Suggestions

E5/E7 Students need to begin their study of circles by exploring patterns and making and verifying conjectures. They might begin their exploration with an activity like the following:

❒ *Activity*

- On a blank sheet of paper (or using technology) place any two points *P* and *Q*. Construct a circle that passes through *P* and *Q* such that *PQ* is not the diameter and explain how you located the centre (C). What type of triangle must *PQC* be? Explain.
- Construct three more different circles that pass through *P* and *Q*. Name their centres *D*, *E*, and *F*.
  - Fold *P* onto *Q*, making a crease to indicate the fold line.
  - What do you notice about the points *C*, *D*, *E*, and *F*?
  - Name the point where the crease intersects *P*, *Q*, as *M*. Is *M* the midpoint of *PQ*? Justify your answer.
  - Is $PQ \perp$ to the fold line? How do you know?
  - Make a conjecture. Verify your conjecture using proof.
  - Take any point A on the fold line, join it to *P* and *Q*. Make a conjecture. Verify your conjecture using proof.

E5/E7/E12 While exploring patterns (as in the previous activity), students might use paper-folding techniques and/or measurement tools like rulers, dividers, compasses, and protractors. In so doing, they will be using both transformational and Euclidean techniques. They will also be using inductive reasoning to make conjectures such as

- any point that is equidistant from two points on a circle must be on the perpendicular bisector of the chord joining those two points,

or its converse:

- any point that is on the perpendicular bisector of a chord of a circle must be equidistant from the end points of that chord.

E11/E4 Moving from conjecturing to proving conjectures in general (e.g., proving theorems), and applying new theorems to calculate or prove other geometric facts, may be a large step for many students. Teachers will need to model the thinking processes necessary to generate proofs. As well, it may be necessary to spend time reaquainting students with the geometric properties and theorems with which they are already familiar (e.g., congruent triangles, angle sum of a triangle, vertically opposite angles, parallel line theorems). See p. 118 for a further elaboration.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E5/E4/E7/E11

Performance

1) Construct two circles with radius *r* so that each circle passes through the centre of the other circle. Label the centres P and Q, and construct the segment PQ. The two circles intersect at A and B.
   a) What is the relationship between the segments AB and PQ? Explain your thinking.
   b) Prove your conjecture in (a).

2) Construct a large circle and two non-parallel congruent chords that are not diameters.
   a) Compare their distance to the centre of the circle.
   b) Write your findings in (a) as a conjecture.
   c) Test your conjecture on other circles.
   d) Prove your conjecture.

E12

Performance

3) a) Restate the conjecture you made in question 2 above in an "if ... then ..." form.
   b) State the converse of this conjecture.
   c) Is the converse true? Explain.

4) State a theorem related to geometry whose inverse is not true.

E4

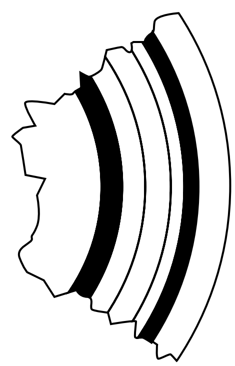
Performance

5) Use a circular object to trace a circle onto your paper. Without using a compass, locate the centre of the circle.

E4/E7

Performance

6) A piece of circular plate was recently dug up on an island in the Mediterranean. The discoverer of the plate wishes to calculate the diameter of the original plate. Describe how he could do this.



Suggested Resources

Hirsch, Christian R. ed. Curriculum and Evaluation Standards for School Matheamtics. Addenda Series. A Cone Curriculum. Reston, VA: NCTM, 1992.

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E12 demonstrate an understanding of the concept of converse

E7 investigate and make and prove conjectures associated with chord properties of circles

E8 investigate and make and prove conjectures associated with angle relationships in circles

Elaboration—Instructional Strategies/Suggestions

E12/E7 Once students begin to articulate conjectures and try to establish them as theorems, it would be appropriate to introduce them to the concept of converse. Essentially, a converse is an opposite, or something reversed in order or action. Consequently, the converse of a conjecture or theorem will be reversed in order and/or action with respect to the original.

Students may find, when they examine the conjectures that they made in their initial explorations, that some are indeed converses of others. They may find this easiest to do if they adopt the conventional, conditional form for statements of conjectures/theorems, e.g., the “if ..., then ...” statement. For example, in this format the statements from p. 114 become

> If a point is equidistant from two points on a circle, then it lies on the perpendicular bisector of the chord joining those two points
> If a point lies on the perpendicular bisector of a chord of a circle, then it is equidistant from the end points of that chord.

E12/E8/E7 In general, students should understand that the converse of any “if *p*, then *q*,”statement is “if *q*, then *p*.” As well, it is critical that students understand that the truth of any theorem does not necessarily imply the truth of its converse. In some cases a converse is also true, in other cases it is not, so students should always test the truth of a converse. For example, they might write the converse of “if an angle is inscribed in a semi-circle, then it is a right angle” and see that the converse is not true in general. The discussions arising out of the examination of these types of examples encourage logical thinking.

Students should know that if a statement and its converse are true, it can be stated as an “if and only if”(iff) situation. This gives rise to sufficient conditions: e.g., points lie on the perpendicular bisector of a chord iff they are equidistant from the end points of the chord. Students should realize that it is sufficient to prove a line is a perpendicular bisector of a chord by proving two points on it are equidistant from the endpoints of the chord.

Converse is not a concept that requires extensive attention in its own right. It should be addressed from time to time as it comes up throughout the unit, with a view to students’ developing a clear understanding of the concept.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E12/E7

Performance

1) State the converse of each of the following. If you think the converse is true, rewrite using ‘if and only if” (iff). If you think it is false, explain why.
   a) If a triangle has two angles of equal measure, then it is isosceles.
   b) If two triangles are congruent, then their corresponding angles are congruent.
   c) A quadrilateral with four axes of symmetry is a square.
   d) Every square has four sides of equal length.
   e) The centre of a circle lies on the perpendicular bisector of a chord of that circle.
   f) A tangent line is perpendicular to the radius of a circle.

2) State and prove the converse of:
   If a point on the hypotenuse of a right triangle is equidistant to all three vertices, then it is the midpoint of the hypotenuse.

3) Prove the converse of this statement:
   If a line passes through the midpoint of a chord on a circle, then it is equidistant from the endpoints of the chord.

4) Make a statement and state its converse so that the original statement is true but the converse is not. State a theorem in geometry that you know is true but whose converse is not.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

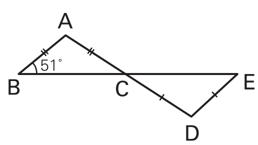
E4 apply properties of circles

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

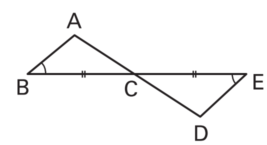
Elaboration—Instructional Strategies/Suggestions

E4/E11 As previously indicated, it may be necessary to refresh students' knowledge of geometric properties and theorems (e.g., congruent triangles) while building strength with respect to writing deductive arguments. This may be accomplished by applying deductive reasoning in situations in which students are asked to apply their knowledge of geometry to i) determine specific angle measures and/or ii) prove geometric facts or theorems in general. Note: This is an opportunity to expose students to both Euclidean and transformational forms of proof.

Sometimes students might be given some angle measures and be asked to find and verify that another angle has a certain measure. For example, given the situation to the left, find and verify that the angle *CDE* must be 78°. Students could begin with the given 51° angle and transfer equality to $\angle CED$ using vertically opposite angles and equal angles in an isosceles triangle, then finally get $m\angle CED = 78^\circ$ using the fact that the angles in a triangle add to 180°.



In a slightly more general situation, students could be asked to prove that *AC* = *DC*. Using 180° rotation, centre *C*, students could state that
$B \leftrightarrow E$ because *B* – *C* – *E* and *BC* = *CE* then $A \to \overline{ED}$
because $\angle ABC = \angle CED$, so $A \to D$, because
AB = *DE* since *C* is the turn centre $AC \to CD$ and
$\therefore AC = CD$ because in a rotation the image must equal the object.



When students are presented with arguments that are not valid they should be able to identify the flaw. For example:

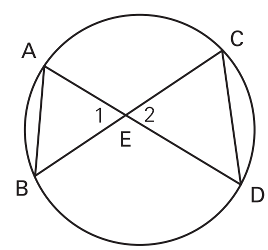
Given: *AD* and *BC* intersect at *E*.

