# Affine-Quadratic Term Structure Models
# – Toward the Understanding of Jumps in Interest Rate

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## Abstract

In this paper, we propose a unifying class of affine-quadratic term structure models (AQTSMs) in the general jump-diffusion framework. Extending existing term structure models, the AQTSMs incorporate random jumps of stochastic intensity in the short rate process. Using information from the Treasury futures market, we propose a GMM approach for the estimation of the risk-neutral process. A distinguishing feature of the approach is that the time series estimates of stochastic volatility and jump intensity are obtained, together with model parameter estimates. Our empirical results suggest that stochastic jump intensity significantly improves the model fit to the term structure dynamics. We identify a stochastic jump intensity process that is negatively correlated with interest rate changes. Overall, negative jumps tend to have a larger size than positive ones. Our empirical results also suggest that, at monthly frequency, while stochastic volatility has certain predictive power of inflation, jumps are neither triggered by nor predictive of changes in macroeconomic variables. At daily frequency, however, we document interesting patterns for jumps associated with informational shocks in the financial market.

# 1 Introduction

Most existing term structure models are diffusion-type models where the underlying state variables driving the yield curve move continuously. Several recent studies, however, have provided strong evidence that interest rates contain jumps, that is, infrequent moves of large magnitude. Das (2002), for example, shows that incorporating jumps captures many empirical features of the Fed Funds rate that can not be explained by the continuous diffusion models. Johannes (2004) develops a test of detecting jumps and finds evidence for the presence of jumps in the 3-month Treasury bill rate. Piazzesi (2005) models the Fed's target rate as a jump process and shows that introducing jumps helps to fit the entire yield curve. An important question arising from these studies is: what causes jumps? Or from a modeling point of view, what determines the arrival rate of jumps, i.e., the jump intensity? In the aforementioned papers, the jump intensity is assumed to be either a constant (e.g. Das (2002)) or of certain specific functional forms. For example, Johannes (2004) specifies the jump intensity as a function of the short rate. Piazzesi (2001, 2005) model jumps based on the state of economy and the Federal Open Market Committee (FOMC) meeting calendar. These assumptions are reasonable but may be too restrictive to fully capture the dynamics of jumps.

In this paper, we propose a general class of affine-quadratic jump-diffusion term structure models (AQTSMs). The key feature of the model is that the jump intensity is by itself a stochastic process, and yet potentially correlated with other state variables such as the short rate and stochastic volatility. This is in contrast to the assumptions on the jump intensity in existing studies. In the spirit of Cox, Ingersoll, and Ross (1985a) and Ahn and Thompson (1988), our model is supported by a representative agent economy and in equilibrium the instantaneous interest rate is an affine-quadratic function of the underlying state variables. This specification extends two existing classes of diffusion-type term structure models: the affine term structure models (ATSMs)[1] and the quadratic term structure models

[1]The literature on the ATSMs goes back to Vasicek (1977), and Cox, Ingersoll, and Ross (1985b). Extensions include Longstaff and Schwartz (1992), Sun (1992), Pearson and Sun (1994), Balduzzi, Das, Foresi, and Sundaram (1996), Chen (1996), Chen and Scott (1996), Duffie and Kan (1996), Andersen and Lund (1997), Dai and Singleton (2000, 2002), and

(QTSMs).[2]

The ATSMs, popularized by Duffie and Kan (1996) and Dai and Singleton (2000), are flexible in describing the dynamic properties of interest rates such as stochastic volatility, while in the meantime are tractable with closed-form or near closed-form bond pricing formula where bond yields are affine functions of the underlying state variables. Among non-affine term structure models, the QTSMs proposed by Ahn, Dittmar, and Gallant (2002) have attracted great interest recently, where the bond yields are quadratic functions of the underlying state variables. Our model contains both classes of models as special cases. In particular, in the absence of jumps our model is a hybrid affine-quadratic model that is examined in Ahn, Dittmar, Gallant, and Gao (2003). This nesting feature is very useful in assessing the importance of incorporating jumps to the models of term structure dynamics. Despite the extra complexity caused by jumps and stochastic jump intensity, our model remains tractable in that, following the approach of Duffie and Kan (1996) and Duffie, Pan, and Singleton (2000), we obtain near closed-form bond pricing formula and conditional characteristic function (CCF) of the state variables. Both bond pricing formula and CCF involve only ordinary differential equations which can be easily solved either analytically or numerically. The availability of the near closed-form solution is crucial as we are able to use the data on the entire yield curve in estimation.

For our empirical analysis, we consider three examples of the general (AQTSMs) jump-diffusion model. The first example, serving as the benchmark, is a two-factor affine stochastic volatility (SV) model. It can be regarded as a restricted version of a three-factor maximally flexible affine model of Dai and Singleton (2000). We have also examined the examples of quadratic models (QTSMs) studied in Ahn, Dittmar, Gallant (2002) and find similar empirical results. Our second example, the SVJ model, generalizes the SV model by including a jump component in the short rate process with a constant

Chacko and Das (2002) among others. See Dai and Singleton (2003) for a review of the literature and the references.

[2] Some early examples of the QTSMs are Longstaff (1989), Beaglehole and Tenney (1992), and Constantinides (1992). More recent studies include Ahn, Dittmar, and Gallant (2002), Gourieroux and Sufana (2003), and Leippold and Wu (2003). Not considered in this paper are other non-affine models such as those in Chan, Karolyi, Longstaff, and Sanders (1992), and Ahn and Gao (1999).

jump intensity. This specification nests the jump-diffusion model of Das (2002) as a restricted case if the volatility of the short rate is also set to be constant. In the third example, the SVJT model, the jump intensity follows a separate stochastic process that is correlated with the processes of the short rate and stochastic volatility. The model belongs to the affine-quadratic family as the bond yields are linear in the short rate but quadratic in the state variables driving the stochastic volatility and jump intensity. It is similar to the affine-quadratic jump-diffusion models considered in Piazzesi (2001), and the main difference is that the jump intensity in our model is itself a stochastic process.

We propose a new GMM approach for the statistical inference of the AQTSMs. Different from the existing literature where model estimation is mainly based on the observed short rates, we rely on information along the entire yield curve (i.e. yields of various maturities) plus information from the T-bill futures market to identify the risk-neutral process. The approach has several distinguishing features and advantages. First, we take advantage of the closed-form solutions of the bond pricing formula derived under the affine-quadratic model framework. The moment conditions based on bond yields are not only robust but also directly measure the model performance of fitting the entire yield curve. Second and more importantly, it identifies the latent state variables using the information, e.g., the variance of yields, from the Treasury futures market. The key identifying restriction is the dynamic relation derived under the model specification between the observed state variables and the unobserved latent state variables. The time series estimates of stochastic volatility and jump intensity help us to understand the dynamics of underlying state variables, and in particular the determinants of jump arrival rate. This procedure also ensures that the model fits directly into the risk-neutral variance of the short rate and the dynamics of short rate volatility. The approach shares with existing studies, such as Pan (2002), the advantage that with proxies of the latent variables, there is no path simulation involved in the model estimation. It thus effectively overcomes the difficulty associated with the unobserved latent state variables.

In our empirical analysis, we use the term structure data and the 3-month T-bill futures data during the period of 1984 through 2002 or the so-called "post-disinflation" period. The yields along the yield curve have maturities ranging from 3 months to 30 years. Our empirical results suggest that the stochastic jump intensity significantly improves the model fit to the term structure dynamics. Incorporating only stochastic volatility or modeling jumps with constant intensity in the short rate process proves to be too restrictive for the dynamics of term structure. We identify a jump intensity process that is negatively correlated with interest rate changes, with the average jump intensity implying 9 to 10 jumps per year. Overall, negative jumps also tend to have a larger size (roughly 17 basis points) than positive jumps (roughly 15 basis points). In addition, the combination of risk-neutral variance and the real-world realized variance allows us to further understand the risk premia of stochastic volatility and random jumps. We document time-varying risk premia that are positively related to the uncertainty of the risk factors.

Recent studies, such as Ang and Piazzesi (2003), Piazzesi (2001, 2005), and Duffee (2005), have focused on the relation between macroeconomic variables and term premia. In this paper, we perform further analysis of the term structure dynamics in relation to economic activities. The time series estimates of stochastic volatility and random jumps allow us to directly examine the relations between term structure dynamics and various macroeconomic variables. We empirically test whether the state variables of the term structure dynamics are predictive of future economic activities, and whether the increase of volatility or in particular jumps are due to shocks in macroeconomic variables. At monthly frequency, while there is evidence that stochastic volatility has certain predictive power of future inflation, we find that jumps are neither triggered by nor predictive of changes in macroeconomic variables. Once focusing on daily frequency, however, we find interesting patterns for jumps associated with various informational shocks in the financial market. While many jumps occur during the scheduled macroeconomic news release dates or FOMC meeting dates, there are a considerable number of jumps that are associated with unanticipated economic news or geopolitical development. Interestingly, interest rate jumps as a result of macroeconomic shocks are dominantly more negative than positive.

The findings have important implications for understanding and managing risk factors associated with random jumps in the bond market.

The rest of the paper is structured as follows. Section 2 presents the affine-quadratic term structure models in a representative agent economy, with solutions for bond price and conditional characteristic functions of the state variables. In section 3, we focus on three specific models that are closely related to those in the existing literature, with the most general model incorporating both stochastic volatility and jumps of stochastic intensity. These models are further examined in our empirical analysis in Section 4. Section 5 concludes.

## 2 Affine-Quadratic Jump-Diffusion Models

Extending the affine and quadratic term structure models (ATSMs and QTSMs), we propose a unifying class of affine-quadratic models (AQTSMs) in the general jump-diffusion framework. The models are supported in equilibrium by a representative agent economy in the spirit of Cox, Ingersoll, and Ross (1985a), and Ahn and Thompson (1988) and includes most existing term structure models as special cases.

Let $X(t) = \binom{X_1(t)}{X_2(t)}$ be an $n \times 1$ vector of state variables with $n_1 \times 1$ and $n_2 \times 1$ sub-vectors $X_1(t)$ and $X_2(t)$ respectively. Under the probability measure $(\Omega, \mathcal{F}, P)$ with an information filtration $(\mathcal{F}_t) = \{\mathcal{F}_t : t \geq 0\}$, the state variables follow a jump-diffusion process:

$$dX = (\mu_X - \lambda \mu_{Q_X})\, dt + \Sigma_X dZ + Q_X dN, \tag{1}$$

where $Z(t) = \binom{Z_1(t)}{Z_2(t)}$, $Z_1(t)$ and $Z_2(t)$ are independent $n_1$- and $n_2$-dimensional standard Brownian motions. $N(t)$ is a Poisson process with arrival intensity $\lambda(t)$, that is,

$$\text{Prob}(dN = 1) = \lambda dt, \tag{2}$$

and $\lambda(t)$ is an affine-quadratic function of the state variables:

$$\lambda = \varphi + \phi^\top X_1 + \psi^\top X_2 + X_2^\top \Psi X_2, \tag{3}$$

where $\varphi \in \mathbb{R}$, $\phi \in \mathbb{R}^{n_1 \times 1}$, $\psi \in \mathbb{R}^{n_2 \times 1}$, and $\Psi \in \mathbb{R}^{n_2 \times n_2}$ is a symmetric matrix. The first $n_1$ components of the jump size $Q_X$ are non-degenerate, i.e., $Q_X = \binom{Q_{X_1}}{0}$, where $Q_{X_1}$ is an i.i.d. $n_1 \times 1$ random vector with mean $\mu_{Q_{X_1}}$. $N$ and $Q_X$ are independent of each other and are independent of the Brownian motions $Z$. We assume that for any vector $k \in \mathbb{R}^{n_1 \times 1}$:

$$\mathbf{E}\left[e^{k^\top Q_{X_1}}\right] = \delta_X(k), \tag{4}$$

where $\delta_X(\cdot)$ is a well defined function and $\mathbf{E}$ is the expectation with respect to the distribution of $Q_{X_1}$. Further, the drift and diffusion terms of the state variables $X$ have the following form:

$$\mu_X = \begin{pmatrix} \mu_{X_1} \\ \mu_{X_2} \end{pmatrix}, \tag{5}$$

$$\Sigma_X \Sigma_X^\top = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12}^\top & \Omega_{22} \end{pmatrix}, \tag{6}$$

where

$$\mu_{X_1} = \kappa_{10} + \kappa_{11} X_1 + \kappa_{12} X_2 + \zeta(X_2), \tag{7}$$

$$\mu_{X_2} = \kappa_{20} + \kappa_{22} X_2, \tag{8}$$

$$\zeta(X_2) = \left(X_2^\top \zeta_1 X_2, \ldots, X_2^\top \zeta_{n_1} X_2\right)^\top, \tag{9}$$

$$(\Omega_{11})_{ij} = (\omega_{10})_{ij} + (\omega_{11})_{ij}^\top X_1 + (\omega_{12})_{ij}^\top X_2 + X_2^\top \nu_{ij} X_2, \quad 1 \le i, j \le n_1, \tag{10}$$

$$(\Omega_{12})_{ij} = (\omega_{20})_{ij} + (\omega_{22})_{ij}^\top X_2, \quad 1 \le i \le n_1, 1 \le j \le n_2. \tag{11}$$

The parameters of the model are defined as: $\kappa_{10} \in \mathbb{R}^{n_1 \times 1}$, $\kappa_{20} \in \mathbb{R}^{n_2 \times 1}$, $\kappa_{11} \in \mathbb{R}^{n_1 \times n_1}$, $\kappa_{12} \in \mathbb{R}^{n_1 \times n_2}$, $\kappa_{22} \in \mathbb{R}^{n_2 \times n_2}$, $\zeta_i$'s $\in \mathbb{R}^{n_2 \times n_2}$ are symmetric matrices with $1 \le i \le n_1$, $\omega_{10} \in \mathbb{R}^{n_1 \times n_1}$, $\omega_{20} \in \mathbb{R}^{n_1 \times n_2}$, $(\omega_{11})_{ij} \in \mathbb{R}^{n_1 \times 1}$, $(\omega_{12})_{ij} \in \mathbb{R}^{n_2 \times 1}$, $(\omega_{22})_{ij} \in \mathbb{R}^{n_2 \times 1}$, $\nu_{ij}$'s $\in \mathbb{R}^{n_2 \times n_2}$ are symmetric matrices, and $\Omega_{22} \in \mathbb{R}^{n_2 \times n_2}$ is a symmetric matrix.

