On the Estimation of the Regional Geoid Error in Canada

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Abstract. The geoid errors for the Canadian Gravimetric Geoid 2005 (CGG05) model are estimated from the error information of the satellite and terrestrial gravity data. Calibration is conducted through the application of variance component estimation (VCE) with GPS-leveling data and their associated covariance matrices. Preliminary results suggest that the error of the geoid heights is generally smaller than 6 cm in Canada, with a range from 6 cm to 31 cm for the Western Cordillera area. Overall, the average error of the CGG05 model is estimated at 5.5 cm.

Keywords. Geoid, gravity, error estimation

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1 Introduction

With the increased use of GPS-based positioning, the demand for directly converting ellipsoidal heights to heights referred to a regional vertical datum with sufficient accuracy is also increasing. For Canada, the use of a gravimetric geoid as the national height reference surface is currently under study. The recent revolutionary development of space gravimetry (i.e., CHAMP, GRACE and the upcoming GOCE mission) offers the opportunity to pursue such a dramatic change in the local vertical datum. However, before a vertical datum based on a gravimetric geoid is adopted, as opposed to one based on conventional leveling networks, it is important to conduct a reliable assessment of the systematic and stochastic errors of the geoid model. This is a challenging task as there is limited or poor information regarding the quality of the terrestrial gravity data, difficulty in quantifying the gravity reduction and interpolation errors, as well as only approximate estimates of the errors associated with the satellite models.

The purpose of this paper is twofold, namely (i) to provide a detailed examination of all error sources of the Canadian Gravimetric Geoid 2005 (CGG05) model (includes the satellite gravity model, terrestrial gravity data, gravity reduction methods and the far-zone contribution error), and (ii) to develop an adequate method for the estimation of the gravimetric geoid error from these error sources. In particular, the iterative almost unbiased estimation (IAUE) scheme is implemented to validate/calibrate the geoid error using existing GPS-leveling data across Canada.

2 Methodology

2.1 Estimation of Gravimetric Geoid Error

The CGG05 model was computed using the degree-banded Stokes integral, which is described in Huang and Véronneau (2005). In order to illustrate how errors propagate into the geoid model, the formula for the geoid determination is simplified as

$$N = N_L^{SG} + \frac{R}{4\pi\gamma}\int_{\Omega_0'} S_{DB}(\psi)(\Delta g^{TG} - \Delta g_L^{SG})\, d\Omega' + F_N \quad (1)$$

where the first term on the right hand side of eq. (1) represents the geoid components below spherical harmonic degree L+1 from a satellite model (SG), R is the mean radius of the Earth, γ is the normal gravity on the reference ellipsoid, and Δg denotes the gravity anomalies. The degree-banded Stokes kernel can be expressed as

$$S_{DB}(\psi) = \sum_{n=L+1}^{m_{TG}} \frac{2n+1}{n-1} P_n(\cos\psi)$$

It is used as a band-pass filter to compute the geoid components from degree L+1 to m_{TG}. The upper limit m_{TG} is dependent on the terrestrial gravity (TG) data spacing.

The Stokes integration is performed within a spherical cap limited to a spherical angular distance of 6 arc-degrees. This implies that the band-pass filtering is incomplete and renders aliasing geoid errors that account for an RMS of approximately 2 cm over Canada.

Finally, F_N is the far-zone contribution outside the Stokes integration. It can be evaluated from a combined global spherical harmonic gravity (GGM) model up to its maximum degree (preferably larger than degree 200):

$$F_N = \frac{R}{2\gamma} \sum_{n=L+1}^{m_{CG}} Q_n^{DB} g_n^{CG}$$

where

$$Q_n^{DB}(\psi_0) = \int_{\psi_0}^{\pi} S_{DB}(\psi) P_n(\cos\psi) \sin\psi d\psi$$