Jon argues that *AB* = *CD*. His argument goes like this:

$\angle 1 = \angle 2$ because opposite angles are equal,

AE = *ED*, *CE* = *ED* because of equal radii.

So the triangles are congruent and *AB* = *CD* since they are corresponding parts of congruent triangles. Find the flaw.



Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E4/E11

Performance

1) Prove that the incentre of a triangle is the centre of the inscribed circle of that triangle.

2) Prove that the circumcentre of a triangle is the centre of the circle circumscribed on that triangle.

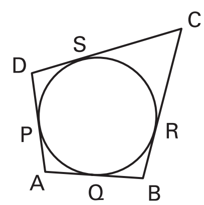
3) Use transformations to prove that the centre of a circle is the intersection of the perpendicular bisectors of two chords.

4) Use transformations to prove that two chords equidistant to the centre have the same length.

5) Use transformations to prove that a tangent to a circle is perpendicular to the radius at the point of tangency.

6) Use congruent triangles to prove that tangent segments to a circle from a point outside the circle are equal in length.

7) If the sides of a quadrilateral ABCD are tangent to the circle, show that AB + DC = AD + BC.



8) Find the flaw(s) in the following proof:

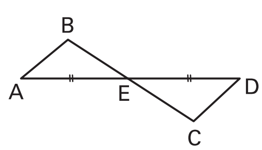
Prove: $\angle B = \angle C$

Proof: for a rotation, centre E

$A \rightarrow D\left(AE = ED\right)$

$B \rightarrow C\left(AB \parallel ED\right)$

$\therefore \angle B = \angle C$



Journal

9) Describe how the following idea could be proved:
The centre of any circle is the intersection of the perpendicular bisectors of any two non-parallel chords in the circle.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

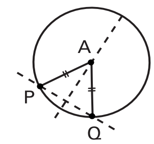
E7 investigate and make and prove conjectures associated with chord properties of circles

Elaboration—Instructional Strategies/Suggestions

E11/E7 Once students have been reaquainted with the necessary geometric ideas, they can move on to applying them in the context of properties of circles, beginning with chord properties.

Students should write proofs using deductive arguments and explain that the validity of the arguments are valid. For example, in proving the conjecture

Any point that is equidistant from two points on a circle must be on the perpendicular bisector of the chord joining those two points



Students might say ... "Since *AP* = *AQ* (*A* is equidistant to two points), then the triangle *APQ* is isosceles and has one line of symmetry passing through the vertex angle, and Ð*A* is the vertex angle.

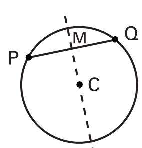
"A property of symmetry says that this line of symmetry must be the perpendicular bisector of the base of an isosceles triangle, thus proving the conjecture."

Alternatively, students might use a Euclidean version of the same proof:

"Join *A* to *M*, where *M* is the midpoint of segment *PQ*; the triangles would now be congruent since there are three pairs of corresponding sides congruent (SSS). Since the triangles are congruent, then the corresponding angles at *M* inside the triangles are congruent, and they are supplementary, so each is a right angle (two 90s make 180). Since the angles at *M* are right angles, then *AM* is perpendicular to *PQ* at its midpoint, and the conjecture is proved."

By way of a second example, consider the following:

Given that *P* and *Q* are points on a circle with centre *C*, students could fold so that



$P \to Q$. Since *PC* = *CQ* (*C* is the centre of the circle), *C* folds onto itself,

$C \to C$, e.g., C must be on the mirror line. Properties of reflection say that the mirror line must be perpendicular to a line joining a point to its image (*P* and *Q*). So if the intersection point is named *M*, then $CM \perp PQ$.

Some students might want to prove $\triangle CMP = \triangle CMQ$ in order to prove $CM \perp PQ$. Their proof may look like this:

Let *M* be the midpoint of *PQ*

$\triangle CMP \cong \triangle CMQ \, (SSS)$

$\angle CMP \simeq \angle CMQ \, (\triangle\text{'s} \cong)$

$\therefore m\angle CMP = 90^\circ = m\angle MCQ$ (a congruent and supplementary)

$\therefore CM \perp PQ$, and CM bisects PQ (M is the midpoint)

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E11/E7

Performance

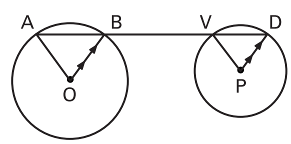
1) Mary said to Beth: “I can draw a segment inside the circle that is equal in length to one half the diameter without measuring it.” Beth was impressed and insisted that Mary show her work. Mary told Beth to draw any circle, with diameter AB then to draw two more chords to make the triangle ABC. She told Beth that, if she connects the two midpoints of the two chords just drawn, the segment joining them would be one half the length of the diameter. Explain how you know that the segment joining the midpoints would be one-half the diameter.

2) Lou, who was listening to and watching Mary and Beth, was startled because now, he exclaimed, he now knew how to prove that the segments joining the midpoints of any quadrilateral form a parallelogram. How can Lou do this?

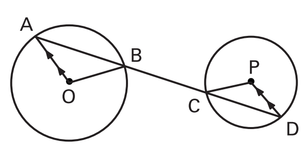
3) Draw a circle. Draw two chords of the circle of unequal length. Which is closer to the centre of the circle? Prove it.

4) In designing various logos that use circles, Frank wanted to make sure that what seemed to look correct really was correct. Help him prove the following relationships:

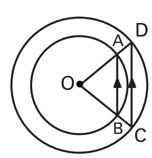
   a) Given: $OB \parallel PD$
      O and *P* are centres of circles
      Prove: $\angle OAB = \angle PCD$



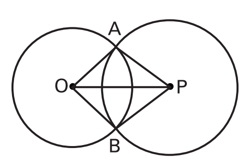
   b) Given: $OA \parallel PD$
      O and *P* are centres of circles
      Prove: $\angle OAB = \angle PCD$



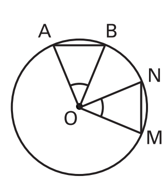
   c) Given: $AB \parallel CD$
      O is the centre of both circles
      Prove: $\angle OAB = \angle C$



5) *O* and *P* are the centres of the circles.
   a) Why is the figure *OBPA* a kite?
   b) Explain why $AB \perp OP$
   c) What other conclusions are valid?



6) Given that O is the centre of the circle, and the two central angles are congruent, prove *AB* = *MN*



Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

D1 develop and apply formulas for distance and midpoint

E5 apply inductive reasoning to make conjectures in geometric situations

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

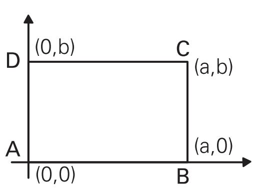
Elaboration—Instructional Strategies/Suggestions

D1 It is common for students to use measurement to attempt to prove conjectures. However, for generalization, they need to develop a way of generating length or distance algebraically. They might begin by contrasting the ease of determining the lengths of horizontal and vertical lines (on a coordinate system) with the difficulty of finding the lengths of oblique lines. (For example, they might compare finding the distance of the point (3, 0) from the origin as compared to finding the distance of (3, 5) from the origin.) The connection between the Pythagorean Theorem and the distance formula needs to be emphasized and understood, so that students can connect this new formula to previous knowledge. In the same way, the calculation of the midpoint can be connected to averaging.

Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ Midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

E5/E11/D1 In circles, there are several opportunities for line segments to be perpendicular. Opportunity should be provided for students to conjecture that when the slopes of two segments are negative reciprocals the two segments are perpendicular. Sometimes in coordinate proof, students can benefit from knowing that the product of the slope of a line and its reciprocal is –1. This may sometimes be easier for students to see than that the fact that they are negative reciprocals of each other. Once students have made such a conjecture, it needs to be proven. Students could then apply (parallel properties) to prove that, given four points, certain segments are parallel and others are perpendicular, and that some quadrilaterals are parallelograms and some are rectangles or rhombi.