The jump-diffusion specification above is very general and nests most existing continuous-time dynamic models as special cases. For example, when the second group of state variables, $X_2$, are not present, the model simplifies to the affine jump-diffusion model of Duffie, Pan, and Singleton (2000). On the other hand, when the first group of state variables, $X_1$, are not present, the model reduces to the quadratic specification of Ahn, Dittmar, and Gallant (2002). An important feature of our model is that the jump intensity $\lambda$ is driven by its own source of uncertainty and yet potentially correlated with other state variables. This differs from most existing jump-diffusion term structural models where the jump intensity is assumed to be either constant (e.g. Das (2002)) or driven by other state variables (e.g. Johannes (2004), and Piazzesi (2005)).

Next, following the approach of Cox, Ingersoll, and Ross (1985a), and Ahn and Thompson (1988), we derive the dynamics of the instantaneous interest rate. Consider an economy in which there is a competitive market for instantaneous borrowing and lending at the endogenously determined spot interest rate $r(t)$. For simplicity, we assume that there is a single risky asset, which one can think of as an equity market index, and a representative utility maximizing investor.[3] The return on the risky asset follows the process:

$$\frac{dS}{S} = (\mu_S - \lambda \mu_{Q_S})\, dt + \sigma_S^\top dZ + Q_S dN, \tag{12}$$

where $Q_S$ is the jump size with mean $\mu_{Q_S}$, and independent of $N$, $Q_X$, and $Z$.[4] $\mu_S$ and $\sigma_S$ are specified as following:

$$\begin{aligned}
\mu_S &= \alpha_1 + \beta_1^\top X_1 + \gamma_1^\top X_2 + X_2^\top \Phi_1 X_2, & (13)\\
\sigma_S^\top \sigma_S &= \alpha_2 + \beta_2^\top X_1 + \gamma_2^\top X_2 + X_2^\top \Phi_2 X_2, & (14)\\
\Sigma_X \sigma_S &= \begin{pmatrix} \alpha_3 + \beta_3^\top X_1 + \gamma_3^\top X_2 + \xi(X_2) \\ \alpha_4 + \gamma_4^\top X_2 \end{pmatrix}, & (15)\\
\xi(X_2) &= \left(X_2^\top \xi_1 X_2, \ldots, X_2^\top \xi_{n_1} X_2\right)^\top, & (16)
\end{aligned}$$

where for $i = 1$ and 2, $\alpha_i \in \mathbb{R}$, $\beta_i \in \mathbb{R}^{n_1 \times 1}$, $\gamma_i \in \mathbb{R}^{n_2 \times 1}$, $\Phi_i \in \mathbb{R}^{n_2 \times n_2}$ are symmetric matrices, $\alpha_3 \in \mathbb{R}^{n_1 \times 1}$, $\alpha_4 \in \mathbb{R}^{n_2 \times 1}$, $\beta_3 \in \mathbb{R}^{n_1 \times n_1}$, $\gamma_3 \in \mathbb{R}^{n_2 \times n_1}$, $\gamma_4 \in \mathbb{R}^{n_2 \times n_2}$, and $\xi_i$'s $\in \mathbb{R}^{n_2 \times n_2}$ are symmetric matrices with $1 \leq i \leq n_1$.

The representative investor has initial wealth $W$ and allocates it between the risky asset and the risk-

[3] The model can be generalized to include multiple assets.

[4] For brevity, we do not consider the case where $Q_S$ and $Q_X$ are correlated in this paper although it can be done in a straightforward way.

free security. Let $w$ denote the fraction of wealth invested in the risky asset. For simplicity, we assume that there is no intermediate consumption and the investor chooses an optimal portfolio to maximize utility on her wealth at a terminal date $T$:

$$\max_{w} \mathbf{E}_t \left[ u(W_T, T) \right], \tag{17}$$

where $\mathbf{E}_t$ is the conditional expectation operator and $u$ is the utility function. We consider power utility function:

$$u(W) = \frac{W^{1-\rho}}{1-\rho}, \tag{18}$$

where $\rho > 1$ is a constant. The process of the spot interest rate under the equilibrium of such economy is given in the following proposition with the proof provided in the Appendix.

*Proposition* 1: Under the equilibrium of the above economy, the instantaneous interest rate, $r(t)$, is an affine-quadratic function of the state variables:

$$r(t) = \alpha + \beta^\top X_1 + \gamma^\top X_2 + X_2^\top \Phi X_2, \tag{19}$$

where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^{n_1 \times 1}$, $\gamma \in \mathbb{R}^{n_2 \times 1}$, and $\Phi \in \mathbb{R}^{n_2 \times n_2}$ a symmetric matrix.[5]

This result highlights the model as a hybrid of ATSMs and QTSMs where the spot interest rate is driven by both affine-type ($X_1$) and quadratic-type ($X_2$) factors. In addition, the affine factors $X_1$ contain jumps. Further, we have the following results under the risk-neutral probability measure. Again, the proof is given in the Appendix.

*Proposition* 2: Given the jump-diffusion process (1), there is an equivalent risk-neutral probability measure $(\Omega, \mathcal{F}, P^*)$ under which the state variables follow the jump-diffusion process:

$$dX \quad = \quad \left( \mu_X^* - \lambda^* \mu_{Q_X}^* \right) dt + \Sigma_X dZ^* + Q_X^* dN^*, \tag{20}$$

[5] Alternatively, we can follow the approach in studies such as Duffie, Pan, and Singleton (2000) for ATSMs, and Ahn, Dittmar, Gallant (2002) for QTSMs by directly assuming $r(t)$ as an affine-quadratic function of the state variables. To obtain the result of Proposition 2, we then need to specify a stochastic discount factor as an affine-quadratic jump-diffusion process. The current approach highlights that the affine-quadratic specification can be obtained in an equilibrium economy.

$$
\begin{aligned}
\text{Prob}(dN^* = 1) &= \lambda^* dt, &(21)\\
\lambda^* &= \varphi^* + \phi^{*\top} X_1 + \psi^{*\top} X_2 + X_2^\top \Psi^* X_2, &(22)\\
\mathbf{E}[e^{k^\top Q^*_{X_1}}] &= \delta^*_X(k). &(23)\\
\mu^*_{X_1} &= \kappa^*_{10} + \kappa^*_{11} X_1 + \kappa^*_{12} X_2 + \zeta^*(X_2), &(24)\\
\mu^*_{X_2} &= \kappa^*_{20} + \kappa^*_{22} X_2, &(25)\\
\zeta(X_2)^* &= \left( X_2^\top \zeta^*_1 X_2, \quad \cdots, \quad X_2^\top \zeta^*_{n_1} X_2 \right)^\top, &(26)
\end{aligned}
$$

where $Q^*_X = \begin{pmatrix} Q^*_{X_1} \\ 0 \end{pmatrix}$ and all the parameters under the risk-neutral probability measure $(\Omega, \mathcal{F}, P^*)$ are similarly defined as those under the real probability measure $(\Omega, \mathcal{F}, P)$.

The proposition shows that there is a transformation from the objective probability measure to the risk-neutral probability measure. In the equilibrium of the above representative agent economy, the dynamics of the model is invariant under the transformation between the two probability measures. The risk-neutral specification of the model allows us to price bonds in closed-form as shown in the next proposition.

*Proposition* 3: Let $P(X_1, X_2, \tau)$ be the price of a zero-coupon bond with maturity $\tau$, as a function of the state variables. And let $\theta$ denote the parameter vector of the model under the risk-neutral probability measure. Then the bond price is given by:

$$
P(X_1, X_2, \tau; \theta) = e^{A^*(\tau) + B^*(\tau)^\top X_1 + C^*(\tau)^\top X_2 + X_2^\top D^*(\tau) X_2}, \tag{27}
$$

where $A^* \in \mathbb{R}$, $B^* \in \mathbb{R}^{n_1 \times 1}$, $C^* \in \mathbb{R}^{n_2 \times 1}$, and $D^* \in \mathbb{R}^{n_2 \times n_2}$ is a symmetric matrix. $A^*$, $B^*$, $C^*$, and $D^*$ are solutions to the ODEs given in the Appendix.

Note that the bond yield is an affine-quadratic function of the state variables. It is also useful for estimation purpose to derive the characteristic function of the state variables. The conditional characteristic function (CCF) is defined as:

$$
f(s, X(t), \tau; \theta) = \mathbf{E}_t \left[ e^{is^\top X(t+\tau)} \right],
$$

with the initial condition:

$$f(s, X(t), 0; \theta) = e^{is^\top X(t)},$$

where $i = \sqrt{-1}$, $s = \binom{s_1}{s_2}$, $s_1 \in \mathbb{R}^{n_1}$ and $s_2 \in \mathbb{R}^{n_2}$. We prove in Appendix the next proposition, which will be used in identifying the latent state variables in our empirical analysis.

*Proposition* 4: The CCF of $X(t)$ under the risk-neutral probability measure is given as follows:

$$f(s, X(t), \tau; \theta) = e^{i\left[A^*(s,\tau)+B^*(s,\tau)^\top X_1+C^*(s,\tau)^\top X_2+X_2^\top D^*(s,\tau)X_2\right]}, \tag{28}$$

where $A^* \in \mathbb{R}$, $B^* \in \mathbb{R}^{n_1\times 1}$, $C^* \in \mathbb{R}^{n_2\times 1}$, and $D^* \in \mathbb{R}^{n_2\times n_2}$ is a symmetric matrix. $A^*$, $B^*$, $C^*$, and $D^*$ are solutions to the ODEs given in the Appendix.

# 3 Examples

In this section, we present three special cases of the general AQTSMs, which we will examine empirically later. The benchmark model is a stochastic volatility model in the class of affine term structural models. The second model extends the first one by incorporating a jump component with constant jump intensity. The third model is a non-trivial example of the AQTSMs. It has both stochastic volatility and random jumps with stochastic intensity. All three examples are specified under the risk-neutral probability measure. The bond price and CCF formulae are given in the Appendix.

## 3.1 Affine SV Model

Within the family of affine term structure models where the state variables $X_2 = \emptyset$ and $X_1$ contains no jumps, we consider a 2-factor stochastic volatility model. The state variables are the instantaneous interest rate $r$ and its volatility $v$, which have the following dynamics:

$$dr = (\mu_r^* + \kappa_{rr}^* r + \kappa_{rv}^* v)\, dt + \sqrt{v} dZ_r^*, \tag{29}$$

$$dv = (\mu_v^* + \kappa_{vr}^* r + \kappa_{vv}^* v)\, dt + \sigma_v \sqrt{v} dZ_v^*, \tag{30}$$

where $dZ_r^*$ and $dZ_v^*$ are Brownian motions with correlation coefficient $\rho$. In the most general form, we allow the interest rate and volatility to enter the drift terms of both variables. Note that this model can be regarded as a restricted version of the three-factor maximally flexible model $A_1(3)$ of Dai and Singleton (2000).

## 3.2 Affine SVJ Model with Constant Jump Intensity

Extending the above affine SV model, we incorporate jumps with constant jump intensity in the instantaneous interest rate process:

$$dr = (\mu_r^* + \kappa_{rr}^* r + \kappa_{rv}^* v - \lambda^* \mu_{Q_r}^*)dt + \sqrt{v} dZ_r^* + Q_r^* dN^*, \tag{31}$$

$$dv = (\mu_v^* + \kappa_{vr}^* r + \kappa_{vv}^* v)\, dt + \sigma_v \sqrt{v} dZ_v^*, \tag{32}$$

$$\text{Prob}(dN^* = 1) = \lambda^* dt, \tag{33}$$

$$\delta^*(k) = \mathbf{E}\left[e^{kQ_r^*}\right] = pe^{\mu_+^* k} + (1-p)e^{\mu_-^* k}, \tag{34}$$

$$\mu_{Q_r}^* = p\mu_+^* + (1-p)\mu_-^*, \tag{35}$$

where $dZ_r^*$ and $dZ_v^*$ are specified similarly as in the previous case, and the jump intensity, $\lambda^*$, is a constant. We assume that the jump size, $Q_r^*$, follows a Bernoulli distribution with probability $p$ to take a positive value $\mu_+^* > 0$ and probability $1-p$ to take a negative value $\mu_-^* < 0$. One major advantage of this specification of the jump distribution is that it allows asymmetric jumps in the instantaneous interest rate. Both of the conditional positive and negative jump sizes $\mu_+^*$ and $\mu_-^*$ are assumed to be constant, instead of following a distribution with infinite support. Given the finite jump size and rare occurrence of jumps, this specification also alleviates the concern of negative interest rates.[6]

[6] In all three models in this section, the instantaneous interest rate $r$ can take negative values. This is an unattractive feature but seems a necessary compromise in order to maintain tractability under the general jump-diffusion framework. Our empirical analysis, however, indicates that the probability of $r$ to be negative is very small based on parameter estimates of these models.

### 3.3 Affine-Quadratic SVJT Model with Stochastic Jump Intensity

In this example, the jump intensity, $\lambda^*$, is no longer constant and follows a stochastic process that is correlated with the interest rate and volatility processes:

$$\begin{aligned} dr &= \Big(\mu_r^* + \kappa_{rr}^* r + \kappa_{rv}^* \sqrt{v} + \kappa_{r\lambda}^* \sqrt{\lambda^*} + \zeta_{vv}^* v \\ &\quad + 2\zeta_{v\lambda}^* \sqrt{v\lambda^*} + \zeta_{\lambda\lambda}^* \lambda^* - \lambda^* \mu_{Q_r}^* \Big) dt + \sqrt{v} dZ_r^* + Q_r^* dN^*, \quad (36) \\ d\sqrt{v} &= \Big(\mu_v^* + \kappa_{vv}^* \sqrt{v} + \kappa_{v\lambda}^* \sqrt{\lambda^*}\Big) dt + \sigma_v dZ_v^*, \quad (37) \\ d\sqrt{\lambda^*} &= \Big(\mu_\lambda^* + \kappa_{\lambda v}^* \sqrt{v} + \kappa_{\lambda\lambda}^* \sqrt{\lambda^*}\Big) dt + \sigma_\lambda dZ_\lambda^*, \quad (38) \\ \text{Prob}(dN^* = 1) &= \lambda^* dt, \quad (39) \\ \delta^*(k) &= \mathbf{E}\left[e^{kQ_r^*}\right] = p e^{\mu_+^* k} + (1-p) e^{\mu_-^* k}, \quad (40) \\ \mu_{Q_r}^* &= p\mu_+^* + (1-p)\mu_-^*, \quad (41) \end{aligned}$$

where $dZ_r^*$, $dZ_v^*$, and $dZ_\lambda^*$ are Brownian motions with constant correlation matrix:

$$\Sigma = \begin{pmatrix} 1 & \rho_{rv} & \rho_{r\lambda} \\ \rho_{rv} & 1 & \rho_{v\lambda} \\ \rho_{r\lambda} & \rho_{v\lambda} & 1 \end{pmatrix}.$$

The jump component, $Q_r^*$, again follows a Bernoulli distribution as in the SVJ model, allowing asymmetry between positive and negative jumps. The stochastic volatility process is specified differently from those of the two earlier examples as the standard deviation $\sqrt{v}$ now follows a Gaussian process while previously the variance $v$ follows a square root process. Similarly, $\lambda^*$ is the square of a Gaussian process $\sqrt{\lambda^*}$.[7] The drift term in the interest rate process is linear in $r$ and quadratic in $\sqrt{v}$ and $\sqrt{\lambda^*}$. Note that if there are no jumps, then the SVJT model becomes a 2-factor quadratic model of Ahn, Dittmar, and Gallant (2002).