The geoid error is primarily comprised of errors from the satellite-only gravity model, the combined global gravity models, and the terrestrial gravity data. The satellite gravity signal usually dominates the low-degree part of the geoid components in a combined model while the terrestrial gravity data complete the GGM for higher degrees and orders (Sideris and Schwarz, 1987). For regional geoid determination, the lower limit, L, must be selected according to the quality of the satellite data. Empirical tests show that it should not exceed 30 for GGMs prior to the CHAMP/GRACE missions, if a decimeter-level accurate geoid is sought. A simplified expression for the geoid error is given by:

$$v_N = v_{SG} + v_{TG} + v_{CG} \tag{2}$$

where

$$v_{SG} = \frac{R}{2\gamma} \sum_{n=2}^{L} \left(\frac{2}{n-1} + Q_n^{DB} \right) \varepsilon_n^{SG} \tag{3}$$

$$v_{TG} = \frac{R}{4\pi\gamma} \sum S_{DB}(\psi) \varepsilon^{TG} \Delta\Omega' \tag{4}$$

$$v_{CG} = \frac{R}{2\gamma} \sum_{n=L+1}^{m_{CG}} Q_n^{DB} \varepsilon_n^{CG} \tag{5}$$

v_{SG}, v_{TG}, and v_{CG} represent the geoid errors from the satellite model, the terrestrial gravity data, and the combined model, respectively. Given the covariance (CV) matrices for each of these three types of errors, the geoid standard deviation (std) can be estimated based on eqs. (3) to (5) via error propagation. In our case, the geoid std may be evaluated only approximately because the CV matrices for the satellite and combined harmonic gravity models are approximate. Furthermore, only approximate error values are available for the terrestrial gravity data in Canada, which inherently contains a combination of errors originating from several sources, including gravity measurements, height measurements at the gravity points, topographic reduction, interpolation of gravity values, digital elevation models (DEM) and actual topographical density distribution.

2.2 Calibration of the Geoid Error

Geoid heights can also be determined at co-located GPS and leveling stations, which provides an independent external means to validate and calibrate the gravimetric geoid model and its precision.

The discrepancies between the GPS/leveling-derived and gravimetric geoid heights can (predominantly) be attributed to a combination of systematic and random errors in the ellipsoidal heights (h), the orthometric heights (H), and the gravimetric geoid heights (N), as discussed in Kotsakis and Sideris (1999). The following general linear functional model was used for the combined (multi-data) least-squares adjustment of the heterogeneous height data:

$$\mathbf{Ax} + \mathbf{Bv} + \mathbf{w} = 0, \quad E\{\mathbf{v}\} = 0 \tag{6}$$

where

$$\mathbf{B} = [\mathbf{I} \quad -\mathbf{I} \quad -\mathbf{I} \quad -\mathbf{I} \quad -\mathbf{I}] \tag{7}$$

$$\mathbf{v} = [\mathbf{v}_h^T \quad \mathbf{v}_H^T \quad \mathbf{v}_{SG}^T \quad \mathbf{v}_{TG}^T \quad \mathbf{v}_{CG}^T]^T \tag{8}$$

$$\mathbf{w} = \mathbf{h} - \mathbf{H} - \mathbf{N} \tag{9}$$

The deterministic term, **Ax,** introduced in eq. (6) represents the parametric model used to approximately model the systematic errors inherent in and among all three types of heights. The selection procedure for the type of model and assessing its validity has been discussed extensively in Fotopoulos (2003). For this particular case a simple four-parameter model was found to be sufficient and therefore incorporated for all calculations.

Individual variance components are estimated using the adjustment model in eq. (6) to (9) and the a-priori CV matrices for each of the data types (see Fotopoulos, 2003 for the detailed procedure). This procedure was followed in this study in order to achieve a more realistic estimate of the geoid model error that incorporates five individual variance components for the ellipsoidal heights and the orthometric heights at the GPS-leveling benchmarks, denoted by σ_h^2 and σ_H^2, respectively. The geoid height errors are separated for the satellite gravity model, terrestrial gravity data, and the combined gravity model, denoted by $\sigma_{SG}^2, \sigma_{TG}^2$ and σ_{CG}^2, respectively.