The coordinate system can also be used to help generalize properties. For example, in proving that the diagonals of a rectangle are congruent, students would assign variable coordinates to the four vertices, then express the length of the diagonals *DB* and *AC* as



$$DB = \sqrt{(a-0)^2 + (0-b)^2} = \sqrt{a^2 + b^2} \qquad AC = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

and conclude that their lengths must be equal.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

D1/E11

Performance

1) A sprinkler head is positioned on the infield so that the spray of the water soaks the entire field that lies within the 10-metre radius of the head. Assume that a Cartesian coordinate system has been placed over the field and the sprinkler head is at the coordinate S (3, –1). Determine whether or not the new trees at the following locations will get wet or not?
   a) A (11, 4) b) B (–3, –9) c) C (12, –6) d) D (–4, –8)

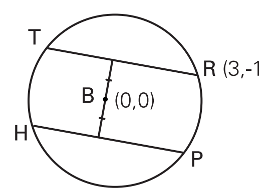
2) Another sprinkler head is positioned at P (–2, 6). A tree is on the circumference at (5.5, 7). What is the area of the ground that gets wet?
3) These trees P (3, 6), Q (6, -1), and R (-1, -4) lie on the circumference of the water circle. Prove that C (1, 1) is the location of the sprinkler head.
4) Determine the coordinates on the field so that the planting of trees A, B, and C lies on the same line and between trees M (–17, –8) and N (47, 20) so that segment MN is divided into four equal parts.

D1/E5/E11

Performance

5) Use Geo-strips to construct a parallelogram. Investigate the many parallelograms and the lengths of their diagonals that can be formed by repositioning the Geo-strips.
   a) Prove that the diagonals of a parallelogram are not congruent.
   b) Prove that the diagonals of a rhombus are congruent.

6) Given C_1: $2y - 6x + 2 = 0$ C_2: $5x + 5y - 15 = 0$ the equations for two chords of a circle, prove that the intersection of these chords is the centre of the circle that contains a diameter that runs from A (–4, 2) to B (6, 2).
7) Given that the radius bisects the central angle in a circle, prove that it bisects the arc subtending the central angle.
8) Given the diagram, ask students to make up a question to find the coordinates for the point P.



Journal

9) Describe how the procedure for calculating the distance between two points, given their coordinates, is similar to the procedure for calculating the slope of the line that joins the two points. Describe how it is different.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

D1 develop and apply formulas for distance and midpoint

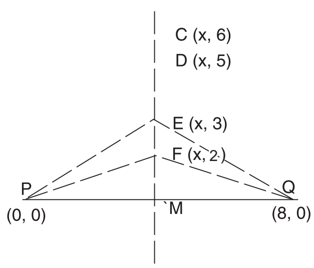
Elaboration—Instructional Strategies/Suggestions

E11/D1 Students might revisit statements that they have already provided using Euclidean or transformational proofs. For example, students have already shown that points equidistant to the two points *P* and *Q* (see p. 120) are on a line ^ to *PQ* and passing through the midpoint. Using coordinate geometry to assign coordinates to *P* and *Q* ask students to find the coordinates for the centre points *C*, *D*, *E*, and *F*, knowing that radii of circles are equal, then find the slope of PQ and CD. This should lead to a conjecture that there is a perpendicular relationship of *PQ* and *CM* through midpoint *M*.

$$PF = \sqrt{(x-0)^2+(2-0)^2} = \sqrt{x^2+4}$$

$$QF = \sqrt{(x-8)^2+(2-0)^2} = \sqrt{x^2-16x+64+4} = \sqrt{x^2-16x+68}$$

Assume F is the centre of the circle:



$$\sqrt{x^2+4} = \sqrt{x^2-16x+68}$$

$\therefore x = 4$, and F's coordinates are (4,2)

$$\text{slope of } PQ = \frac{rise}{run} = \frac{0}{8} = 0$$

$$\text{slope of } FC = \frac{4}{0} = \infty$$

$\therefore FC \perp PQ$

They will re-examine these statements and attempt to prove them using coordinate arguments.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E11/D1

Performance

1) A triangle has vertices *R* (17, 16), *S* (1, 4), and *T* (7, –4).
   a) Prove that the triangle is right-angled.
   b) *M* is the midpoint of *RT*. Prove that a circle with centre *M* and passing through *R*, also passes through *T* and *S*.

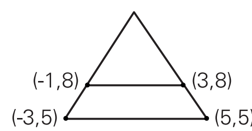
2) Find the equation for the set of all points that are equidistant from (6, 2) and (1, –5).

3) Given a right triangle with vertices *A* (1, 4) and *B* (9, 3). The third vertex *C* is on the x-axis. If side *AB* is the hypotenuse, find the coordinates of *C*. Is there more than one answer? Explain.

4) Ralph decided to use coordinate geometry to help him solve his problem. He located the middle of a beam at *M* (2, 3). He was able to locate approximate positions for the endpoints *P* and *Q*. He knew the *x*-value for the point *P* (–4, *y*), and the *y*-value for the point *Q* (*x*, -2). Find the values for *x* and *y*.

5) A ceiling support beam is constructed from several congruent isosceles triangles. When the midpoints of one of the triangles are joined will the new triangle also be isosceles?

6) Create a real-life problem using the diagram on the right.



7) Using transformations, prove that the midpoint, *M*, of a line segment *PQ* with endpoints *P* (*a*, *b*), and *Q* (*c*, *d*) is $M\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$.

8) Two designers are using a Cartesian plane to design a large plus sign to hang in the operations room. The plus sign is made using two perpendicular lines whose equations are almost completely determined: $x – 2y = -8$, and $kx – y = –3 (x + 1)$. Help them determine the value for *k*. Is there more than one value for *k*? Explain.

9) Determine the ratio of the sum of the lengths of the three altitudes to the perimeter of the triangle whose vertices are P (0, 0), Q (4, 3), and R (–2, 5). Do you think that this ratio is the same for any triangle? Investigate.

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

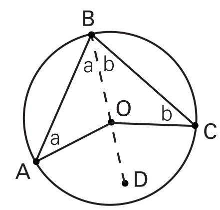
E8 investigate and make and prove conjectures associated with angle relationships in circles

E9 investigate and make and prove conjectures associated with tangent properties of circles

Elaboration—Instructional Strategies/Suggestions

E11/E8/E9 Students will continue to apply their knowledge of axioms and algebra to construct logical arguments with respect to other properties of circles, in particular angles in circles and tangents to circles. For example, to prove that the inscribed angle is half the central angle, students might use synthetic geometry:

Draw the line through *BO*. In triangle *AOB*: *OB* = *OA* because of equal radii $m\angle A = m\angle OBA$ = "*a*" because triangle *AOB* is isosceles. Thus, $m\angle AOD = 2a$ since the exterior angle of a triangle is the sum of the two remote interior angles.



Similarly, $m\angle DOC = 2b$

So, $m\angle AOC = 2a + 2b$

$= 2(a + b)$

$m\angle ABC = a + b$

$\therefore m\angle AOC = 2m\angle ABC$

Students could be asked to make the logical argument that an angle in a semicircle is 90°. Some students might say that an inscribed angle is half of the central angle and since the central angle is 180°, the inscribed angle must be 90°. Other students might construct two isosceles triangles as in the above diagram and show that when 2*a* + 2*b* = 180° results the equation *a* + *b* = 90° is logical for $\angle ABC$. The "proofs" need not be long and involved. What is important is that they are based on agreed-upon axioms within the geometry system (synthetic, transformational, coordinate) being used.

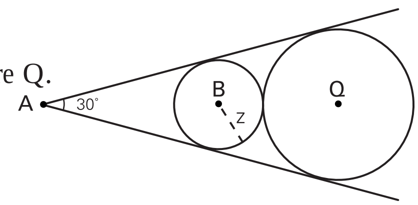
Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E11/E8/E9

Performance

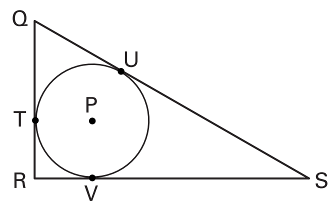
1) Find the diameter for the circle with centre Q.



2) Given: $\angle R$ is a right angle *T*, *U*, and *V* are points of tangency with the circle centre *P*.

   Prove: $r = \frac{1}{2}(QR + RS - QS)$

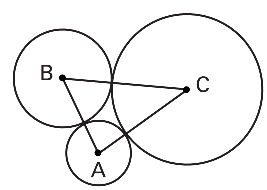
   when *r* is the radius of circle *P*.