[7] Note that in the current specification, $\sqrt{v}$ and $\sqrt{\lambda^*}$ can be negative but $v$ and $\lambda^*$ are always non-negative because they are squares of their corresponding square-roots. Technically speaking, this is different from directly restricting $v$ and $\lambda^*$ to be non-negative processes. The main difference is the boundary properties at $v = 0$ and $\lambda^* = 0$. For the exact effect of such restrictions, please refer to the results in Ball and Roma (1994).

# 4 Empirical Analysis

## 4.1 GMM Estimation of the Affine-Quadratic Models

It is well known that the statistical inference of continuous-time jump-diffusion models with latent state variables presents a great challenge, at least for the following two reasons. First, the transition density of the continuous-time jump-diffusion process is, with the exception of a few special cases, generally unknown or unavailable in closed-form so that the conventional likelihood-based inference is inapplicable. Second, any inference procedure has to deal with the unobserved variables, which typically involves integrating out the latent variables in the likelihood function (see, e.g., Jacquier, Polson, and Rossi (1994)). Since the high-dimensional integral in the likelihood function mostly can not be reduced to one (or substantially lower) dimensional integrals, the numerical integration has to resort to path simulation of the discretized continuous-time process and is thus computationally intensive. In the option pricing literature, the risk-neutral process is often implied from the observed market option prices or quotes through the fit of option pricing formula. The option pricing model is then evaluated based on in-sample fitting and/or out-of-sample forecasting performance. In general, such a procedure is feasible since there is a large panel of observed option prices or quotes in the market with different strike prices and maturities. For multivariate interest rate term structure models such as the ones proposed in this paper, the implied procedure is infeasible due to the fact that there are too many unobserved state variables and model parameters, compared to a limited number of bond yield observations along the yield curve. In this paper, we propose a generalized method of moments (GMM) approach for the estimation of the AQTSMs.

The distinguishing features of the proposed GMM approach are as follows. First, we take advantage of the closed-form solution of the bond pricing formula derived under the affine-quadratic model framework. The corresponding bond yields allow us to construct robust moment restrictions in model estimation and at the same time to use information from the entire yield curve (i.e. yields of different maturities). Second and more importantly, we identify the latent state variables using the information in the futures market of Treasury bills. The intraday observations of T-bill yields in the futures market contain important information of the term structure, such as the level of volatility and unusual large changes or jumps in the short rate. The key identifying restriction is the dynamic relation derived directly under the model specification between the observed state variables and the unobserved latent state variables. These dynamic relations enable us to imply the latent state variables using various moments of T-bill yields in the futures market. The approach is similar to the implied state generalized method of moments (IS-GMM) proposed by Pan (2002) where the implied volatility from option prices is used for model estimation. The advantage of the approach is that with proxies or estimates of the latent variables, there is no path simulation involved in the estimation. It thus effectively overcomes the difficulty associated with the unobserved latent state variables.

Specifically, the dynamic relations used in the identification of the latent variables, namely stochastic volatility and random jumps, are based on various moments of the short rate. The intuition of the identifying procedure is as follows. Since the changes of yields in the immediate future reflect the current level of instantaneous variance and stochastic jump intensity, observations of yields naturally contain information of the state variables. Such information can thus be used for the identification and estimation of these state variables. Formally, under the affine-quadratic model framework with the CCF given in Proposition 4, the following relation between the cumulants of the spot interest rate and the state variables can be derived:

*Proposition* 5: Given the CCF of the state variables $f(s, r(t+\tau), X_1(t+\tau), X_2(t+\tau)|r(t) = r, X_1(t) = X_1, X_2(t) = X_2) = e^{i\left[A^*(s,\tau)+B_r^*(s,\tau)r+B_1^*(s,\tau)^\top X_1+C^*(s,\tau)^\top X_2+X_2^\top D^*(s,\tau)X_2\right]}$ with the initial conditions $A^*(s,0) = 0, B_r^*(s,0) = s_r, B_1^*(s,0) = s_1, C^*(s,0) = s_2,\ D^*(s,0) = 0$, where $s = (s_r, s_1, s_2)$, we have the following relation:

$$K^{(l)}(r(t+\tau)|\mathcal{F}_t) = \frac{\partial^l A^*}{\partial s_r^l}|_{s=0} + \frac{\partial^l B_r^*}{\partial s_r^l}|_{s=0} r + \frac{\partial^l B_1^{*\top}}{\partial s_r^l}|_{s=0} X_1 + \frac{\partial^l C^{*\top}}{\partial s_r^l}|_{s=0} X_2 + X_2^\top \frac{\partial^l D^*}{\partial s_r^l}|_{s=0} X_2, \quad (42)$$

where $l = 1, 2, \cdots, L$, and $K^{(l)}(r(t+\tau)|\mathcal{F}_t)$ denotes the $l$-th order cumulant of $r(t+\tau)$ conditional on $\mathcal{F}_t$ under the $P^*$-measure.

The above proposition is a direct application of Proposition 4 with interest rate $r(t)$ specified explicitly as an affine-type state variable in our general affine-quadratic jump-diffusion model. The rest of the variables in $(X_1, X_2)$ are unobserved or latent state variables of dimension $n_1 - 1$ and $n_2$. They can be identified from a set of moment conditions of $r(t + \tau)$, i.e. $\{K^{(l)}(r(t + \tau)|\mathcal{F}_t)\}_{l=1}^{L}$, given the parameter values. For instance, for the SV model, the risk-neutral variance of the short rate can be calculated from the intraday observations of the T-bill futures. Through the relation in (42) with $l = 2$, the spot variance ($V_t$) can be backed out from the risk-neutral variance as a function of the model parameters. The only conditions required on $A^*$, $B_r^*$, $B_1^*$, $C^*$, and $D^*$ are that there exist solutions to a system of affine-quadratic equations given by (42). The model parameters are in the end estimated via the iterative GMM procedure. That is, combining the above identification restrictions of the state variables with moment conditions, the GMM procedure results in consistent estimates of the model parameters. With the final estimates of model parameters, the time series estimates of latent variables are also obtained. For the asymptotic properties of parameter estimates under the standard GMM procedure please refer to Hansen (1982), and for the details of the implied state generalized method of moments (IS-GMM) please refer to Pan (2002).

## 4.2 The Data

The data used in our empirical analysis consist of daily US T-bill, T-note and T-bond yields with maturities 3-, 6-month, 1-, 2-, 3-, 5-, 7-, 10-, and 30-year from January 3, 1984 to February 15, 2002. This period belongs to the so-called "post-disinflation" period (see Duffee (2005)). Note that 2002 was the year the long term bond with 30-year maturity was discontinued. We use the 3-month T-bill yields as proxy of the short rate. Information in the futures market contains the intraday 3-month T-bill quotes with a maturity cycle of March, June, September, and December. Figure 1 plots the time series of the daily 3-month Treasury yields and the daily changes in panels A and B, respectively. The time series plot reflects the wide range of observations of the 3-month T-bill yields over the sampling period, and the first difference reflects some large changes of the 3-month T-bill yields from day to day. Descriptive statistics of the data are reported in Table 1. The average yields of different maturities suggest that the yield curve is overall upward sloping, and the standard deviations of the daily yield changes suggest that the yield curve is more volatile over the short end than the long end. Both skewness and kurtosis statistics indicate that interest rates are non-normally distributed. As in Aït-Sahalia (1996), we also report the first five orders of autocorrelations of the monthly interest rates as well as monthly interest rate changes in Table 1. The autocorrelation coefficients are in general very high, reflecting high persistence of interest rate over time.

Table 2 reports the principal components of daily interest rate changes. The principal component analysis reports similar results as in Litterman and Scheinkman (1991) and other existing studies. Namely, there are mainly three factors driving the dynamics of yield curve, the level, the slope and the curvature. In particular, the short rate explains more than 80% of the total variation of the yield curve dynamics, suggesting the importance of modeling the short rate process. The first three factors explain almost 96% variation of the yield curve.

## 4.3 Estimation Results of Alternative Models

The models specified in Section 3 are estimated following the GMM procedure proposed in Section 4.1. Dai and Singleton (2000), in their study of the ATSMs, point out that highly parameterized models may overfit the data, and they suggest using some restricted models to reduce the high degree of freedom. We follow their suggestion and restrict $\kappa^*_{vr} = 0$ for the SV and SVJ models. For the three-factor SVJT model, we impose more restrictions for parsimonious model specifications, namely $\kappa^*_{rv} = \kappa^*_{r\lambda} = \kappa^*_{v\lambda} = \kappa^*_{\lambda v} = \zeta^*_{v\lambda} = \zeta^*_{\lambda\lambda} = 0$. The moment conditions include the daily changes of yields with nine different maturities, including the 3-month T-bill yields.[8] The lagged yields are used as the instrumental variable.

[8]Note that both the moment conditions based on bond yields and the identifying restrictions for latent variables involve numerical solutions to ODEs. In this paper, we employ a procedure based on the implicit backward-difference methods (for details, please refer to, e.g., Aiken (1985)). The effectiveness and accuracy of the numerical procedure is verified for the CIR model using the closed-form bond pricing formula, and the affine SV model using the Monte Carlo simulations.

This results in a total of 18 moment conditions. These moment conditions capture the dynamics of the entire yield curve. Using the daily changes of yields, instead of daily yields, has the following advantages. First, the yields are known to be highly persistent over time, especially the yields with short maturities. Daily changes of yields, on the other hand, are much less persistent and thus provide more robust moment conditions. Second, the yields of different maturities along the yield curve are also highly correlated with each other. Third, the conditional information set of the above moment conditions involves the latent variables identified from the information in the futures market. The latent variable estimates thus inevitably contain measurement errors. Taking the difference of yields can remove the systematic component of such error and improve the robustness of estimation results. Finally, the use of daily changes of yields as moment conditions also directly measures the model fit to the yield curve with lagged yield as conditioning information.

Although the model is not estimated using a conventional likelihood method, the dynamic structure of the model is explicitly incorporated in the estimation procedure. The latent state variables are identified from the 3-month T-bill yields in the futures market using the dynamic relations derived directly from the model. Specifically, for the SV model, we use the conditional variance calculated from intraday changes of yields in the futures market to identity the stochastic volatility. For the SVJ and SVJT models, we use the conditional variance and kurtosis calculated from intraday changes of yields in the futures market to identity the stochastic volatility and jump intensity. The only difference between the SVJ model and SVJT model is that the jump intensity $\lambda^*$ is a constant in SVJ, but time varying in SVJT. It is known that, compared to stochastic volatility, jumps have a distinctively impact on higher order moments of interest rate changes (such as skewness and kurtosis). Our simulations of the sampling paths suggest that, compared to skewness, kurtosis is a more robust moment. The use of kurtosis thus helps to disentangle jumps from stochastic volatility in the T-bill futures yields. We restrict both stochastic volatility and jump intensity to be non-negative in our estimation.

The futures data we use are the 15-minute 3-month T-bill futures quotes. We choose the futures contract with the shortest maturity among the March, June, September, and December cycle but with more than one week remaining to maturity. For the long maturity and very short maturity contracts, there may be liquidity and market microstructure concerns. The choice of sampling frequency follows Andersen, Bollerslev, Diebold, and Labys (2003). This is to further avoid market microstructure effect associated with high frequency data. The main concern of the high frequency data is the serial correlations induced by the market microstructure noise. Various studies have examined the impact of the market microstructure noise on the estimation of return variance, see, e.g., Aït-Sahalia, Mykland and Zhang (2005). The issue of market microstructure noise is in general less severe for future prices than for spot prices, such as stock prices and stock indexes. It is known, for example, that stock returns tend to have negative serial correlations due to bid-ask bounce, whereas stock index returns tend to have positive serial correlations due to the infrequent trading of the stocks in the index. Sampled at 15-minute frequency, the changes of the 3-month T-bill future yields used in our analysis have, respectively, a first order autocorrelation of -5.79% and second order autocorrelation of 2.58%. Higher orders of autocorrelation are in general diminishing and negligible. To ensure the robustness of our analysis, we also follow Hansen and Lunde (2004) and Zhou (1996) to correct for the first order autocorrelation in the variance estimation, the results are not materially affected.

The estimation results of alternative models are reported in Table 3.[9] All parameter estimates are based on daily annualized bond yields. Since the estimates of stochastic volatility and jump intensity rely on the model parameter values, in each iteration of the GMM optimization procedure the parameter values, the volatility and jump intensity estimates are updated accordingly. The updating procedure is very similar to the implied state generalized method of moments (IS-GMM) proposed in Pan (2002). The results of all three models suggest that the daily interest rate process is highly persistent with very low mean reversion. On the other hand, the stochastic variance process implied from the term structure

[9]To gauge the extent of negative interest rates, we simulate the sampling pathes of the SVJT model based on the parameter estimates. Using a Euler scheme with 100 intervals per day and starting value $r_0 = 5.6\%$ (the sample mean), we find that out of 10,000 sampling paths over a 1-year period, there are 61 of them reaching negative values.

dynamics for the SV model has a high mean reverting parameter and a very high volatility of volatility. That is, the variance process is less persistent and very volatile. The Hansen $J$-test, with a $p$-value of $0.02\%$, suggests that the bond yields of the SV model fit poorly to the yield curve.