3 Gravity and GPS-leveling data

The lower degrees (2 to 90) of the GRACE-based GGM02C model (Tapley et al., 2005) are used for the determination of the long wavelength components of the geoid while the higher degrees (91 to 200) determine the far-zone contribution of the Stokes integration. EGM96 (Lemoine et al., 1998) is used to extend the GGM02C up to degree and order 360. The local residual terrestrial gravity data, i.e., ground, airborne, shipboard (including satellite altimetry-derived), are used to compute the geoid components above degree 90. These terrestrial data are the same as those used for the CGG2000 model (Véronneau, 2002). The latest model, CGG05, is a high-resolution geoid model for North America with a geographical spacing of 2 minutes of arc along latitudes and longitudes. Its reference ellipsoid is GRS80 and the reference frame is ITRF (no specific realization).

Canadian GPS surveys after year 1994 were used in a least-squares adjustment to compute the ellipsoidal heights with respect to the GRS80 reference ellipsoid and their associated variances/covariances (Craymer and Lapelle, 2004; pers. comm.). The reference frame is ITRF97.

The geopotential numbers are determined from a minimally constrained least-squares adjustment (via Helmert-blocking) of the geodetic leveling observations after year 1981. The single fixed station is a benchmark located along the St-Lawrence River in Rimouski, Québec, which is the same constraint used for the North American Vertical Datum of 1988 (NAVD88). Gravity values are interpolated at each benchmark from local measurements and converted to mean values along the plumbline (from geoid to topography) by correcting for the variable terrain. The variances and covariance within each Helmert block are implemented for the calibration of the geoid error, while the correlation between neighboring Helmert blocks have been omitted at this stage.

The CGG05 geoid model is validated using 430 co-located GPS-leveling stations with a distribution as depicted in Figure 1. The computed h-H-N residuals for the 430 stations plotted as a function of longitude and latitude are shown in Figure 2. The overall standard deviation of these residuals is 10.2 cm. A negative mean value of -40 cm indicates that the zero-height point of the leveling network is approximately 40 cm lower than the CGG05 geoid model.

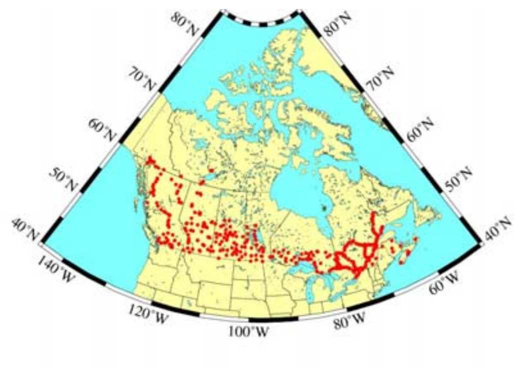

Figure 1. 430 GPS-leveling stations

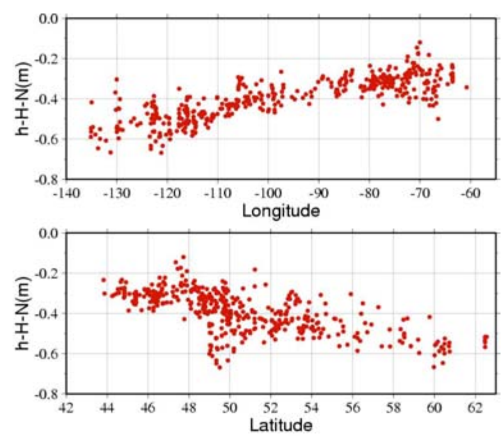

Figure 2. Height residuals (h-H-N) versus longitude and latitude

The observed north-south and east-west slopes evident in Figure 2 result in a standard deviation of about 8 cm and are caused by systematic errors attributed to all three types of heights. In general, the uncertainty in the location of the ITRF97 reference frame geocenter can generate systematic errors of a few centimeters in the ellipsoidal heights (Altamimi et al., 2002), while the GRACE data may introduce systematic errors of less than a few centimeters in the low-degree components of the gravimetric geoid heights. However, the leveling data are most likely the major source of systematic errors that accumulate over the 6000 km separation between the east and west coasts. Current knowledge about the mathematical and physical characteristics of the systematic errors in the leveling data is limited and therefore it is difficult to accurately model and correct for these discrepancies.