3) Prove:
   a) The central angle of a circle is twice the measure of the inscribed angle on the same arc.
   b) The angle inscribed in a semi-circle is a right angle.

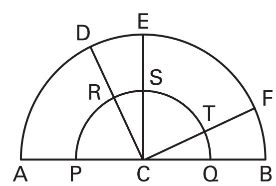
4) Two circles with centres *P* and *Q* are tangent at *S*. Prove that *P*, *S*, and *Q* are collinear points.

5) In the diagram, each circle is tangent to the other two. If *AB* = 10 cm, *AC* = 14 cm, and *BC* = 18 cm, find the radius of each circle.



6) $\overset{\frown}{AB}$ is a semicircle with centre C, $\overset{\frown}{PQ}$ is the concentric with $\overset{\frown}{AB}$, $EC \perp AB$, and $CD \perp CF$.

   Prove $m\overset{\frown}{AD} + m\overset{\frown}{QT} = m\overset{\frown}{EF} + m\overset{\frown}{RS}$.



7) Construct two non-parallel, non-congruent chords on a circle of any radius. Connect the endpoints of the chords with segments so that the segments intersect. Measure the four angles formed at the circumference
   a) Make a conjecture about two angles subtended by the same arc.
   b) Construct a central angle and compare the measures of a central angle with the inscribed angle subtended by the same arc. Make a conjecture and check it with other central angles in the diagram.
   c) Prove the conjecture in (b), then use that proof to prove the conjecture in (a).

Suggested Resources

McKillop, David W. et al. *Pre- Calculus Mathematics One.* Toronto: Nelson, 1992

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E4 apply properties of circles

E15 solve problems involving the equations and characteristics of circles and ellipses

Elaboration—Instructional Strategies/Suggestions

E4/E15 Many problems can be solved with the drawing of a circle or circles. Students will explore how technology (especially using the Draw Mode on graphing calculators) can be used to construct circles.

All students should be able to attempt to solve problems involving circles such as the following:

The planners of the arena want to have a smaller rectangular grassed area on which games like croquet, badminton, tennis, dodge ball, etc., might be played. Suppose the field might have dimensions 75 m by 50 m. The groundskeeper for the field wants to position three sprinklers. Each sprinkler throws water out over a semi-circular path. He wants to position the three water heads on three of the four boundaries of the field so that every place in the field will be exposed to water. He places one at the centre of the front 50 m wall, and one on each of the 75-m walls, 55 m back from the front wall. Will sprinkler heads that cover semi-circles up to 30 m be adequate, or will he have to purchase the expensive 35-m size or even more expensive 40-m size?

Students might begin to explore this problem by drawing circles (or by using their graphing calculators). If using calculators, they would set their window to represent the field dimensions, then draw circles using the three sprinkler head locations as centres. In the Draw menu, select 9: circle (0, 25, 30), then push Enter. This will draw a circle centre (0, 25) radius 30 m.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E4/E15

Performance

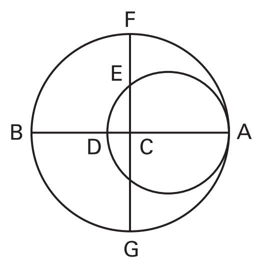
1) How many circular pipes, each with an inside diameter of 10 cm, will carry the same amount of water as a pipe with an inside diameter of 60 cm?

2) By how much does the radius of a circle increase when the circumference is increased from 20 cm to 25 cm?

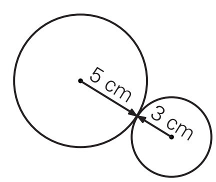
3) A 16-cm chord is 15 cm from the centre of a circle. What is the radius of this circle?

4) An 18-cm chord is perpendicular to the radius of a circle. The distance from the intersection of the chord and the radius to the outer end of the radius is 3 cm. What is the length of the radius?

5) Two circles are internally tangent at *A*. *C* is the centre of the larger circle. BA is perpendicular to *CF*, *EF* = 5 cm and *BD* = 9 cm. What is the length of the diameter of the smaller circle?



6) Two gears have radii 5 cm and 3 cm. How many times must the smaller gear be turned in order for the arrows on each gear to align again?



7) For the past several years on opening day of lobster season, the weather and other circumstances have caused life-threatening incidents. This year the air sea rescue helicopters are to be placed so that every point in the 90-km square region can be reached within 20 minutes. In 20 minutes these helicopters can travel up to 36 km. Three plans are proposed.
Assuming that you will use an appropriate window on your calculator:
Plan A
  Five helicopters
  One in the middle
  One at each corner

Plan B
  Four helicopters at (20, 20), (20, 70), (70, 20), (70, 70)

Plan C
  Four helicopters at (25, 25), (25, 65), (65, 25), (65, 65)

How can you use the circle-drawing feature of your graphing calculator to evaluate these three plans? Do it. Which is best? Which is worst? Why? Can you create a better plan?

Suggested Resources

McKillop, David W. et al. *Pre- Calculus Mathematics One*. Toronto: Nelson, 1992

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E15 solve problems involving the equations and characteristics of circles and ellipses

E4 apply properties of circles

D1 develop and apply formulas for distance and midpoint

Elaboration—Instructional Strategies/Suggestions

E15/E4/D1 In exploring the sprinkler system described in the elaboration for E15, p. 128, it would be helpful to have other ways to decide whether or not given points are within the circular boundary of the spray given off by the sprinkler heads. For example, have students determine if the following points are within the 10 unit boundary, on the boundary, or outside the boundary of the circular spray when the sprinkler head is given the coordinates (0, 0):

a) (9, 4) b) (8, 5) c) (8, 6) d) (9, 6)

Some students might do this using the graphing calculator approach, while others may draw diagrams with compasses or use the distance formula.

It might help to have students focus on exactly what it means to have a circle with radius 10 units. They should come up with a description of such a circle that generalizes to "A circle is the set of all those points in a plane that are a given distance (the radius) from a given point (the centre).

Using the distance formula, students should determine if specific points are within the boundaries of the circle with radius 10. Ask students what equation could be written using coordinate variables *x* and *y*, so that the graph of the equation is a circle.

Students should test whether the point P (8, 6) lies on the circle using the distance formula:

$$OP = \sqrt{(8-0)^2 + (6-0)^2} = \sqrt{8^2 + 6^2} = \sqrt{64+36} = \sqrt{100} = 10$$

When (*x*, *y*) is used to replace (8,6), the equation describes any point on the circle.

$$OP_1 = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

They know the radius (OP_1) is 10, so:

$$\sqrt{x^2 + y^2} = 10 \text{ or } x^2 + y^2 = 10^2$$

The equation of a circle with radius 10 can be expressed as $x^2 + y^2 = 10^2$. Emphasize with the students, that this equation produces a circle, Students can now conjecture that

1) If the coordinates of a point satisfy the equation, then the point is on the circle.
2) If a point is on the circle, then its coordinates satisfy the equation.

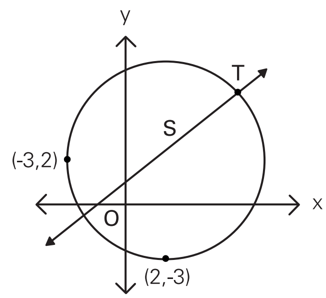
This is another opportunity to use the term converse.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E15/E4/D1

Pencil and Paper



1) A ranger station, S, is located on the line $2x - 3y = -2$. The area of sight from the station is bounded by the circle.
   a) Find the equation of the boundary.
   b) If a second station is located at T, find its coordinates.

2) A circle is defined by $x^2 + y^2 = 10$.
   a) Show that *AB* is a chord of the circle where *A* (1, –3) and *B* (–3, 1).
   b) Find the equation of the right bisector of *AB*.
   c) Show that the right bisector of *AB* passes through the centre of the circle.

3) The equation of a circle is given by $x^2 + y^2 - 4x + my - 18 = 0$. If *A* (7, 3) is a point on the circle, find the value of *m*.

4) Prove that the line $x - 3y + 24 = 0$ passes through the centre of the circle given by $x^2 + y^2 + 12x - 12y + 27 = 0$.

5) a) Graph the region defined by the inequalities

   $x^2 + y^2 - 2x + 4y - 5 \leq 0$ and $x + y - 1 \geq 0$.

   b) Determine the area of the region defined in (a).