The SVJ model also has a poor fit to the yield curve, as suggested by the p-value of 0.19% for the Hansen J-test. However, it does suggest a significant jump component in the interest rate process. The jump frequency is statistically significant and the daily jump frequency or the arrival rate of jumps implies approximately, on average, 9 to 10 jumps per year. This high jump frequency is most likely driven by the large changes of short rates in the 80's. The results also suggest that whenever a jump occurs, there is a slightly higher chance (but statistically insignificant) of negative jump than positive jump ($1 - p = 51.01\%$). In addition, there is on average a larger size for negative jumps than for positive jumps, with $\mu_+^* = 0.1536\%$ or roughly 15 basis points and $\mu_-^* = -0.1694\%$ or roughly 17 basis points. With the addition of the jump term, the variance process implied from the term structure for the SVJ model becomes more persistent and less volatile. There is a significant but low level of negative correlation between stochastic volatility and interest rate changes.

The SVJT model represents a significant improvement in model fit to the yield curve. The Hansen J-test has a p-value of 2.64% and much higher than those of the SV and SVJ models. It should be noted that other than the additional jump intensity process in the SVJT model, both the variance process and jump intensity process are specified for their square roots. This also likely helps to improve model specification. It is clear that the volatility (square root of variance) process implied from the term structure dynamics for the SVJT model is well behaved with relatively low conditional volatility (volatility of volatility) over time. The jump intensity process, however, is very volatile with a high volatility parameter. There is also a strong mean reversion in the jump intensity process. Similar to the SVJ model, we identify an on average larger size for negative jumps ($\mu_-^* = -0.1689\%$) than for positive jumps ($\mu_+^* = 0.1518\%$). Both stochastic volatility process and the jump intensity process are negatively correlated with the changes of short rate. There is a non-negligible correlation between the jump intensity and interest rate changes at $-4.331\%$. A direct interpretation of these results is that when interest rate is moving downward, there is a higher chance of large changes (or jumps). In addition, a negative jump tends to be of a larger magnitude than a positive jump.

## 4.4 Risk Premia of Stochastic Volatility and Random Jumps

A by-product of the estimation approach proposed in Section 4.1 is that the time series estimates of unobserved latent state variables are also obtained as a result of model estimation. In this section, we investigate the risk premia associated with the stochastic volatility and jump volatility. We focus on the SVJT model as it has a reasonable fit to the term structure dynamics. For the SVJT model, although both stochastic volatility and jump intensity are estimated from the futures market, the risk premium of each individual factor can not be measured. We thus measure the total risk premium of both risk factors as follows:

$$RP_t = E_t^{P^*}[\int_t^{t+\tau} V_u du] + E_t^{P^*}[\int_t^{t+\tau} JV_u du] - \sum_{t \leq t_{i-1} < t_i \leq t+\tau} (r_{t_i} - r_{t_{i-1}})^2 \tag{43}$$

where $E_t^{P^*}[\int_t^{t+\tau} V_u du]$ and $E_t^{P^*}[\int_t^{t+\tau} JV_u du]$ denote the risk-neutral stochastic variance and jump variance, respectively, and the last term is the variance of the short rate in the real world measure which is calculated from daily observations of the 3-month T-bill yields. We focus on monthly realized variance since we only have daily observations of short term interest rate. With daily interest rates, the monthly variance calculation is reliable with reasonable accuracy.

The monthly volatility series are plotted in Figure 2, with the total volatility (stochastic volatility plus the jump volatility) under both risk-neutral and real world measures in Panel A, and the stochastic volatility and jump volatility in Panels B and C, respectively. The time series plot in Panel A shows that the real world realized volatility is mostly lower than its risk-neutral counterpart. This is an indication that there is an overall positive risk premium for the risk factors. For the SVJT model, the difference between the total risk-neutral variance and the realized variance, or the risk premium in (44), is equal to 6.8$\times 10^{-5}$ with a $t$-value of 5.32. More importantly, the risk premium is clearly time varying. To further understand the effect of stochastic variance and random jump on the risk premium, we perform the following regression:

$$RP_t = \alpha + \beta_1 \text{SteDev}(E_t^{P^*}[\int_t^{t+\tau} V_u du]) + \beta_2 \text{SteDev}(E_t^{P^*}[\int_t^{t+\tau} JV_u du]) + \epsilon_t, \tag{44}$$

The above regression tests whether the risk premium is correlated with the uncertainty of the risk factors, as measure by the standard deviations of risk neutral stochastic volatility and jump volatility. The regression results can be interpreted as the loadings or decomposition of total risk premium into the uncertainty associated with each individual factor. The regression has significantly positive estimates for both $\beta_1$ and $\beta_2$, with $\hat{\beta}_1 = 0.556\ (8.058), \hat{\beta}_2 = 0.499\ (8.133)$ and an adjusted $R^2 = 0.832$. The numbers in the brackets next to the parameter estimates are the Newey-West (1987) $t$-statistics with 2 lags. The results suggest that the total risk premium is positively correlated with the standard deviation of both factors, but relatively more sensitive to the uncertainty of stochastic volatility.

## 4.5 Stochastic Volatility, Jumps, and Macroeconomic Variables

Recent works of Ang and Piazzesi (2003), Piazzesi (2001, 2005), and Duffee (2005) consider the joint dynamics of bond yields and macroeconomic variables. They find that the macroeconomic variables have strong power in explaining the variations in bond yields. In particular, Ang and Piazzesi (2003) find that macroeconomic variables primarily explain movements at the short end and middle end of the yield curve while unobservable factors still account for most of the movements at the long end of the yield curve. Note that the interest rate models used in Ang and Piazzesi (2003) are of the affine structure with no jumps, while Duffee (2005) integrates out latent variables of the term structure model in his analysis. Our model extends the affine structure and incorporates jumps of stochastic intensity in the short rate. More importantly, the time series estimates of the latent variables, namely the stochastic volatility and random jumps, allow us to directly investigate the relation between term structure dynamics and the information in macroeconomic variables.

We follow Ang and Piazzesi (2003) and choose two groups of macroeconomic variables. The first group consists of inflation measures: the consumer price index (CPI) and producer price index (PPI)[10], and the second group contains variables that capture real activity: the index of Help Wanted Advertising in Newspapers (HELP), unemployment rate (UE), the growth rate of employment (EMPLOY), and the growth rate of industrial production (IP). All growth rates (including inflation) are measured as the difference in logs of the index in months $t$ and $t-1$.[11] The monthly time series data of macro variables are obtained from the Bureau of Labor Statistics of U.S. Department of Labor for the sample period of January, 1984 to February, 2002.

As in Ang and Piazzesi (2003), we extract the first principal component from the inflation group and the first principal component from the real activity group, and call them "inflation" or IN and "real activity" or RA, respectively. The variations accounted by the principal components in the inflation group and real activity group are, respectively, 0.842, 0.158, and 0.654, 0.327, 0.017, 0.001. This suggests that it is reasonable to use the first principal component in our analysis. The correlation between the two first principal components is low at 4.05%.

We focus on two empirical questions in our analysis. One is whether the state variables that drive the dynamics of term structure have predictive power of economic activities, and the other is whether the change of volatility or in particular jumps are due to shocks in macroeconomic variables. To answer the above questions, we estimate standard VAR models for the macroeconomic variables as well as stochastic volatility and jump volatility, except that the contemporaneous macroeconomic variables are also included in the stochastic volatility and jump volatility equations. This is to capture the response of stochastic volatility and jump volatility to the current information in the macroeconomic variables. We include lags up to three months or a quarter in our analysis. Note that the estimates of both stochastic volatility and jump intensity are obtained under the risk-neutral measure, and thus directly related to the term structure dynamics. The results reported in Table 4 suggest that while the stochastic volatility

[10] Ang and Piazzesi (2003) also use spot market commodity prices (PCOM).

[11] Ang and Piazzesi (2003) define growth rate as the difference in logs of index in months $t$ and $t-12$. Our focus is the relation between stochastic volatility/random jumps and monthly changes in macroeconomic variables.

is predictive of next month's inflation with a coefficient significant at the 10% critical level, it is not predictive of real activity. Interestingly, random jump is not predictive of either inflation or real activity. The results also indicate a significant coefficient for the contemporaneous inflation in the SV equation. That is, an increase in inflation tends to drive up the volatility in the short rate. A jump in the short rate also tends to drive up the level of volatility. The results for the jump volatility equation suggest that other than the autoregressive structure, no variables are predictive of random jumps.[12] In other words, jumps are not triggered by the informational shocks in macroeconomic variables.

The plots of impulse response functions in Figure 3 provide further information on the relation between the latent variables and macroeconomic variables. The figure plots impulse response of the inflation (IN) and real activity (RA) to shocks in stochastic volatility (SV) and jump volatility (JV), as well as the impulse responses of stochastic volatility (SV) and jump volatility (JV) to shocks in inflation (IN) and real activity (RA). The lags are up to 12 months. It is clear that volatility shock has a positive initial effect on inflation and the impact dies off gradually. In comparison, jumps have no substantial effect on inflation. Neither the shock in stochastic volatility nor that in jump volatility has a significant impact on the real activity. Turning to the response of stochastic volatility and jump volatility to shocks in macroeconomic variables, a shock in inflation has a contemporaneous positive effect on the stochastic volatility but no clear effect on jumps. Informational shocks in the real activity do not have clear effect on either stochastic volatility or jumps. That is, overall the term structure dynamics is more related to inflation at the monthly frequency.

The finding that random jumps are neither predictive of nor triggered by the information in macroeconomic variables may not be surprising. This is because we focus on monthly frequency in our analysis, at which the macroeconomic variables are observed. On the one hand, the aggregation of daily information over a month period can reduce the impact of random noises, on the other hand it also

[12] Note also that the jump volatility is the least predictable process with an adjusted $R^2 = 0.071$. Since there are more time periods with no jumps than those with jumps in our sample, the autoregressive structure of the jump volatility process should be interpreted as more of the predictability of no jumps rather than jumps.

smoothes out day-to-day informational shocks in the market place. Johannes (2004) identifies jump intensity and size at daily frequency, and performs anecdotal analysis of the macro or geopolitical events associated with jumps. He finds that interest rate jumps often reflect the market reaction to surprising economic news or geopolitical development. Our results show that such dynamic link between jumps and economic news disappears when information is measured at monthly frequency.

### 4.6 What Causes the Jumps in Interest Rate?

The results in the above section and those in Johannes (2004) suggest that to understand what drives jumps in short rate or term structure dynamics, we need to examine a finer information set at a higher frequency than monthly. In this section, we relate jumps at a daily frequency to the release of macroeconomic news and other information. In particular, we are interested in how often jumps in short rate or term structure dynamics are associated with scheduled macroeconomic news release. Again, we focus on the two groups of macro variables related to inflation and real activities. We collect the historical announcement dates for CPI, PPI, industrial production, and employment. The dates of PPI and industrial production cover the entire time period of our sample, while those of CPI and employment only go back to the beginning of 1994. We also collect the historical meeting dates of Federal Open Market Committee (FOMC). As reported in Table 5, there are 218 announcement dates for PPI and industrial production during the period of January 1984 to February 2002, and 97 announcement dates for CPI and employment during the period of January 1994 to February 2002. There are 145 scheduled FOMC meetings during the period of January 1984 to February 2002. In addition to the scheduled meetings, there are also 3 unscheduled FOMC meetings during this period.

Similar to Johannes (2004), we base on the jump intensity estimates to identify the days with jumps. In particular, we focus on the days with jump intensity estimate of the SVJT model higher than its sample mean. This results in 105 days with jumps. Note that the jump intensity estimates are obtained under the risk-neutral measure. Since we focus on days with jump intensity above the sample mean, it is likely that these are also the days with jump intensity estimates above its sample mean under the objective measure. A couple of patterns are noted. First, there is a higher frequency of jumps toward the beginning of the sample period during the 80's than in the second half of the sample period. For example, after 1994 there are only 26 jumps. Second, there tends to be clustering of jumps in short rate. In particular, jumps can happen in consecutive days.

Panel A of Table 5 reports the number of jumps on the scheduled announcement or FOMC meeting days. Out of the 729 unique announcement or meeting days, there are 32 unique days with jumps. Note that we are not attempting to isolate whether there is any confounding news or information that causes the jump. Instead, we simply document the patterns of jumps associated with the scheduled news release or meeting dates. Still, a couple of interesting observations are noted. First, taking away the 9 common jumps between PPI and industrial production, jump seems to be more likely associated with PPI information than with other information. This is consistent with the results in previous section that inflation tends to have more impact on the dynamics of interest rate. Second, among all jumps associated with news announcement dates, there are dominantly more negative ones than positive ones, where the sign of jumps is based on the skewness of intraday T-bill future yield changes. The only exception is the FOMC meeting dates where the shocks appear to be symmetric. To take into account of the pre-announcement anticipation or post-announcement reaction, we also count the number of jumps one day before or after the scheduled announcement or FOMC meeting dates. Out of the 20 unique jumps on the day before announcement or meeting dates, 5 overlap with those on the announcement dates of other news. The shocks appear to be less asymmetric for the pre-announcement anticipation. There are also total 15 unique jumps on the day after announcement or meeting dates. The asymmetry of shocks surfaces again for the post-announcement effect.

Overall, there are total 56 unique jumps associated with scheduled news announcements or FOMC meetings, including one day lead or lag. This is roughly half of the jumps identified in our sample period. The other half could be associated with the release of other macroeconomic news that is not included in our study. There is also clear evidence that a significant portion of the jumps may well not be associated with scheduled news release at all. Instead, they are driven by some unanticipated economic news or geopolitical development, as documented in Johannes (2004). Panel B of Table 5 provides a sample list of such events during the period of January 1998 to February 2002. These observations suggest that the uncertainty of timing is also an important component of jump risk. The findings on jumps associated with daily informational shocks has important implications for measuring and managing jump risk in the bond market.

# 5 Conclusion

We propose a unifying class of affine-quadratic term structure models (AQTSMs) in a jump-diffusion framework, which extends most existing term structure models. The model incorporate random jumps of stochastic intensity in the short rate process. We propose a GMM approach for the estimation of the affine-quadratic term structure models that uses information from the entire yield curve and the treasury futures market. We identify a jump intensity process that is negatively correlated with interest rate changes. Negative jumps have, on average, larger size than positive jumps.