4 Geoid Error from the GGM02C Model

The error model of the GGM02C model propagates into the low-degree components and far-zone contribution of the Canadian geoid model. The error information of the GGM02C model is provided in terms of error coefficients obtained from the diagonal elements of the covariance matrix. Figures 3a and 3b show the low-degree geoid error estimates from the diagonal-only terms and from the fully-populated covariance matrix of the GGM02C model, respectively. In this case, the use of diagonal-only elements does not provide sufficient information for the estimation of the geoid errors.

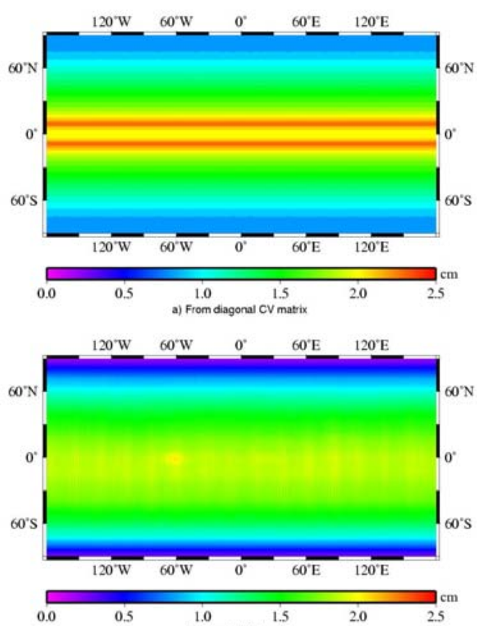

Figure 3. GGM02C geoid error estimates for degrees 2 to 90 based on (a) diagonal-only CV matrix and (b) fully populated CV matrix

The far-zone contribution error was found to be closely approximated by a constant of 1.6 cm according to the diagonal CV matrix and 0.7 cm to 1.8 cm if the fully-populated form of the CV matrix is utilized. Again, it was determined that the fully-populated covariance matrix is needed to evaluate the far-zone contribution error.

Initial covariance matrices of the low-degree and far-zone contribution components of the geoid heights at the 430 GPS-leveling stations have been estimated from the CV matrix of the harmonic coefficients of GGM02C.

5 Geoid Error from the Terrestrial Data

The terrestrial gravity error is comprised of measurement, datum, data reduction and interpolation errors. A fully populated covariance matrix for this data is not available; however, the standard deviation at each gravity station can be estimated from the measurement and elevation standard deviations. By neglecting the covariance between any two gravity stations, and the datum and interpolation errors, the initial geoid error standard deviations can be estimated via simple error propagation of Stokes integration (Li and Sideris, 1994). Figure 4 depicts the computed geoid error standard deviations based on the terrestrial gravity data, which provides an average error of 1.5 cm across Canada and a maximum of 15.7 cm in the western region. These values are most likely too optimistic due to the obvious omissions mentioned previously. However, since this is the best information currently available, these values were used to construct the initial CV matrix for the terrestrial gravity component at the 430 GPS-levelling points.

6 Estimation of Variance Components

Given the initial covariance matrices corresponding to the h, H, N_{SG}, N_{TG} and N_{CG} data, the geoid errors estimated from the GGM02C model and from the terrestrial data can be verified and calibrated at the GPS-leveling stations as per the procedure described in section 2.2.

Figure 5 shows the mean covariances with respect to the spherical distance computed from the initial CV matrices for the (a) ellipsoidal heights, (b) orthometric heights, (c) satellite and (d) terrestrial geoid errors at the 430 GPS-leveling stations, respectively.