Suggested Resources

Circle Geometry

Outcomes

SCO: In this course, students will be expected to

E13 analyse and translate between symbolic, graphical, and written representations of circles and ellipses

E3 write the equations of circles and ellipses in transformational form and as mapping rules to visualize and sketch graphs

E16 demonstrate the transformational relationship between the circle and the ellipse

E14 translate between different forms of the equations of circles and ellipses

E15 solve problems involving the equations and characteristics of circles and ellipses

E7 investigate and make and prove conjectures associated with chord properties of circles

E11 write proofs using various axiomatic systems and assess the validity of deductive arguments

Elaboration—Instructional Strategies/Suggestions

E13/E3/E16 Reconsider the sprinkler system problem. Suppose the sprinkler head is moved to a new location *Q* (3, –1). Keep the radius 10 and find the equation of the circle. From their earlier work:

$$RQ = \sqrt{(x-3)^2 + (y+1)^2}$$
$$\Rightarrow (x-3)^2 + (y+1)^2 = 10^2$$

From this centre-radius form of the equation, students can see the coordinates for the centre of the circle. Just as they would see the vertex for the parabola in a quadratic equation. This circle equation could be expressed in transformational form as

$\left[\frac{1}{10}(x-3)\right]^2 + \left[\frac{1}{10}(y+1)\right]^2$ where the stretch factors give the radius of the circle. As a mapping rule the image equation can be expressed as $(x.y) \to (10x+3,\ 10y-1)$.

If the water pressure was turned up and the sprinkler head could now throw water 12 units, the equation would become $\left[\frac{1}{12}(x-3)\right]^2 + \left[\frac{1}{12}(y+1)\right]^2 = 1.$

Extending the understanding of the equation of a circle to that of the equation of an ellipse is a simple matter of understanding that the equation of the ellipse is generated from the equation of a circle by having different vertical and horizontal stretches. Its equation, in transformational form, would be

$\left[\frac{1}{m}(x-h)\right]^2 + \left[\frac{1}{n}(y-v)\right]^2 = 1.$

E14 Completing the square is used to manipulate of equations of circles and ellipses into transformational form, or in centre—radius form. For example, when students are asked to graph $x^2 + 2x + y^2 - 4y = 12$, they would write the following:

$$x^2 + 2x + y^2 - 4y + \_\_ = 20$$
$$x^2 + 2x + \_\_ + y^2 - 4y + 4 = 2 - +1 + 4$$
$$(x+1)^2 + (y-2)^2 = 25 \text{ (this is the centre-radius form)}$$
$$\left[\tfrac{1}{5}(x+1)\right]^2 + \left[\tfrac{1}{5}(y-2)\right]^2 = 1 \text{ (this is the transformational form)}$$

From this form they can describe the transformations of $x^2 + y^2 = 1$: “The centre(0, 0) is translated left 1, and up 2, and the radius is 5 units.”

E15/E7/E11 Students should solve problems using the equations for circles and ellipses. For example: A circle is defined by $x^2 + y^2 - 10x - 10y + 25 = 0$. Show that the line joining *A* (2, 1) to *B* (5, 0) is a chord of the circle, and that the right bisector of *AB* passes through the centre of the circle.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

E3/E16

Pencil and Paper

1) Find the equation of the image of $x^2 + y^2 = 1$ under each mapping. Write the equation in standard form.

   a) $(x, y) \to \left(3x - 2,\ \frac{1}{2}y + 1\right)$ b) $(x, y) \to (5x,\ y - 5)$

   c) $(x, y) \to (x + 3,\ y - 2)$ d) $(x, y) \to \left(\sqrt{3}x + 5,\ \sqrt{3}y\right)$

   e) a vertical stretch of $\frac{1}{2}$, a horizontal stretch of 3, centre $(-2, 5)$

   f) $(x, y) \to \left(\frac{1}{3}x - 2,\ \frac{1}{3}y - 1\right)$

2) What is the centre for the circle or ellipse defined by each equation above? If any of the above are circles, state the radius. If any are ellipses, state the length of the major and minor axis.

E13/E3/E14

Pencil and Paper

3) Express each equation in transformational form state the transformation of $x^2 + y^2 = 1$, and the mapping rules then sketch the graph.

   a) $x^2 + y^2 - 8x - 9 = 0$ b) $x^2 + y^2 - 10x - 6y - 2 = 0$

   c) $2x^2 + 2y^2 + 5y = 0$ d) $2x^2 + 2y^2 - 4x + 6y - 2 = 0$

   e) $x^2 + y^2 - 8x + 6y - 11 = 0$

E15

Pencil and Paper

4) The dome of an arena is elliptical in shape. If the height of the dome is 28 m, and it has a span of 75 m, find a possible equation for this ellipse.

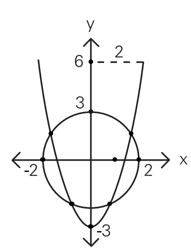
5) An ellipse is given by the equation $25x^2 + 4y^2 + 100x - 16y + 16 = 0$.
   a) What are the coordinates of its centre?
   b) Define the mapping applied to the circle $x^2 + y^2 = 1$ to obtain this ellipse.

6) The points $P\left(-1, \frac{8\sqrt{2}}{3}\right)$ and $Q\left(2, \frac{-4\sqrt{5}}{3}\right)$ are on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ Find the values of a^2 and b^2.

E15/E7/E11

Pencil and Paper

7) Given the circle and ellipse as in the graph
   a) Determine the equation for both the circle and the ellipse.
   b) Determine the length of the two chords.
   c) Determine the larger ratio:
      i) the length of the longer chord to the centre of the ellipse
      ii) the length of the shorter chord to the centre of the circle



Suggested Resources

Circle Geometry

Outcomes

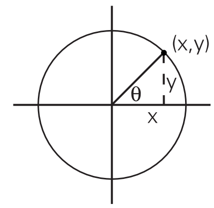
SCO: In this course, students will be expected to

C20 (111) represent circles using parametric equations

C36 (111) demonstrate an understanding of the relationship between angle rotation and the coordinates of a rotating point

Elaboration—Instructional Strategies/Suggestions

C20(111) Students should investigate how technology "draws" circles. For example, a calculator calculates coordinates for points and so turns numbers into shapes on the screen.



C36(111) Students should understand that if the cursor traced a circle on the screen, it would report coordinates (x, y) along the circle. To simulate this, students should draw a unit circle (radius 1) with centre (0, 0). Try $\theta = 30^\circ$, where the radius meets the circle drop a perpendicular to the x-axis and calculate the value of x, or the length from the origin to the intersection point $\cos\theta = \frac{x}{radius} \Rightarrow \cos 30^\circ = \frac{x}{1} = \cos 30^\circ$.

Similarly, $\sin 30^\circ \Rightarrow \frac{y}{1} \Rightarrow y = \sin 30^\circ$.

If $\theta = 135^\circ$, the coordinate where the radius meets the circle is (cos 135°, sin 135°). Likewise, if $\theta = 238^\circ$, the calculator would plot the point (cos 238°, sin 238°) or approximately (–.53, –.85).

Also students should then be able to find the angle of rotation (θ) given a coordinate on the unit circle such as (.34, .94). They should approximate θ to be about 70°.

C20(Z) The angle θ is called the parameter of $x = \cos\theta,\ y = \sin\theta$. It is a variable used to describe another variable. Equations that contain parameters are called parametric equations. So, if students wanted to draw a circle using parametric equations, they would assign θ the values 0° to 360° and graph $x = \cos\theta,\ y = \sin\theta$.

On the TI-83 calculator, go to parametric mode and in the window, set $T = 0^\circ \rightarrow 360^\circ$, T-step at 5, x-values from –3 to 3, and y-values from –2 to 2. Then at "*y* =," students would enter x_{1T}= cos (*T*) and y_{1T} = sin (*T*). Push Enter to graph a circle with radius one.

Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

C36(111)

Paper and Pencil

1) Without using trigonometric ratios, calculate the coordinate of the point P as it rotates through the following degrees on a unit circle, centre (0, 0). Express your answer using exact values.

a) 45°	e) 30°	i) 60°
b) 135°	f) 150°	j) 120°
c) 225°	g) 210°	k) 240°
d) 315°	h) 330°	l) 300°

2) Describe the patterns you see in the above results.
3) Find the same values in (1) above using trigonometric ratios and a calculator.
4) Find the values cos 50° and sin 50°. Explain the value of each in terms of a unit circle.
5) On a unit circle, the coordinates of the image of (1, 0) after a rotation are (–0.6282, 0.7781).
   a) find the angle of rotation correct to two decimal places.
   b) What would be the arc measure from (-1, 0) to (-0.6282, 0.7781)?

C20(111)

Paper and Pencil

6) Using parametric mode, write the equations that will produce a unit circle with centre (0, 0). State your 'window' settings.
7) How would you change your equations in # 6) above to increase the radius, and produce a circle with radius 5 units. State your window settings.

Suggested Resources

Circle Geometry

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Outcomes

SCO: In this course, students will be expected to

C37 (111) describe and apply parameter changes within parametric equations of circles

Elaboration—Instructional Strategies/Suggestions

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* C37(111) Students should be encouraged to investigate all the parameters in the list below:

What happens as T-step is increased? Explain this behaviour. (Fewer points are being calculated, which because of the pixel display has a positive effect on the appearance of the circles up to t-step about 20).

With x set to –6 to 6, and y to –4 to 4, investigate circles produced by 2 cos T, 2 sin T; 4cos T, 4 sin T, and explain how the "2" and "4" affect the circle (the "2" and "4" are the actual radii of the circles).

Now explore the graphs of 2 + cos T, 2 + sin T; 4 + cos T, 4 + sin T (the "2" moves the centre to (2, 2), the "4" moves the centre (4, 4), and a small T-step is better. Also, a better window is needed—add two of each of the previous values.

Explore adding a value to T in the argument: cos (T + 5), sin (T + 5); cos (T +1), sin (T + 1). (Same circle—the "T" value has been incremented, which doesn't affect the size or position of the circle.)

Explore multiplications of T in the argument: cos 2T, sin 2T; cos 5T, sin 5T. (Again, this produces the same circle—affects T-step—not the position or size of the circle.)

In summary, when the variable T is multiplied by numbers, the radius is affected, but when numbers are added to T, the location of the centre of the circle is transformed.

Students will want to justify their conjectures by making up examples, estimating position and size, then graphing to check.

For example: $3\cos\theta,\ 3\sin\theta,\ 7+3\cos\theta,\ 4+3\sin\theta.$

Finally students should change the range of T-step to various numbers less than 360° to see partial circles or pieces of circles.

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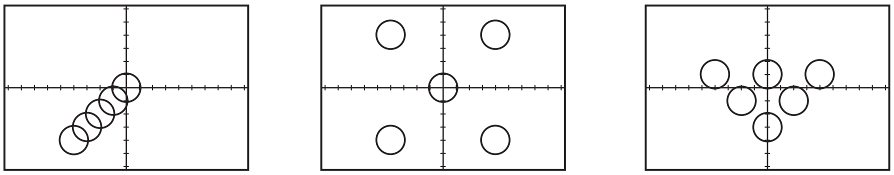
Circle Geometry

Worthwhile Tasks for Instruction and/or Assessment

C37(111)

Paper and Pencil

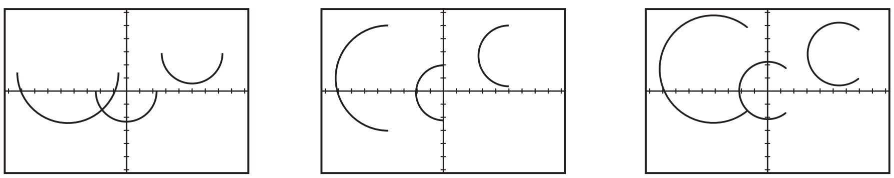
1) Write parametric equations to represent the circles in each of these graphs.



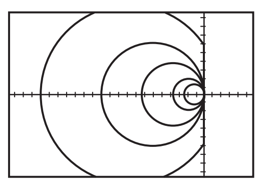
C37(111)

Performance

2) Explain how to get each of these graphs displayed on your screen.



3) This is a design created with parametric equations and a graphing calculator. Create your own design or picture and record the equations, settings, and proper ranges to duplicate it.



4) a) Set maximum T = 180°; use the equations $X = \cos(T)$ and $Y = \sin(T)$.
   b) Set maximum T = 360°; use the equations $X = \cos(0.5T)$ and $Y = \sin(0.5T)$.
   c) Set maximum T = 90°; use the equations $X = \cos(2T)$ and $Y = \sin(2T)$.
   How do these different descriptions affect the way the graphing calculator plots the points? Which is the fastest? Which is the most accurate? Explain your answer.

Suggested Resources

Appendix A:
Assessing and Evaluating Student Learning

Assessing and Evaluating Student Learning

In recent years there have been calls for change in the practices used to assess and evaluate students' progress. Many factors have set the demands for change in motion, including the following:

- *new expectations for mathematics education as outlined in* Curriculum and Evaluation Standards for School Mathematics (NCTM 1989)

The *Curriculum Standards* provide educators with specific information about what students should be able to do in mathematics. These expectations go far beyond learning a list of mathematical facts; instead, they emphasize such competencies as creative and critical thinking, problem solving, working collaboratively, and the ability to manage one's own learning. Students are expected to be able to communicate mathematically, to solve and create problems, to use concepts to solve real-world applications, to integrate mathematics across disciplines, and to connect strands of mathematics. For the most part, assessments used in the past have not addressed these expectations. New approaches to assessment are needed if we are to address the expectations set out in *Curriculum and Evaluation Standards for School Mathematics* (NCTM 1989).

- *understanding the bonds linking teaching, learning, and assessment*

Much of our understanding of learning has been based on a theory that viewed learning as the accumulation of discrete skills. Cognitive views of learning call for an active, constructive approach in which learners gain understanding by building their own knowledge and developing connections between facts and concepts. Problem solving and reasoning become the emphases rather than the acquisition of isolated facts. Conventional testing, which includes multiple choice or having students answer questions to determine if they can recall the type of question and the procedure to be used, provides a window into only one aspect of what a student has learned. Assessments that require students to solve problems, demonstrate skills, create products, and create portfolios of work reveal more about the student's understanding and reasoning of mathematics. If students are expected to develop reasoning and problem-solving competencies, then teaching must reflect such, and in turn, assessment must reflect what is valued in teaching and learning. Feedback from assessment directly affects learning. The development of problem-solving, and higher, order thinking skills will become a realization only if assessment practices are in alignment with these expectations.

- *limitations of the traditional methods used to determine student achievement*

Do traditional methods of assessment provide the student with information on how to improve performance? We need to develop methods of assessment that provide us with accurate information about students' academic achievement and information to guide teachers in decision making to improve both learning and teaching.

What Is Assessment?

Assessment is the systematic process of gathering information on student learning. Assessment allows teachers to communicate to students what is really valued—what is worth learning, how it should be learned, what elements of quality are considered most important, and how well students are expected to perform. In order for teachers to assess student learning in a mathematics curriculum that emphasizes applications and problem solving, they need to employ strategies that recognize the reasoning involved in the process as well as in the product. *Assessment Standards for School Mathematics* (NCTM 1995, p. 3) describes assessment practices that enable teachers to gather evidence about a student's knowledge of, ability to use, and disposition toward, mathematics and of making inferences from that evidence for a variety of purposes.

Assessment can be informal or formal. Informal assessment occurs while instruction is occurring. It is a mind-set, a daily activity that helps the teacher answer the question, Is what is taught being learned? Its primary purpose is to collect information about the instructional needs of students so that the teacher can make decisions to improve instructional strategies. For many teachers, the strategy of making annotated comments about a student's work is part of informal assessment. Assessment must do more than determine a score for the student. It should do more than portray a level of performance. It should direct teachers' communication and actions. Assessment must anticipate subsequent action.

Formal assessment requires the organization of an assessment event. In the past, mathematics teachers may have restricted these events to quizzes, tests, or exams. As the outcomes for mathematics education broaden, it becomes more obvious that these assessment methods become more limited. Some educators would argue that informal assessment provides better quality information because it is in a context that can be put to immediate use.

Why Should We Assess Student Learning?