The time series estimates of latent state variables, namely stochastic volatility and jump intensity, allow us to investigate their relation with other economic variables. Our empirical results suggest that while stochastic volatility is related to inflation at monthly frequency, jumps are neither predictive of nor triggered by informational shocks in macroeconomic variables. Once focusing on daily frequency, however, we document interesting patterns for jumps associated with various events in the financial market. We find that jumps associated with informational shocks in macroeconomic news are dominantly more negative than positive. In addition, while a majority of jumps are related to scheduled news announcement dates, there is a large number of jumps driven by unanticipated economic news or geopolitical development. The findings have important implications for understanding and managing jump risk in the bond market.

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# Appendix

This appendix provides technical details of our results. In particular, we derive the equilibrium interest rate, and the ODEs that determine the bond price and conditional characteristic function formulae for the general model and the special cases.

## A.1 Change of Probability Measure and Interest Rate Process in Equilibrium

The representative investor's wealth, $W$, follows the process:

$$\frac{dW}{W} = (\mu_W - \lambda\mu_{Q_W})\,dt + \sigma_W dZ + Q_W dN, \tag{A.1}$$

where

$$\mu_W = w\mu_S + (1-w)r, \tag{A.2}$$
$$\sigma_W = w\sigma_S, \tag{A.3}$$
$$Q_W = wQ_S. \tag{A.4}$$

Define the value function of the investor:

$$J(W_t, X_t, t) \equiv \max_w \mathbf{E}_t\left[u(W_T, T)\right], \tag{A.5}$$

which depends only on $W$, $X$, and $t$ since they are the state variables. Following Merton(1973) and using subscripts to denote the partial derivative of $J$, a solution to the optimization problem (17) satisfies the Bellman equation:

$$0 = \max_w\left[J_t + \mathcal{L}(J)\right], \tag{A.6}$$

with

$$\begin{aligned}\mathcal{L}(J) &= (\mu_W - \lambda\mu_{Q_W})\,WJ_W + (\mu_X - \lambda\mu_{Q_X})\,J_X + \frac{1}{2}\sigma_W^\top\sigma_W W^2 J_{WW} \\ &\quad + \frac{1}{2}\text{tr}\left(\Sigma_X\Sigma_X^\top J_{XX^\top}\right) + \sigma_W^\top\Sigma_X^\top WJ_{WX} + \lambda\mathbf{E}\left[\Delta J\right],\end{aligned} \tag{A.7}$$

where $\Delta J \equiv J(W(1+Q_W), X+Q_X, t) - J(W,X,t)$ denotes jump in the value function. The first order condition to (A.6) and the fact that in equilibrium the representative investor needs to hold all her wealth in the stock ($w^* = 1$) imply that:

$$r = \mu_S + \sigma_S^\top\sigma_S W\frac{J_{WW}}{J_W} + \sigma_S^\top\Sigma_X^\top\frac{J_{WX}}{J_W} + \lambda\mathbf{E}\left[\frac{\Delta J_W}{J_W}Q_S\right], \tag{A.8}$$

where $\Delta J_W = J_W(W(1+Q_S), X+Q_X, t) - J_W(W,X,t)$. We guess a solution to (A.6):

$$J(W,X,t) = g(X,t)\frac{W^{1-\rho}}{1-\rho}, \tag{A.9}$$

where $g$ is a function of $X$ and $t$ with the boundary condition:

$$g(X,T) = 1. \tag{A.10}$$

From (A.6) and the equilibrium condition $w^* = 1$, we have:

$$
\begin{aligned}
0 &= g_t + \left[(1-\rho)(\mu_S - \lambda\mu_{Q_S}) - \frac{1}{2}\rho(1-\rho)\sigma_S^\top\sigma_S\right] g + \left[\mu_X - \lambda\mu_{Q_X} + (1-\rho)\Sigma_X\sigma_S\right]^\top g_X \\
&\quad + \frac{1}{2}\text{tr}\left(\Sigma_X\Sigma_X^\top g_{XX^\top}\right) + \lambda\left(\mathbf{E}\left[(1+Q_S)^{1-\rho}\right]\mathbf{E}\left[g(X+Q_X,t)\right] - g\right), \qquad \text{(A.11)}
\end{aligned}
$$

where we have used the fact that $Q_S$ is independent of $Q_X$. We guess a solution of $g$:

$$
g(X,\tau) = e^{a(\tau)+b(\tau)^\top X_1 + c(\tau)^\top X_2 + X_2^\top d(\tau) X_2}, \qquad \text{(A.12)}
$$

where $\tau \equiv T - t$, $a \in \mathbb{R}$, $b \in \mathbb{R}^{n_1\times 1}$, $c \in \mathbb{R}^{n_2\times 1}$, and $d \in \mathbb{R}^{n_2\times n_2}$ is a symmetric matrix with initial values $a(0) = 0$, $b(0) = 0_{n_1\times 1}$, $c(0) = 0_{n_2\times 1}$, $d(0) = 0_{n_2\times n_2}$ respectively.

Note that (A.11) is a Riccati type partial differential equation with coefficients linear in $X_1$ and quadratic in $X_2$. Defining $\chi \equiv \mathbf{E}\left[(1+Q_S)^{1-\rho}\right]$, substituting (A.12) into (A.11), collecting terms of similar powers in $X_1$ and $X_2$, and setting the coefficients to zero, we derive the following ODEs that are satisfied by $a$, $b$, $c$, and $d$:[13]

$$
\begin{aligned}
\frac{da}{d\tau} &= (1-\rho)\left(\alpha_1 - \varphi\mu_{Q_S} - \frac{1}{2}\rho\alpha_2\right) + \left(\kappa_{10}^\top - \varphi\mu_{Q_{X_1}}^\top + (1-\rho)\alpha_3^\top\right) b \\
&\quad + \left(\kappa_{20}^\top + (1-\rho)\alpha_4^\top\right) c + \frac{1}{2} b^\top\omega_{10} b + b^\top\omega_{20} c + \frac{1}{2} c^\top\Omega_{22} c \\
&\quad + \text{tr}(\Omega_{22} d) + \varphi\left(\chi\delta_X(b) - 1\right), \qquad \text{(A.13)} \\
\frac{db}{d\tau} &= (1-\rho)\left(\beta_1 - \phi\mu_{Q_S} - \frac{1}{2}\rho\beta_2\right) + \left(\kappa_{11}^\top - \phi\mu_{Q_{X_1}}^\top + (1-\rho)\beta_3\right) b \\
&\quad + \frac{1}{2}\sum_{i,j} b_i b_j (\omega_{11})_{ij} + \phi\left(\chi\delta_X(b) - 1\right), \qquad \text{(A.14)} \\
\frac{dc}{d\tau} &= (1-\rho)\left(\gamma_1 - \psi\mu_{Q_S} - \frac{1}{2}\rho\gamma_2\right) + \left(\kappa_{12}^\top - \psi\mu_{Q_{X_1}}^\top + (1-\rho)\gamma_3\right) b \\
&\quad + \left(\kappa_{22}^\top + (1-\rho)\gamma_4\right) c + 2d\left(\kappa_{20} + (1-\rho)\alpha_4\right) + 2d\omega_{20}^\top b + 2d\Omega_{22} c \\
&\quad + \frac{1}{2}\sum_{i,j} b_i b_j (\omega_{12})_{ij} + \sum_{i,j} b_i c_j (\omega_{22})_{ij} + \psi\left(\chi\delta_X(b) - 1\right), \qquad \text{(A.15)} \\
\frac{dd}{d\tau} &= (1-\rho)\left(\Phi_1 - \Psi\mu_{Q_S} - \frac{1}{2}\rho\Psi_2\right) + \sum_i b_i\left(\zeta_i + (1-\rho)\zeta_i\right) - \Psi\mu_{Q_{X_1}}^\top b \\
&\quad + \left(d\kappa_{22} + \kappa_{22}^\top d\right) + (1-\rho)\left(d\gamma_4^\top + \gamma_4 d\right) + 2d\Omega_{22} d \\
&\quad + \frac{1}{2}\sum_{i,j} b_i b_j \nu_{ij} + \sum_{i,j} b_i\left(d_j(\omega_{22})_{ij}^\top + (\omega_{22})_{ij} d_j^\top\right) + \Psi\left(\chi\delta_X(b) - 1\right), \qquad \text{(A.16)}
\end{aligned}
$$

where we have defined $b \equiv (b_1, \ldots, b_{n_1})^\top$, $c \equiv (c_1, \ldots, c_{n_2})^\top$, and $d \equiv (d_1, \ldots, d_{n_2})$. Now Equation (19) is obtained by combining (A.8), (A.9), (A.12), (13), (14), and (15). And this proves Proposition 1.

[13] A similar and more detailed proof is presented for bond pricing.

We next show that under the risk-neutral measure the dynamics of the state variables are given by (20)-(26). An argument similar to Cox, Ingersoll, and Ross (1985a) shows that the price of any security, $F(W,X,t)$, satisfies the PDE:

$$\begin{aligned} rF &= F_t + \left(r - \lambda \mathbf{E}\left[\frac{J_W(W(1+Q_S), X+Q_X, t)}{J_W} Q_S\right]\right) W F_W \\ &+ \left(\mu_X - \lambda \mu_{Q_X} + \Sigma_X \sigma_S W \frac{J_{WW}}{J_W} + \Sigma_X \Sigma_X^\top \frac{J_{WX}}{J_W}\right)^\top F_X \\ &+ \frac{1}{2}\sigma_S^\top \sigma_S W^2 F_{WW} + \sigma_S^\top \Sigma_X^\top F_{WX} + \frac{1}{2}\text{tr}\left(\Sigma_X \Sigma_X^\top F_{XX^\top}\right) \\ &+ \lambda \mathbf{E}\left[\frac{J_W(W(1+Q_S), X+Q_X, t)}{J_W} \Delta F\right], \end{aligned} \tag{A.17}$$

where $\Delta F = (F(W(1+Q_S), X+Q_X, t) - F(W,X,t))$. Using (A.9) and (A.12), the above equation is simplified to:

$$\begin{aligned} rF &= F_t + \left(r - \lambda \mathbf{E}\left[(1+Q_S)^{1-\rho} e^{b^\top Q_{X_1}} Q_S\right]\right) W F_W \\ &+ \left[\mu_X - \lambda \mu_{Q_X} - \rho \Sigma_X \sigma_S + \Sigma_X \Sigma_X^\top \left({}_{c+2dX_2}^{\;\;\;b}\right)\right]^\top F_X \\ &+ \frac{1}{2}\sigma_S^\top \sigma_S W^2 F_{WW} + \sigma_S^\top \Sigma_X^\top F_{WX} + \frac{1}{2}\text{tr}\left(\Sigma_X \Sigma_X^\top F_{XX^\top}\right) \\ &+ \lambda \mathbf{E}\left[(1+Q_S)^{1-\rho} e^{b^\top Q_{X_1}} \Delta F\right]. \end{aligned} \tag{A.18}$$

Observe that this equation can also be obtained if the state variables follow (20)-(26) and we define:

$$\mu_X^* \equiv \mu_X - \rho \Sigma_X \sigma_S + \Sigma_X \Sigma_X^\top \left({}_{c+2dX_2}^{\;\;\;b}\right), \tag{A.19}$$

$$\lambda^* \equiv \lambda \chi \delta_X(b), \tag{A.20}$$

$$\delta_X^*(k) \equiv \frac{\delta_X(b+k)}{\delta_X(b)}, \tag{A.21}$$

$$h^*(q) \equiv \frac{e^{b^\top q}}{\delta_X(b)} h(q), \tag{A.22}$$

where $h$ and $h^*$ are the probability density functions of $Q_{X_1}$ and $Q_{X_1}^*$ respectively. Hence we have proved Proposition 2.

## A.2 Bond Price and Conditional Characteristic Function

In this subsection, we first derive the bond price formula and then derive formula for the characteristic function under the risk-neutral probability measure. In the absence of arbitrage the bond price $P(X_1, X_2, \tau)$ satisfies the partial differential equation (PDE):

$$\begin{aligned} 0 &= P_t - rP + \left[\kappa_{10}^* + \kappa_{11}^* X_1 + \kappa_{12}^* X_2 + \zeta^*(X_2) - \lambda^* \mu_{Q_{X_1}}^*\right]^\top P_{X_1} \\ &+ \left(\kappa_{20}^* + \kappa_{22}^* X_2\right)^\top P_{X_2} + \frac{1}{2}\text{tr}\left[\Omega_{11} P_{X_1 X_1^\top} + 2\Omega_{12}^\top P_{X_1 X_2^\top} + \Omega_{22} P_{X_2 X_2^\top}\right] \\ &+ \lambda^* \mathbf{E}\left[P(X_1 + Q_{X_1}^*, X_2, \tau) - P(X_1, X_2, \tau)\right], \end{aligned} \tag{A.23}$$

with the initial condition $P(X_1, X_2, 0) = 1$. We guess a solution of $P$ that has the form $P = e^{A(\tau)+B(\tau)^\top X_1+C(\tau)^\top X_2+X_2^\top D(\tau)X_2}$ with initial values $A(0) = 0$, $B(0) = 0_{n_1\times 1}$, $C(0) = 0_{n_2\times 1}$, $D(0) = 0_{n_2\times n_2}$. The partial derivatives of $P$ are:

$$
\begin{aligned}
P_t &= -P\left(\frac{dA}{d\tau} + \frac{dB}{d\tau}^\top X_1 + \frac{dC}{d\tau}^\top X_2 + X_2^\top \frac{dD}{d\tau} X_2\right), && \text{(A.24)}\\
P_{X_1} &= PB, && \text{(A.25)}\\
P_{X_2} &= P\left(C + 2DX_2\right), && \text{(A.26)}\\
P_{X_1X_1^\top} &= PBB^\top, && \text{(A.27)}\\
P_{X_2X_2^\top} &= P\left[\left(C + 2DX_2\right)\left(C + 2DX_2\right)^\top + 2D\right], && \text{(A.28)}\\
P_{X_1X_2^\top} &= PB\left(C + 2DX_2\right)^\top. && \text{(A.29)}
\end{aligned}
$$