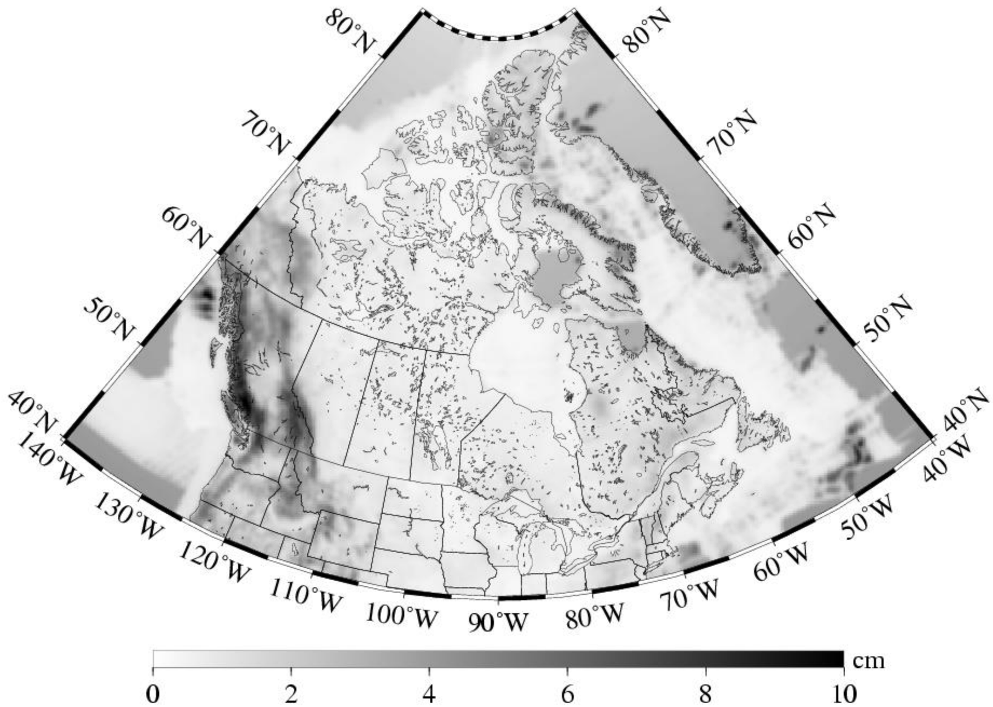

Figure 4. Regional geoid error propagated from estimated errors of the terrestrial gravity data through the degree-banded Stokes integration

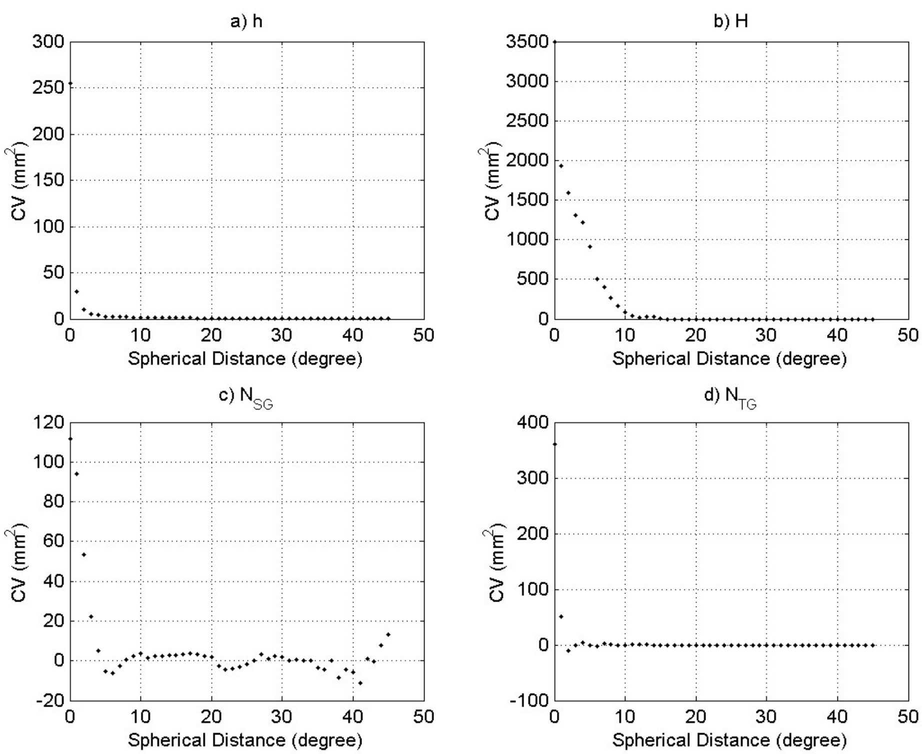

Figure 5. Initial mean covariance (in mm^2) via spherical distance for (a) ellipsoidal heights, (b) orthometric heights, (c) satellite geoid heights, and (d) residual terrestrial geoid heights