We should assess student learning in order to

- improve instruction by identifying successful instructional strategies
- identify and address specific sources of the students' misunderstandings
- inform the students about their strengths in skills, knowledge, and learning strategies
- inform parents of their child's progress so that they can provide more effective support
- determine the level of achievement for each outcome

As an integral and ongoing part of the learning process, assessment must give each student optimal opportunity to demonstrate what he/

she knows and is able to do. It is essential, therefore, that teachers develop a repertoire of assessment strategies.

Assessment Strategies

Some assessment strategies that teachers may employ include the following.

Documenting classroom behaviours

In the past teachers have generally made observations of students' persistence, systematic working, organization, accuracy, conjecturing, modeling, creativity, and ability to communicate ideas, but often failed to document them. Certainly the ability to manage the documentation played a major part. Recording information signals to the student those behaviours that are truly valued. Teachers should focus on recording only significant events, which are those that represent a typical student's behaviour or a situation where the student demonstrates new understanding or a lack of understanding. Using a class list, teachers can expect to record comments on approximately four students per class. The use of an annotated class list allows the teacher to recognize where students are having difficulties and identify students who may be spectators in the classroom.

Using a portfolio or student journal

Having students assemble on a regular basis responses to various types of tasks is part of an effective assessment scheme. Responding to open-ended questions allows students to explore the bounds and the structure of mathematical categories. As an example, students are given a triangle in which they know two sides or an angle and a side and they are asked to find out everything they know about the triangle. This is preferable to asking students to find a particular side, because it is less prescriptive and allows students to explore the problem in many different ways and gives them the opportunity to use many different procedures and skills. Students should be monitoring their own learning by being asked to reflect and write about questions such as the following:

- What is the most interesting thing you learned in mathematics class this week?
- What do you find difficult to understand?
- How could the teacher improve mathematics instruction?
- Can you identify how the mathematics we are now studying is connected to the real world?

In the portfolio or in a journal, teachers can observe the development of the students' understanding and progress as a problem solver. Students should be doing problems that require varying lengths of time and represent both individual and group effort. What is most important is that teachers discuss with their peers what items are to be part of a meaningful portfolio, and that students also have some input into the assembling of a portfolio.

Projects and investigative reports

Students will have opportunities to do projects at various times throughout the year. For example, they may conduct a survey and do a statistical report, they may do a project by reporting on the contribution of a mathematician, or the project may involve building a complex three-dimensional shape or a set of three-dimensional shapes which relate to each other in some way. Students should also be given investigations in which they learn new mathematical concepts on their own. Excellent materials can be obtained from the National Council of Teachers of Mathematics, including the *Student Math Notes.* (These news bulletins can be downloaded from the Internet.)

Written tests, quizzes, and exams

Some critics allege that written tests are limited to assessing a student's ability to recall and replicate mathematical facts and procedures. Some educators would argue that asking students to solve contrived applications, usually within time limits, provides us with little knowledge of the students' understanding of mathematics.

How might we improve the use of written tests?

- Our challenge is to improve the nature of the questions being asked, so that we are gaining information about the students' understanding and comprehension.
- Tests must be designed so that questions being asked reflect the expectations of the outcomes being addressed.
- One way to do this is to have students construct assessment items for the test. Allowing students to contribute to the test permits them to reflect on what they were learning, and it is a most effective revision strategy.
- Teachers should reflect on the quality of the test being given to students. Are students being asked to evaluate, analyse, and synthesize information, or are they simply being asked to recall isolated facts from memory? Teachers should develop a table of specifications when planning their tests.
- In assessing students, teachers have a professional obligation to ensure that the assessment reflects those skills and behaviours that are truly valued. Good assessment goes hand-in-hand with good instruction and together they promote student achievement.

Appendix B:
SCOs for Grades 9 and 10

GCO A: Students will demonstrate number sense and apply number theory concepts.

Elaboration: Number sense includes understanding of number meanings, developing multiple relationships among numbers, recognizing the relative magnitudes of numbers, knowing the relative effect of operating on numbers, and developing referents for measurement. Number theory concepts include such number principles as laws (e.g., commutative and distributive), factors and primes, number system characteristics (e.g., density), etc.

By the end of grade 9, students will be expected to

A1 solve problems involving square root and principal square root

A2 graph, and write in symbols and in words, the solution set for equations and inequalities involving integers and other real numbers

A3 demonstrate an understanding of the meaning and uses of irrational numbers

A4 demonstrate an understanding of the inter-relationships of subsets of real numbers

A5 compare and order real numbers

A6 represent problem situations using matrices

By the end of grade 10, students will be expected to

A1 relate sets of numbers to solutions of inequalities

A2 analyse graphs or charts of situations to derive specific information

A3 demonstrate an understanding of the role of irrational numbers in applications

A4 approximate square roots

A5 demonstrate an understanding of the zero product property and its relationship to solving equations by factoring

A6 apply properties of numbers when operating upon expressions and equations

A7 demonstrate and apply an understanding of discrete and continuous number systems

A8 demonstrate an understanding of and apply properties to operations involving square roots

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration: Operation sense consists of recognizing situations in which a given operation would be useful, building awareness of models and the properties of an operation, seeing relationships among operations and acquiring insights into the effects of an operation on a pair of numbers. Operation principles and procedures would include such items as the effect of identity elements, computational strategies and mental mathematics.

By the end of grade 9, students will be expected to

B1 model, solve, and create problems involving real numbers

B2 add, subtract, multiply, and divide rational numbers in fractional and decimal forms, using the most appropriate method

B3 apply the order of operations in rational number computations

B4 demonstrate an understanding of, and apply the exponent laws, for integral exponents

B5 model, solve, and create problems involving numbers expressed in scientific notation

B6 judge the reasonableness of results in problem situations involving square roots, rational numbers, and numbers written in scientific notation

B7 model, solve, and create problems involving the matrix operations of addition, subtraction, and scalar multiplication

B8 add and subtract polynomial expressions symbolically to solve problems

B9 factor algebraic expressions with common monomial factors concretely, pictorially, and symbolically

B10 recognize that the dimensions of a rectangular model of a polynomial are its factors

B11 find products of two monomials, a monomial and a polynomial, and two binomials, concretely, pictorially, and symbolically

By the end of grade 10, students will be expected to

B1 model (with concrete materials and pictorial representations) and express the relationships between arithmetic operations and operations on algebraic expressions and equations

B2 develop algorithms and perform operations on irrational numbers

B3 use concrete materials, pictorial representations, and algebraic symbolism to perform operations on polynomials

B4 identify and calculate the maximum and/or minimum values in a linear programming model

B5 develop, analyse, and apply procedures for matrix multiplication

B6 solve network problems using matrices

GCO B: Students will demonstrate operation sense and apply operation principles and procedures in both numeric and algebraic situations.

Elaboration: Operation sense consists of recognizing situations in which a given operation would be useful, building awareness of models and the properties of an operation, seeing relationships among operations and acquiring insights into the effects of an operation on a pair of numbers. Operation principles and procedures would include such items as the effect of identity elements, computational strategies and mental mathematics.

By the end of grade 9, students will be expected to

B12 find quotients of polynomials with monomial divisors

B13 evaluate polynomial expressions

B14 demonstrate an understanding of the applicability of commutative, associative, distributive, identity, and inverse properties to operations involving algebraic expressions

B15 select and use appropriate strategies in problem situations

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally.