Substitute these into the PDE to derive:

$$
\begin{aligned}
0 \;=\; & -\frac{dA}{d\tau} - \frac{dB}{d\tau}^\top X_1 - \frac{dC}{d\tau}^\top X_2 - X_2^\top \frac{dD}{d\tau} X_2 - \left(\alpha + \beta^\top X_1 + \gamma^\top X_2 + X_2^\top \Phi X_2\right)\\
& + \left[\kappa_{10}^* + \kappa_{11}^* X_1 + \kappa_{12}^* X_2 + \zeta^*(X_2) - \left(\varphi^* + \phi^{*\top} X_1 + \psi^{*\top} X_2 + X_2^\top \Psi^* X_2\right)\mu_{Q_{X_1}}^*\right]^\top B\\
& + \left(\kappa_{20}^* + \kappa_{22}^* X_2\right)^\top \left(C + 2DX_2\right)\\
& + \frac{1}{2}\text{tr}\left\{\Omega_{11} BB^\top + 2\Omega_{12}^\top B\left(C + 2DX_2\right)^\top + \Omega_{22}\left[\left(C + 2DX_2\right)\left(C + 2DX_2\right)^\top + 2D\right]\right\}\\
& + \left(\varphi^* + \phi^{*\top} X_1 + \psi^{*\top} X_2 + X_2^\top \Psi^* X_2\right)\left(\delta_X^*(B) - 1\right). \qquad \text{(A.30)}
\end{aligned}
$$

We collect the terms with the same powers of the state variables, and define $B \equiv (B_1, \ldots, B_{n_1})^\top$, $C \equiv (C_1, \ldots, C_{n_2})^\top$, and $D \equiv (D_1, \ldots, D_{n_2})$. Since the PDE holds for any values of $X_1$ and $X_2$, these coefficients must be zero. And that leads to the following ODEs:

$$
\begin{aligned}
\frac{dA}{d\tau} &= -\alpha + \left(\kappa_{10}^{*\top} - \varphi^* \mu_{Q_{X_1}}^{*\top}\right) B + \kappa_{20}^{*\top} C + \frac{1}{2} B^\top \omega_{10} B\\
&\quad + B^\top \omega_{20} C + \frac{1}{2} C^\top \Omega_{22} C + \text{tr}(\Omega_{22} D) + \varphi^*\left(\delta_X^*(B) - 1\right), && \text{(A.31)}\\
\frac{dB}{d\tau} &= -\beta + \left(\kappa_{11}^{*\top} - \phi^* \mu_{Q_{X_1}}^{*\top}\right) B + \frac{1}{2}\sum_{i,j} B_i B_j (\omega_{11})_{ij} + \phi^*\left(\delta_X^*(B) - 1\right), && \text{(A.32)}\\
\frac{dC}{d\tau} &= -\gamma + \left(\kappa_{12}^{*\top} - \psi^* \mu_{Q_{X_1}}^{*\top}\right) B + \kappa_{22}^{*\top} C + 2D\kappa_{20}^* + 2D\omega_{20}^\top B + 2D\Omega_{22} C\\
&\quad + \frac{1}{2}\sum_{i,j} B_i B_j (\omega_{12})_{ij} + \sum_{i,j} B_i C_j (\omega_{22})_{ij} + \psi^*\left(\delta_X^*(B) - 1\right), && \text{(A.33)}\\
\frac{dD}{d\tau} &= -\Phi + \sum_i B_i \zeta_i^* - \Psi^* \mu_{Q_{X_1}}^{*\top} B + \left(D\kappa_{22}^* + \kappa_{22}^{*\top} D\right) + 2D\Omega_{22} D\\
&\quad + \frac{1}{2}\sum_{i,j} B_i B_j \nu_{ij} + \sum_{i,j} B_i\left(D_j (\omega_{22})_{ij}^\top + (\omega_{22})_{ij} D_j^\top\right) + \Psi^*\left(\delta_X^*(B) - 1\right). && \text{(A.34)}
\end{aligned}
$$

This concludes the proof of Proposition 3.

To derive the characteristic function $f(s,X,\tau)$, observe that $f$ is a martingale and hence it satisfies a PDE similar to (A.23):

$$
\begin{aligned}
0 \;=\;& f_t + \left(\kappa_{10}^* + \kappa_{11}^* X_1 + \kappa_{12}^* X_2 + \zeta^*(X_2) - \lambda^* \mu_{Q_{X_1}}^*\right)^\top f_{X_1} + \left(\kappa_{20}^* + \kappa_{22}^* X_2\right)^\top f_{X_2} \\
&+ \frac{1}{2}\mathrm{tr}\left[\Omega_{11} f_{X_1 X_1^\top} + 2\Omega_{12}^\top f_{X_1 X_2^\top} + \Omega_{22} f_{X_2 X_2^\top}\right] \\
&+ \lambda^* \mathbf{E}\left[f(s, X_1 + Q_{X_1}^*, X_2, \tau) - f(s, X_1, X_2, \tau)\right], \qquad \text{(A.35)}
\end{aligned}
$$

with the initial condition $f(s,X,0) = e^{is^\top X}$. We again guess a solution of the functional form $f = e^{i\left[A^*(s,\tau) + B^*(s,\tau)^\top X_1 + C^*(s,\tau)^\top X_2 + X_2^\top D^*(s,\tau) X_2\right]}$ with the initial conditions $A^*(s,0) = 0$, $B^*(s,0) = s_1$, $C^*(s,0) = s_2$, $D^*(s,0) = 0$. Then $A^*$, $B^*$, $C^*$, and $D^*$ solve the following ODEs:

$$
\begin{aligned}
\frac{dA^*}{d\tau} \;=\;& \left(\kappa_{10}^{*\top} - \varphi^* \mu_{Q_{X_1}}^{*\top}\right) B^* + \kappa_{20}^{*\top} C^* + \frac{1}{2} B^{*\top} \omega_{10} B^* \\
&+ B^{*\top} \omega_{20} C^* + \frac{1}{2} C^{*\top} \Omega_{22} C^* + \mathrm{tr}(\Omega_{22} D^*) + \varphi^* \left(\delta_X^*(B^*) - 1\right), \qquad \text{(A.36)} \\
\frac{dB^*}{d\tau} \;=\;& \left(\kappa_{11}^{*\top} - \phi^* \mu_{Q_{X_1}}^{*\top}\right) B^* + \frac{1}{2}\sum_{i,j} B_i^* B_j^* (\omega_{11})_{ij} + \phi^* \left(\delta_X^*(B^*) - 1\right), \qquad \text{(A.37)} \\
\frac{dC^*}{d\tau} \;=\;& \left(\kappa_{12}^{*\top} - \psi^* \mu_{Q_{X_1}}^{*\top}\right) B^* + \kappa_{22}^{*\top} C^* + 2D^* \kappa_{20}^* + 2D^* \omega_{20}^\top B^* + 2D^* \Omega_{22} C^* \\
&+ \frac{1}{2}\sum_{i,j} B_i^* B_j^* (\omega_{12})_{ij} + \sum_{i,j} B_i^* C_j^* (\omega_{22})_{ij} + \psi^* \left(\delta_X^*(B^*) - 1\right), \qquad \text{(A.38)} \\
\frac{dD^*}{d\tau} \;=\;& \sum_i B_i^* \zeta_i^* - \Psi^* \mu_{Q_{X_1}}^{*\top} B^* + \left(D^* \kappa_{22}^* + \kappa_{22}^{*\top} D^*\right) + 2D^* \Omega_{22} D^* + \frac{1}{2}\sum_{i,j} B_i^* B_j^* \nu_{ij} \\
&+ \sum_{i,j} B_i^* \left(D_j^* (\omega_{22})_{ij}^\top + (\omega_{22})_{ij} D_j^{*\top}\right) + \Psi^* \left(\delta_X^*(B^*) - 1\right), \qquad \text{(A.39)}
\end{aligned}
$$

where $B^* \equiv \left(B_1^*, \ldots, B_{n_1}^*\right)^\top$, $C^* \equiv \left(C_1^*, \ldots, C_{n_2}^*\right)^\top$, and $D^* \equiv \left(D_1^*, \ldots, D_{n_2}^*\right)$. This proves Proposition 4.

## A.3 SV Model

The bond prices in affine models have the form $P = e^{A + B^\top X}$, where $X = \binom{r}{v}$, and $A$ and $B = \binom{B_r}{B_v}$ satisfy the ODEs:

$$
\begin{aligned}
\frac{dA}{d\tau} &= \Pi^{*\top} B, \qquad \text{(A.40)} \\
\frac{dB}{d\tau} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \Lambda^{*\top} B + \begin{pmatrix} 0 \\ \frac{1}{2} B^\top \Gamma^* B \end{pmatrix}, \qquad \text{(A.41)}
\end{aligned}
$$

with initial values $A(0) = 0$, $B(0) = \binom{0}{0}$, and $\Pi^* \equiv \begin{pmatrix} \mu_r^* \\ \mu_v^* \end{pmatrix}$, $\Lambda^* \equiv \begin{pmatrix} \kappa_{rr}^* & \kappa_{rv}^* \\ \kappa_{vr}^* & \kappa_{vv}^* \end{pmatrix}$, and $\Gamma^* \equiv \begin{pmatrix} 1 & \rho\sigma_v \\ \rho\sigma_v & \sigma_v^2 \end{pmatrix}$.

Using the notation $s = \binom{s_r}{s_v}$, the CCF has the form $f(s) = e^{i\left(A^* + B^{*\top} X\right)}$ where $A^*$ and $B^* = \begin{pmatrix} B_r^* \\ B_v^* \end{pmatrix}$ satisfy the ODEs:

$$\frac{dA^*}{d\tau} = \Pi^{*\top} B^*, \tag{A.42}$$

$$\frac{dB^*}{d\tau} = \Lambda^{*\top} B^* + \begin{pmatrix} 0 \\ \frac{1}{2} B^{*\top} \Gamma^* B^* \end{pmatrix}, \tag{A.43}$$

with initial values $A^*(0) = 0$ and $B^*(0) = \binom{s_r}{s_v}$.

## A.4 SVJ Model

Again we can write the bond price as $P = e^{A + B^\top X}$, where $X = \binom{r}{v}$, and $A$ and $B = \binom{B_r}{B_v}$ satisfy the ODEs:

$$\frac{dA}{d\tau} = \Pi^{*\top} B + \lambda^* \left( p e^{\mu_+^* B_r} + (1-p) e^{\mu_-^* B_r} - 1 \right), \tag{A.44}$$

$$\frac{dB}{d\tau} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \Lambda^{*\top} B + \begin{pmatrix} 0 \\ \frac{1}{2} B^\top \Gamma^* B \end{pmatrix}, \tag{A.45}$$

with initial values $A(0) = 0$, $B(0) = \binom{0}{0}$, and $\Pi^*$, $\Lambda^*$, and $\Gamma^*$ defined as in the previous example.

Using the notation $s = \binom{s_r}{s_v}$, the CCF has the form $f = e^{i\left(A^* + B^{*\top} X\right)}$ where $A^*$ and $B^* = \begin{pmatrix} B_r^* \\ B_v^* \end{pmatrix}$ satisfy the ODEs:

$$\frac{dA^*}{d\tau} = \Pi^{*\top} B^* + \lambda^* \left( p e^{\mu_+^* B_r} + (1-p) e^{\mu_-^* B_r} - 1 \right), \tag{A.46}$$

$$\frac{dB^*}{d\tau} = \Lambda^{*\top} B^* + \begin{pmatrix} 0 \\ \frac{1}{2} B^{*\top} \Gamma^* B^* \end{pmatrix}, \tag{A.47}$$

with initial values $A^*(0) = 0$ and $B^*(0) = \binom{s_r}{s_v}$.

## A.5 SVJT Model

For this model, $X_1 = r$ and $X_2 = \begin{pmatrix} \sqrt{v} \\ \sqrt{\lambda^*} \end{pmatrix}$. Writing the bond price as $P = e^{A + B^\top X_1 + C^\top X_2 + X_2^\top D X_2}$, then $A$, $B$, $C = \binom{C_v}{C_\lambda}$, and $D = \begin{pmatrix} D_{vv} & D_{v\lambda} \\ D_{v\lambda} & D_{\lambda\lambda} \end{pmatrix}$ satisfy the ODEs:

$$\frac{dA}{d\tau} = \mu_r^* B + \Pi^{*\top} C + \frac{1}{2} C^\top \Gamma^* C + \text{tr}\left(\Gamma^* D\right), \tag{A.48}$$

$$\frac{dB}{d\tau} = -1 + \kappa_{rr}^* B, \tag{A.49}$$

$$\frac{dC}{d\tau} = B \Xi^* + 2 D \Pi^* + \left( \Lambda^{*\top} + B \Upsilon^* + 2 D \Gamma^* \right) C, \tag{A.50}$$

$$\frac{dD}{d\tau} = \Theta^* + B \zeta^* + \left( \Lambda^{*\top} + B \Upsilon^* \right) D + D \left( \Lambda^* + B \Upsilon^{*\top} \right) + 2 D \Gamma^* D, \tag{A.51}$$

with initial values $A(0) = 0$, $B(0) = 0$, $C(0) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, and $D(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $\zeta^* \equiv \begin{pmatrix} \zeta^*_{vv} & \zeta^*_{v\lambda} \\ \zeta^*_{v\lambda} & \zeta^*_{\lambda\lambda} \end{pmatrix}$, $\Pi^* \equiv \begin{pmatrix} \mu^*_v \\ \mu^*_\lambda \end{pmatrix}$, $\Xi^* \equiv \begin{pmatrix} \kappa^*_{rv} \\ \kappa^*_{r\lambda} \end{pmatrix}$, $\Lambda^* \equiv \begin{pmatrix} \kappa^*_{vv} & \kappa^*_{v\lambda} \\ \kappa^*_{\lambda v} & \kappa^*_{\lambda\lambda} \end{pmatrix}$, $\Theta^* \equiv \begin{pmatrix} \frac{1}{2}B^2 & 0 \\ 0 & -\mu^*_{Q_r}B + pe^{\mu^*_+ B_r} + (1-p)e^{\mu^*_- B_r} - 1 \end{pmatrix}$, $\Upsilon^* \equiv \left(\begin{smallmatrix} \rho_{rv}\sigma_v & \rho_{r\lambda}\sigma_\lambda \\ 0 & 0 \end{smallmatrix}\right)$, and $\Gamma^* \equiv \begin{pmatrix} \sigma^2_v & \rho_{v\lambda}\sigma_v\sigma_\lambda \\ \rho_{v\lambda}\sigma_v\sigma_\lambda & \sigma^2_\lambda \end{pmatrix}$.