The ellipsoidal heights are weakly correlated across the GPS-leveling network with a range in standard deviations from 0.2 cm to 7.6 cm and an average value of 1.3 cm. In general, these values obtained directly from the results of the post-processing software tend to be on the optimistic side.

The orthometric heights are strongly spatially correlated within each of the Helmert blocks. The corresponding standard deviations increase as the distance with respect to the 'fixed' station in Rimouski increases. These values reach a maximum of approximately 9 cm with a mean standard deviation of 5.4 cm.

The low-degree geoid components from the GRACE model values are correlated because they are evaluated from the same set of spherical harmonic coefficients. Standard deviations for this data vary from 0.5 cm to 1.4 cm with a mean of 0.9 cm. The covariance matrix for N_{CG} is similar to that of N_{SG} (Figure 5c) and therefore not shown.

The residual geoid components evaluated from the degree-banded Stokes integral exhibit a high correlation only when common data have been used between any two computational points. In this study, the integration cap radius is 6 arc-degrees, which indicates a correlation for any two computational points located within a spherical angular distance of less than 6 arc-degrees. The standard deviations of the residual components of the geoid model are evaluated from the errors of the gravity anomaly data and range from 0.3 cm to 8.8 cm with a mean value of 1.4 cm.

Using the iterative almost unbiased variance component estimation scheme (Horn and Horn, 1975; Fotopoulos, 2003) and the a-priori CV matrices described above, five individual variance components were estimated. These values are tabulated in Table 1 for two scenarios, namely (i) diagonal-only CV matrices and (ii) fully-populated CV matrices (where available). The sensitivity of the estimated variance factors to the a-priori covariance information is evident from the differences between the estimated variance factors in each scenario. As expected, if only the diagonal information of the matrices is used as an approximation, the computed variance components are (in general) low for data where correlation is evident (e.g., orthometric heights).

The estimates in Table 1 suggest that the a-priori CV matrices corresponding to the ellipsoidal, orthometric and geoid heights are too optimistic, with final estimated variance components suggesting a re-scaling of the a-priori CV matrices of more than 3. The result for the far-zone contribution is less conclusive. In all cases, the number of iterations remained constant at approximately 70.

Table 1. Estimated variance factors using fully-populated and diagonal a-priori covariance matrices (n is the number of iterations)

CV	$\hat{\sigma}^2_h$	$\hat{\sigma}^2_H$	$\hat{\sigma}^2_{SG}$	$\hat{\sigma}^2_{TG}$	$\hat{\sigma}^2_{CG}$	n
Diagonal	2.24	0.03	7.71	2.69	6.00	72
Full	9.09	5.85	3.19	3.61	0.01	69

7 Total Geoid Error

The total geoid error is finally computed from the scaled CV matrices (after variance component estimation) of the three estimated components corresponding to SG, TG and CG. Assuming that the variance factors in Table 1 are applicable for non-GPS/leveling points (albeit a bold assumption), the total calibrated geoid error for CGG05 is illustrated in Figure 6 on a 2' × 2' grid. This assumption will be further tested using additional data that was not implemented in this study.

The calibrated geoid error ranges from 1 cm to 32 cm, with a mean error of 5.5 cm across the entire Canadian landmass. The mountainous areas of western Canada (up to Alaska) exhibit the largest errors due to sparse gravity anomaly error information (i.e., 2' in the mountains is insufficient). The geoid error in central Canada is generally smaller than 6 cm, with the hilly regions in eastern Canada showing slightly larger geoid errors. In particular, the geoid errors in the Fox Basin and Ungava Bay are significantly larger than those of the surrounding regions, due to the lack of terrestrial gravity data.