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 9, students will be expected to

C1 represent patterns and relationships in a variety of formats and use these representations to predict and justify unknown values

C2 interpret graphs that represent linear and non - linear data

C3 construct and analyse tables and graphs to describe how changes in one quantity affect a related quantity

C4 determine the equations of lines by obtaining their slopes and y-intercepts from graphs, and sketch graphs of equations using y-intercepts and slopes

C5 explain the connections among different representations of patterns and relationships

C6 solve single-variable equations algebraically, and verify the solutions

C7 solve first-degree single-variable inequalities algebraically, verify the solutions, and display them on number lines

C8 solve, and create problems involving linear equations and inequalities

By the end of grade 10, students will be expected to

C1 express problems in terms of equations and vice versa

C2 model real-world phenomena with linear, quadratic, exponential, and power equations, and linear inequalities

C3 gather data, plot the data using appropriate scales, and demonstrate an understanding of independent and dependent variables, and domain and range

C4 create and analyse scatter plots using appropriate technology

C5 sketch graphs from words, tables, and collected data

C6 apply linear programming to find optimal solutions to real-world problems

C7 model real-world situations with networks and matrices

C8 identify, generalize, and apply patterns

C9 construct and analyse graphs and tables relating two variables

C10 describe real-world relationships depicted by graphs, tables of values, and written descriptions

C11 write an inequality to describe its graph

C12 express and interpret constraints using inequalities

C13 determine the slope and y-intercept of a line from a table of values or a graph

C14 determine the equation of a line using the slope and y-intercept

C15 develop and apply strategies for solving problems

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally. (continued)

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 10, students will be expected to

C16 interpret solutions to equations based on context

C17 solve problems using graphing technology

C18 investigate and find the solution to a problem by graphing two linear equations with and without technology

C19 solve systems of linear equations using substitution and graphing methods

C20 evaluate and interpret non-linear equations using graphing technology

C21 explore and apply functional relationships notation, both formally and informally

C22 analyse and describe transformations of quadratic functions and apply them to absolute value functions

C23 express transformations algebraically and with mapping rules

C24 rearrange equations

C25 solve equations using graphs

C26 solve quadratic equations by factoring

C27 solve linear and simple radical, exponential, and absolute value equations and linear inequalities

C28 explore and describe the dynamics of change depicted in tables and graphs

C29 investigate, and make and test conjectures concerning, the steepness and direction of a line

C30 compare regression models of linear and non-linear functions

GCO C: Students will explore, recognize, represent, and apply patterns and relationships, both informally and formally. (continued)

Elaboration: Patterns and relationships run the gamut from number patterns and those made from concrete materials to polynomial and exponential functions. The representation of patterns and relationships will take on multiple forms, including sequences, tables, graphs, and equations, and these representations will be applied as appropriate in a wide variety of relevant situations.

By the end of grade 10, students will be expected to

C31 graph equations and inequalities and analyse graphs both with and without graphing technology

C32 determine if a graph is linear by plotting points in a given situation

C33 graph by constructing a table of values, by using graphing technology, and when appropriate, by the slope y-intercept method

C34 investigate and make and test conjectures about the solution to equations and inequalities using graphing technology

C35 expand and factor polynomial expressions using perimeter and area models

C36 explore, determine, and apply relationships between perimeter and area, surface area, and volume

C37 represent network problems using matrices and vice versa

GCO D: Students will demonstrate an understanding of and apply concepts and skills associated with measurement.

Elaboration: Concepts and skills associated with measurement include making direct measurements, using appropriate measurement units and using formulas (e.g., surface area, Pythagorean Theorem) and/or procedures (e.g., proportions) to determine measurements indirectly.

By the end of grade 9, students will be expected to

D1 solve indirect measurement problems by connecting rates and slopes

D2 solve measurement problems involving conversion among SI units

D3 relate the volumes of pyramids and cones to the volumes of corresponding prisms and cylinders

D4 estimate, measure, and calculate dimensions, volumes, and surface areas of pyramids, cones, and spheres in problem situations

D5 demonstrate an understanding of and apply proportions within similar triangles

By the end of grade 10, students will be expected to

D1 determine and apply formulas for perimeter, area, surface area, and volume

D2 apply the properties of similar triangles

D3 relate the trigonometric functions to the ratios in similar right triangles

D4 use calculators to find trigonometric values of angles and angles when trigonometric values are known

D5 apply trigonometric functions to solve problems involving right triangles, including the use of angles of elevation

D6 solve problems involving measurement using bearings and vectors

D7 determine the accuracy and precision of a measurement

D8 solve problems involving similar triangles and right triangles

D9 determine whether differences in repeated measurements are significant or accidental

D10 determine and apply relationships between the perimeters and areas of similar figures, and between the surface areas and volumes of similar solids

D11 explore, discover, and apply properties of maximum areas and volumes

D12 solve problems using the trigonometric ratios

D13 demonstrate an understanding of the concepts of surface area and volume

D14 apply the Pythagorean Theorem

GCO E: Students will demonstrate spatial sense and apply geometric concepts, properties, and relationships.

Elaboration: Spatial sense is an intuitive feel for one's surroundings and the objects in them and is characterized by such geometric relationships as (i) the direction, orientation, and perspectives of objects in space; (ii) the relative shapes and sizes of figures and objects; and (iii) how a change in shape relates to a change in size. Geometric concepts, properties, and relationships are illustrated by such examples as the concept of area, the property that a square maximizes area for rectangles of a given perimeter, and the relationships among angles formed by a transversal intersecting parallel lines.

By the end of grade 9, students will be expected to

E1 investigate, and demonstrate an understanding of, the minimum sufficient conditions to produce unique triangles

E2 investigate, and demonstrate an understanding of, the properties of, and the minimum sufficient conditions to guarantee, congruent triangles

E3 make informal deductions, using congruent triangle and angle properties

E4 demonstrate an understanding of and apply the properties of similar triangles

E5 relate congruence and similarity of triangles

E6 use mapping notation to represent transformations of geometric figures, and interpret such notations

E7 analyse and represent combinations of transformations, using mapping notation

E8 investigate, determine, and apply the effects of transformations of geometric figures, on congruence, similarity, and orientation

By the end of grade 10, students will be expected to

E1 explore properties of, and make and test conjectures about 2- and 3-dimensional figures

E2 solve problems involving polygons and polyhedra

E3 construct and apply altitudes, medians, angle bisectors, and perpendicular bisectors to examine their intersection points

E4 apply transformations when solving problems

E5 use transformations to draw graphs

E6 represent network problems as digraphs

E7 demonstrate an understanding of, and write a proof for, the Pythagorean Theorem

E8 use inductive and deductive reasoning when observing patterns, developing properties, and making conjectures

E9 use deductive reasoning and construct logical arguments and be able to determine, when given a logical argument, if it is valid

GCO F: Students will solve problems involving the collection, display and analysis of data.

Elaboration: The collection, display and analysis of data involves (i) attention to sampling procedures and issues, (ii) recording and organizing collected data, (iii) choosing and creating appropriate data displays, (iv) analysing data displays in terms of broad principles (e.g., display bias) and via statistical measures (e.g., mean), and (v) formulating and evaluating statistical arguments.

By the end of grade 9, students will be expected to

F1 determine characteristics of possible relationships shown in scatter plots

F2 sketch lines of best fit and determine their equations

F3 sketch curves of best fit for relationships that appear to be non-linear

F4 select, defend, and use the most appropriate methods for displaying data

F5 draw inferences and make predictions based on data analysis and data displays

F6 demonstrate an understanding of the role of data management in society

F7 evaluate arguments and interpretations that are based on data analysis

By the end of grade 10, students will be expected to

F1 design and conduct experiments using statistical methods and scientific inquiry

F2 demonstrate an understanding of the concerns and issues that pertain to the collection of data

F3 construct various displays of data

F4 calculate various statistics using appropriate technology, analyse and interpret data displays, and describe relationships

F5 analyse statistical summaries, draw conclusions, and communicate results about distributions of data

F6 solve problems by modeling real-world phenomena

F7 explore non-linear data using power and exponential regression to find a curve of best fit

F8 determine and apply the line of best fit using the least squares method and median-median method with and without technology, and describe the differences between the two methods

F9 demonstrate an intuitive understanding of correlation

F10 use interpolation, extrapolation and equations to predict and solve problems

F11 describe real-world relationships depicted by graphs and tables of values

F12 explore measurement issues using the normal curve

F13 calculate and apply mean and standard deviation using technology, to determine if a variation makes a difference

GCO G: Students will represent and solve problems involving uncertainty.

Elaboration: Representing and solving problems involving uncertainty entails (i) determining probabilities by conducting experiments and/or making theoretical calculations, (ii) designing simulations to determine probabilities in situations which do not lend themselves to direct experiment, and (iii) analysing problem situations to decide how best to determine probabilities.

In grade 9, students will be expected to

By the end of grade 9, students will be expected to

G1 make predictions of probabilities involving dependent and independent events by designing and conducting experiments and simulations

G2 determine theoretical probabilities of independent and dependent events

G3 demonstrate an understanding of how experimental and theoretical probabilities are related

G4 recognize and explain why decisions based on probabilities may be combinations of theoretical calculations, experimental results, and subjective judgements