Using the notation $s = \begin{pmatrix} s_r \\ s_v \\ s_\lambda \end{pmatrix}$, the CCF has the form

$$f = e^{i\left(A^* + B^{*\top} X_1 + C^{*\top} X_2 + X_2^\top D^* X_2\right)},$$

where $A^*$, $B^*$, $C^* = \begin{pmatrix} C^*_v \\ C^*_\lambda \end{pmatrix}$, and $D^* = \begin{pmatrix} D^*_{vv} & D^*_{v\lambda} \\ D^*_{v\lambda} & D^*_{\lambda\lambda} \end{pmatrix}$ satisfy the ODEs:

$$\frac{dA^*}{d\tau} = \mu^*_r B^* + \Pi^{*\top} C^* + \frac{1}{2} C^{*\top} \Gamma^* C^* + \text{tr}\left(\Gamma^* D^*\right), \tag{A.52}$$

$$\frac{dB^*}{d\tau} = \kappa^*_{rr} B^*, \tag{A.53}$$

$$\frac{dC^*}{d\tau} = B\Xi^* + 2D^*\Pi^* + \left(\Lambda^{*\top} + B^*\Upsilon^* + 2D^*\Gamma^*\right) C^*, \tag{A.54}$$

$$\frac{dD^*}{d\tau} = \Theta^* + B^*\zeta^* + \left(\Lambda^{*\top} + B^*\Upsilon^*\right) D^* + D^*\left(\Lambda^* + B^*\Upsilon^{*\to p}\right) + 2D^*\Gamma^* D^*, \tag{A.55}$$

with initial values $A^*(0) = 0$, $B^*(0) = s_r$, $C^*(0) = \left(\begin{smallmatrix} s_v \\ s_\lambda \end{smallmatrix}\right)$, and $D^*(0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

**Table 1**

**Summary Statistics of Interest Rates**

| | Mean* | StDev | Skew | Kurt | Min | Max | Autocorrelations of Monthly Series | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | $\rho_1$ | $\rho_2$ | $\rho_3$ | $\rho_4$ | $\rho_5$ |
| Panel A: Summary statistics of daily interest rates | | | | | | | | | | | |
| R3M | 5.597 | 1.780 | 0.406 | 0.029 | 1.55 | 10.67 | 0.98 | 0.96 | 0.94 | 0.92 | 0.89 |
| R6M | 5.708 | 1.789 | 0.429 | 0.182 | 1.59 | 10.77 | 0.98 | 0.96 | 0.94 | 0.91 | 0.89 |
| R1Y | 6.190 | 1.980 | 0.585 | 0.429 | 1.93 | 12.34 | 0.98 | 0.96 | 0.93 | 0.91 | 0.88 |
| R2Y | 6.635 | 1.989 | 0.772 | 0.678 | 2.32 | 13.17 | 0.98 | 0.95 | 0.93 | 0.90 | 0.87 |
| R3Y | 6.834 | 1.990 | 0.877 | 0.775 | 2.70 | 13.49 | 0.98 | 0.95 | 0.92 | 0.89 | 0.86 |
| R5Y | 7.120 | 1.956 | 1.031 | 0.991 | 3.47 | 13.84 | 0.97 | 0.95 | 0.92 | 0.89 | 0.85 |
| R7Y | 7.341 | 1.938 | 1.060 | 0.970 | 3.95 | 13.95 | 0.97 | 0.95 | 0.92 | 0.89 | 0.85 |
| R10Y | 7.435 | 1.932 | 1.026 | 0.872 | 4.16 | 13.99 | 0.97 | 0.95 | 0.92 | 0.89 | 0.85 |
| R30Y | 7.672 | 1.817 | 1.044 | 0.931 | 4.70 | 13.94 | 0.97 | 0.95 | 0.92 | 0.89 | 0.86 |
| Panel B: Summary statistics of daily interest rate changes | | | | | | | | | | | |
| ΔR3M | -1.609 | 0.059 | -0.569 | 11.53 | -0.54 | 0.48 | 0.12 | 0.04 | 0.00 | 0.07 | 0.05 |
| ΔR6M | -1.626 | 0.057 | -0.967 | 13.39 | -0.78 | 0.31 | 0.08 | 0.04 | 0.04 | 0.04 | 0.06 |
| ΔR1Y | -1.746 | 0.063 | -0.751 | 10.57 | -0.83 | 0.36 | 0.08 | 0.03 | 0.02 | 0.05 | 0.04 |
| ΔR2Y | -1.743 | 0.068 | -0.530 | 8.11 | -0.84 | 0.36 | 0.09 | 0.03 | 0.02 | 0.03 | 0.04 |
| ΔR3Y | -1.677 | 0.069 | -0.347 | 6.34 | -0.79 | 0.40 | 0.08 | 0.04 | 0.03 | 0.01 | 0.03 |
| ΔR5Y | -1.615 | 0.069 | -0.247 | 5.38 | -0.77 | 0.41 | 0.06 | -0.08 | 0.03 | -0.00 | 0.03 |
| ΔR7Y | -1.574 | 0.069 | -0.179 | 5.27 | -0.77 | 0.42 | 0.04 | -0.02 | 0.03 | -0.00 | 0.02 |
| ΔR10Y | -1.545 | 0.067 | -0.188 | 5.46 | -0.75 | 0.39 | 0.02 | -0.02 | 0.03 | -0.01 | 0.03 |
| ΔR30Y | -1.448 | 0.059 | -0.433 | 7.58 | -0.76 | 0.32 | -0.04 | -0.02 | 0.03 | -0.00 | 0.02 |

Panels A reports the summary statistics of the daily interest rates with maturities of 3-month, 6-month, 1-year, 2-year, 3-year, 5-year, 7-year, 10-year and 30-year from January 3, 1984 to February 15, 2002. Panel B reports the summary statistics of the daily interest rate changes. The mean for the daily change of interest rate has a magnitude of $10^{-4}$. The autocorrelations are calculated from the monthly interest rate data.

**Table 2**

**Principal Components of Daily Interest Rate Changes**

| Factor | Factor Loadings | | | | | | | | | Variation (%) |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2.41 | 3.02 | 3.91 | 4.40 | 4.56 | 4.57 | 4.48 | 4.25 | 3.55 | 0.822 |
| 2 | -2.82 | -1.99 | -1.15 | -0.27 | 0.16 | 0.65 | 1.10 | 1.20 | 1.33 | 0.106 |
| 3 | 1.27 | -0.04 | -0.81 | -0.92 | -0.66 | -0.14 | 0.39 | 0.61 | 1.00 | 0.030 |
| 4 | 0.72 | -1.10 | -0.38 | 0.37 | 0.48 | 0.29 | 0.02 | -0.08 | -0.52 | 0.015 |
| 5 | -0.11 | 0.65 | -0.95 | 0.04 | 0.26 | 0.31 | 0.11 | 0.02 | -0.39 | 0.010 |
| 6 | 0.01 | 0.02 | -0.33 | 0.66 | 0.06 | -0.37 | -0.31 | -0.20 | 0.55 | 0.007 |
| 7 | -0.00 | -0.00 | -0.02 | 0.45 | -0.59 | -0.14 | 0.31 | 0.27 | -0.31 | 0.005 |
| 8 | 0.00 | -0.01 | -0.00 | 0.09 | -0.38 | 0.63 | -0.24 | -0.22 | 0.14 | 0.004 |
| 9 | 0.00 | -0.00 | -0.00 | -0.01 | -0.02 | -0.02 | 0.46 | -0.48 | 0.07 | 0.003 |

This table reports the principal components of daily interest rate changes. The rows represent 9 principal components while the first 9 columns represent the loadings of principal components on the 9 factors (changes in yields). The last column represents the percentage of total variation of the yield curve explained by each of the individual principal components.

**Table 3**

**Estimation Results of Alternative Term Structure Models**

| | SV | SVJ | SVJT |
|---|---|---|---|
| | | Panel A: GMM estimation results | |
| $\mu_r^*$ | 3.258 $10^{-5}$ (4.157 $10^{-5}$) | 3.412 $10^{-5}$ (4.258 $10^{-5}$) | 3.344 $10^{-5}$ (4.346 $10^{-5}$) |
| $\kappa_{rr}^*$ | -8.459 $10^{-4}$ (8.401 $10^{-4}$) | -5.591 $10^{-4}$ (8.688 $10^{-4}$) | -4.993 $10^{-4}$ (8.897 $10^{-4}$) |
| $\kappa_{rv}^*$ or $\zeta_{vv}^*$ | -4.770 $10^{-3}$ (2.164 $10^{-3}$) | -3.143 $10^{-3}$ (3.882 $10^{-3}$) | -2.816 $10^{-2}$ (5.067 $10^{-2}$) |
| $\zeta_{\lambda\lambda}^*$ | | | -6.163 $10^{-5}$ (1.005 $10^{-4}$) |
| $\mu_v^*$ | 4.233 $10^{-4}$ (4.578 $10^{-5}$) | 2.492 $10^{-4}$ (5.239 $10^{-5}$) | 1.761 $10^{-3}$ (3.769 $10^{-4}$) |
| $\kappa_{vv}^*$ | -1.201 $10^{-1}$ (6.037 $10^{-2}$) | -8.422 $10^{-2}$ (4.225 $10^{-2}$) | -3.306 $10^{-2}$ (2.513 $10^{-2}$) |
| $\sigma_v$ | 5.645 $10^{-2}$ (3.706 $10^{-3}$) | 3.199 $10^{-2}$ (1.434 $10^{-3}$) | 6.445 $10^{-3}$ (2.090 $10^{-4}$) |
| $\rho_{rv}$ | -5.923 $10^{-3}$ (2.107 $10^{-3}$) | -3.645 $10^{-4}$ (1.308 $10^{-4}$) | -3.133 $10^{-3}$ (1.432 $10^{-3}$) |
| $\mu_\lambda^*$ | | | 1.631 $10^{-2}$ (3.421 $10^{-3}$) |
| $\kappa_{\lambda\lambda}^*$ | | | -2.077 $10^{-1}$ (9.325 $10^{-2}$) |
| $\sigma_\lambda$ | | | 1.389 $10^{-1}$ (9.271 $10^{-3}$) |
| $\rho_{r\lambda}$ | | | -4.331 $10^{-2}$ (2.198 $10^{-2}$) |
| $\lambda^*$ | | 3.807 $10^{-2}$ (9.052 $10^{-3}$) | |
| $p$ | | 4.899 $10^{-1}$ (1.761 $10^{-1}$) | 4.932 $10^{-1}$ (1.619 $10^{-1}$) |
| $\mu_+^*$ | | 1.536 $10^{-3}$ (6.159 $10^{-4}$) | 1.518 $10^{-3}$ (5.431 $10^{-4}$) |
| $\mu_-^*$ | | -1.694 $10^{-3}$ (8.314 $10^{-4}$) | -1.689 $10^{-3}$ (6.655 $10^{-4}$) |
| | | Panel B: GM M test of overidentifying restrictions | |
| $\chi^2$ | 34.99 | 22.76 | 9.23 |
| d.o.f. | 11 | 7 | 3 |
| p-value | 0.02% | 0.19% | 2.64% |

This table reports the GMM estimation results for the three alternative term structure models, namely the SV, SVJ and SVJT models. The moment conditions used for the GMM estimation are the daily changes of yields of different maturities, with the lagged yields as instrumental variable. Standard errors are reported in parenthesis next to the parameter estimates. The blank cell indicates that the parameter is restricted to be zero. The Hansen $J$-test statistic is also reported for each model together with the degree of freedom and the $p$-value.

**Table 4**

**Term Structure Dynamics and Macroeconomic Variables**

| | Inflation (IN) | | Real activity (RA) | | Volatility (SV) | | Jump Volatility (JV) | |
|---|---|---|---|---|---|---|---|---|
| Constant | 1.838* | (0.721) | 0.849 | (0.746) | 4.811* | (1.477) | 1.314 | (0.710) |
| $IN_0$ | | | | | 0.188** | (0.101) | 0.028 | (0.105) |
| $IN_{-1}$ | 0.357* | (0.069) | 0.008 | (0.072) | -0.040 | (0.109) | -0.023 | (0.113) |
| $IN_{-2}$ | -0.106 | (0.073) | -0.313* | (0.075) | 0.116 | (0.113) | 0.082 | (0.117) |
| $IN_{-3}$ | 0.149* | (0.070) | 0.119 | (0.072) | 0.031 | (0.112) | -0.018 | (0.116) |
| $RA_0$ | | | | | -0.028 | (0.130) | 0.020 | (0.124) |
| $RA_{-1}$ | 0.074 | (0.058) | 0.154* | (0.061) | -0.025 | (0.112) | -0.004 | (0.107) |
| $RA_{-2}$ | 0.118* | (0.057) | 0.223* | (0.059) | -0.115 | (0.112) | 0.054 | (0.107) |
| $RA_{-3}$ | -0.014 | (0.058) | 0.474* | (0.060) | 0.005 | (0.132) | 0.009 | (0.126) |
| $SV_{-1}$ | 0.070** | (0.038) | -0.006 | (0.047) | -0.108 | (0.095) | -0.090 | (0.091) |
| $SV_{-2}$ | -0.038 | (0.036) | -0.078 | (0.049) | -0.082 | (0.088) | 0.005 | (0.084) |
| $SV_{-3}$ | -0.010 | (0.035) | 0.019 | (0.048) | 0.186** | (0.088) | 0.087 | (0.084) |
| $JV_{-1}$ | -0.029 | (0.051) | 0.003 | (0.052) | 0.318* | (0.100) | 0.172** | (0.095) |
| $JV_{-2}$ | -0.019 | (0.052) | 0.033 | (0.054) | 0.080 | (0.097) | -0.004 | (0.093) |
| $JV_{-3}$ | 0.017 | (0.052) | -0.007 | (0.054) | 0.033 | (0.097) | 0.290* | (0.094) |
| Adj. $R^2$ | 0.120 | | 0.297 | | 0.197 | | 0.071 | |

This table reports the VAR estimation results for the two groups of macroeconomic variables, namely inflation (IN) and real activity (RA), stochastic volatility (SV), and jump volatility (JV). The results are based on monthly observations from January, 1984 to February, 2002. Standard errors are reported in parenthesis next to the parameter estimates. The * and ** indicate the coefficient significantly different from 0 at the 5% and 10% critical levels, respectively.