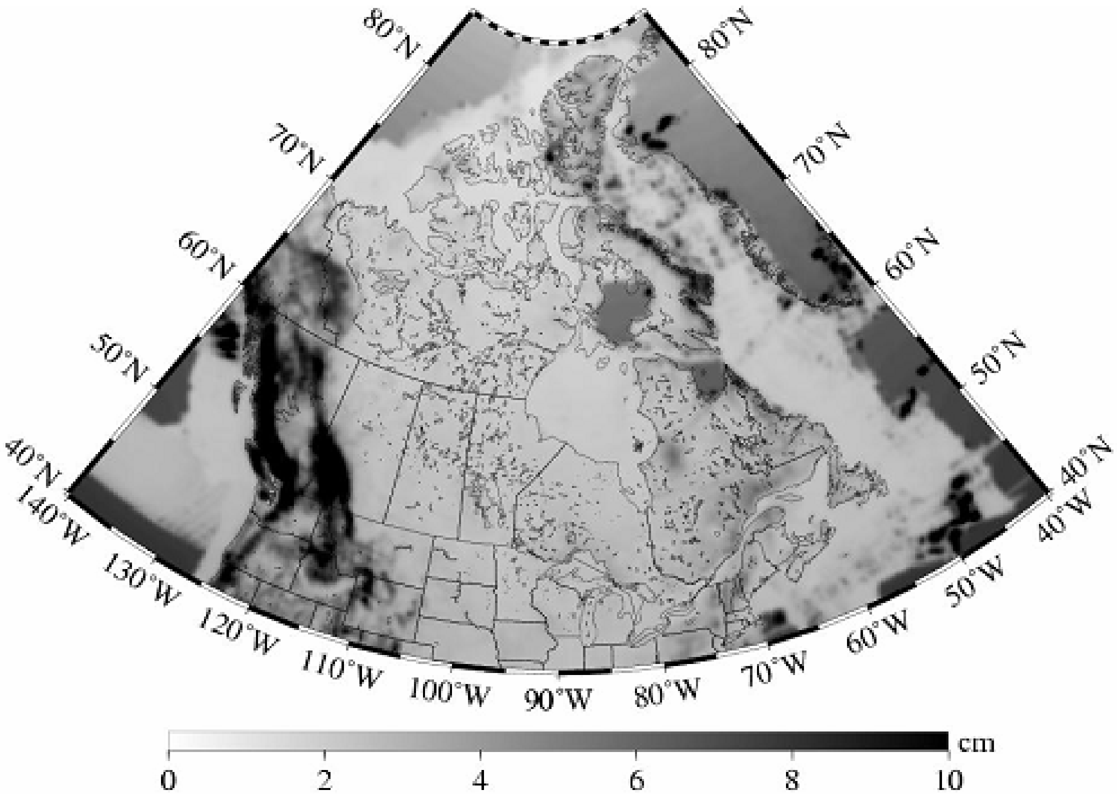

Figure 6. Regional geoid error map for Canada estimated using GPS-leveling data

8 Discussion of Future Work

The progress made in this study represents a significant step forward to achieving realistic error estimates for the Canadian gravimetric geoid model. However, it should be stated that the total geoid errors shown in Figure 6 are preliminary and refinements are ongoing. In particular, major improvements are expected on three fronts, namely (i) the inclusion of additional GPS-leveling data for a regional calibration based on the geographical heterogeneity of the data, (ii) the incorporation of the correlation between the terrestrial gravity data and (iii) the verification of the reliability of the estimated variance components, through an external validation process. Further tests will also be conducted to re-evaluate the suitability of the deterministic term introduced in the functional model for the systematic errors (i.e., type of parametric model).

Acknowledgements

The authors thank Dr. Mike Craymer and Dr. Joe Henton for their constructive comments. Appreciation is also extended to Dr. Robert Tenzer and an anonymous reviewer for their critical review.

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