**Table 5**

**News that Drives the Jumps in Interest Rate**

Panel A: Jumps associated with scheduled news announcements or FOMC meetings

| | Jan 1984 – Feb 2002 | | | Jan 1994 – Feb 2002 | |
|---|---|---|---|---|---|
| | PPI | Industrial Production | FOMC | CPI | Employment |
| Total Announcements[1] | 218 | 218 | 145 | 97 | 97 |
| Jumps on the day[2] | 17 | 13 | 8 | 2 | 4 |
| Positive | 5 | 3 | 4 | 0 | 1 |
| Negative | 12 | 10 | 4 | 2 | 3 |
| Jumps the day before[3] | 10 | 8 | 5 | 0 | 1 |
| Positive | 5 | 2 | 2 | 0 | 0 |
| Negative | 5 | 6 | 3 | 0 | 1 |
| Jumps the day after[4] | 3 | 5 | 7 | 2 | 0 |
| Positive | 0 | 2 | 2 | 0 | 0 |
| Negative | 3 | 3 | 5 | 2 | 0 |

Panel B: A sample of jumps associated with unscheduled events (Jan 1998 – Feb 2002)

| Date | Event |
|---|---|
| 10/08/1998 | Federal Reserve Chairman Alan Greenspan warns the economy is slowing. |
| 10/28/1998 | US reports $70 billion budget surplus for fiscal '98, dwarfing May estimate. |
| 12/29/1999 | Possibly a millennium effect. |
| 12/05/2000 | Election 2000: state court denies Al Gore's challenge to Florida's vote tally. |
| 01/03/2001 | Unscheduled FOMC meeting (cut overnight rate by 50 bps). |
| 04/18/2001[5] | Unscheduled FOMC meeting (cut Fed Funds rate by 50 bps). |
| 09/13/2001[5] | Post September 11, 2001, bond market reopens in attack aftermath. |

[1]: There are total 729 unique announcement or meeting dates. For example, IP and PPI share 44 dates, employment and PPI share 2 dates, there are 9 FOMC meetings dates overlap with IP dates, and 2 FOMC meetings dates overlap with PPI dates.

[2]: There are total 32 unique jumps on the announcement or meeting dates. For example, IP and PPI share 9 dates, and IP and CPI share 1 date.

[3]: There are total 20 unique jumps on the day before announcement or meeting dates, of which 5 overlap with those on the announcement dates of other news.

[4]: There are total 15 unique jumps on the day after announcement or meeting dates, of which 4 overlap with those on the announcement dates of other news, and 2 overlap with those the day before the announcement of other news. This results in total unique 56 jumps associated with scheduled news announcements or FOMC meetings, including one day lead or lag.

[5]: These dates are also one day before the announcement of other news. However, there are no jumps on the news announcement dates.

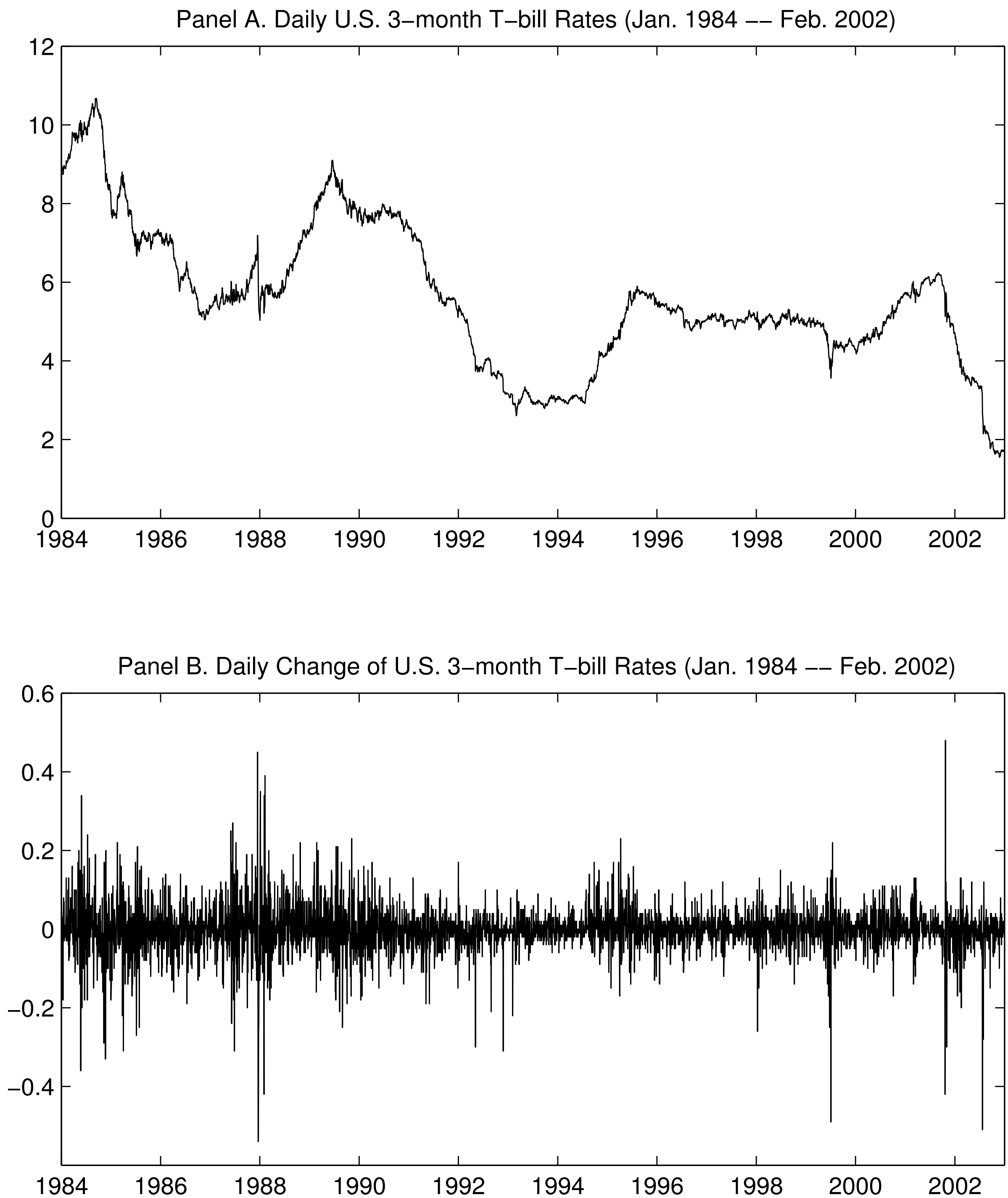


**Figure 1**

**Daily and daily changes of US 3-month T-bill yields**

This figure plots the time series of the daily 3-month Treasury bill yields and the daily changes in panels A and B respectively. The sample period is from January 3, 1984 to February 15, 2002.

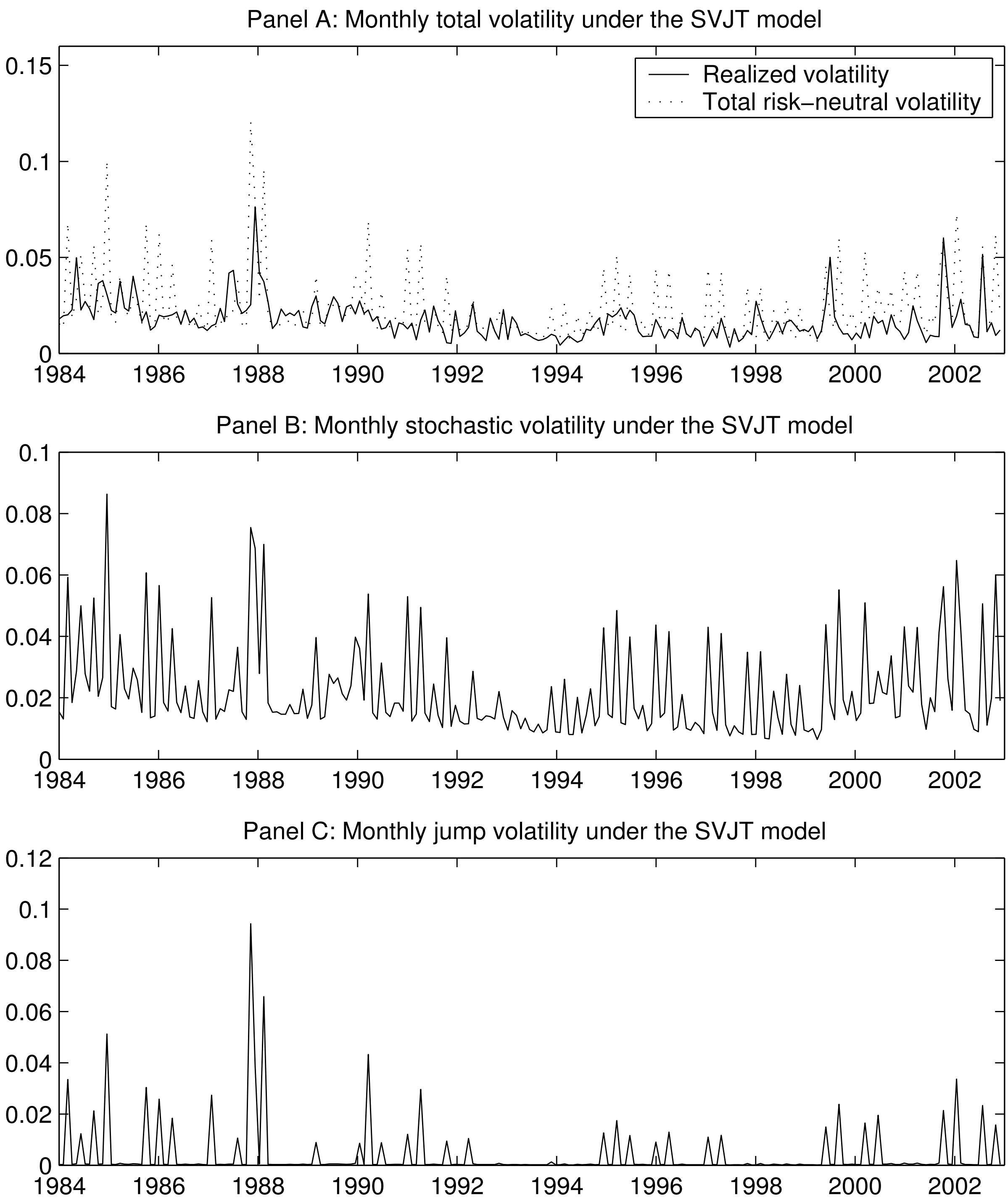


**Figure 2**

**Monthly Stochastic Volatility and Jump Volatility**

This figure plots the time series of the monthly total volatility (stochastic volatility plus jump volatility) of the SVJT model in Panel A, and the stochastic volatility and jump volatility in Panels B and C, respectively. Panel A also plots the monthly realized volatility calculated from the daily 3-month T-bill yields.

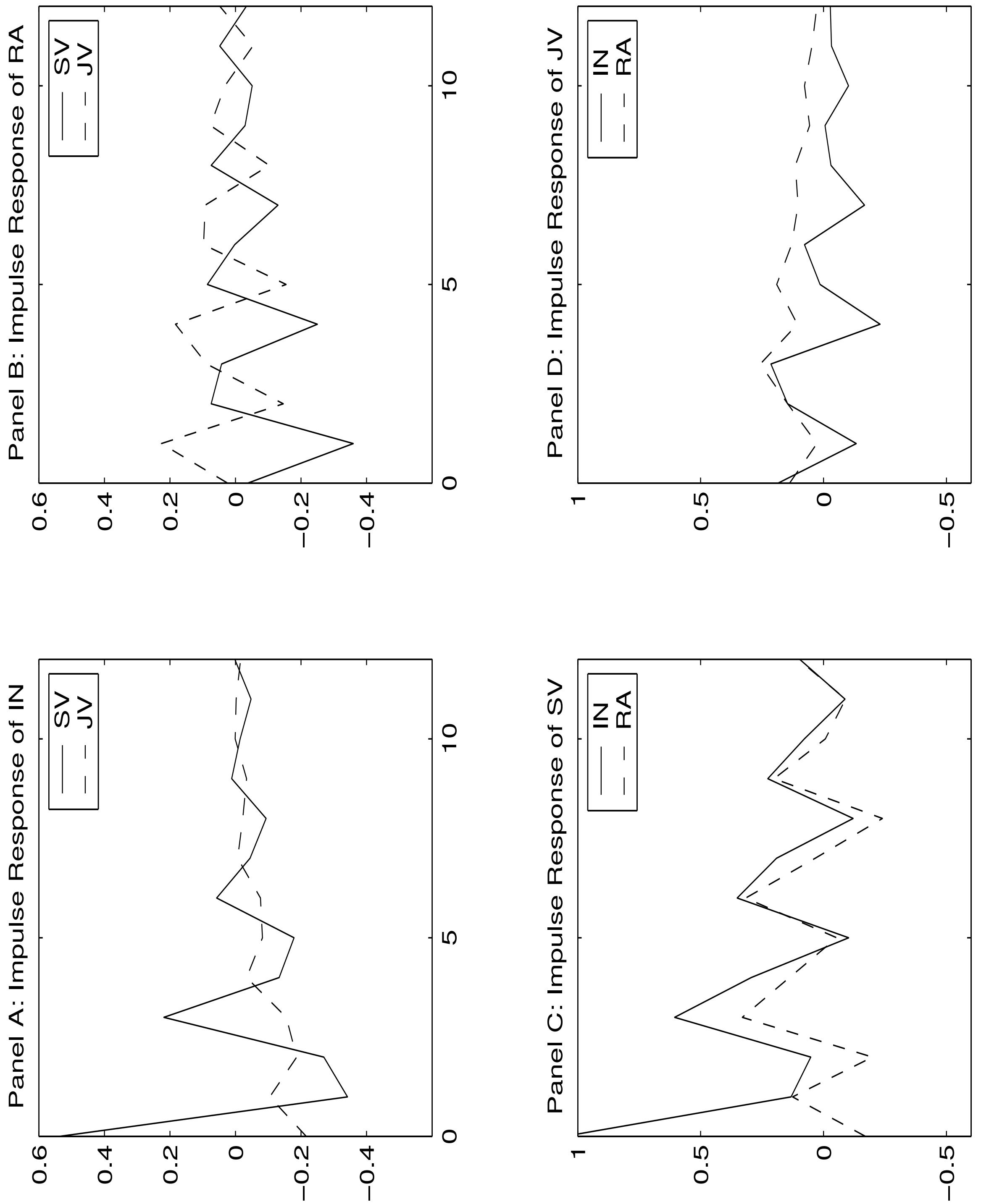


**Figure 3**

**Impulse Response Functions**

This figure plots the impulse responses of the inflation (IN) and real activity (RA) to shocks in stochastic volatility (SV) and jump volatility (JV), as well as the impulse responses of stochastic volatility (SV) and jump volatility (JV) to shocks in inflation (IN) and real activity (RA). The lags are up to 12 months.