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Time-Varying Risk, Interest Rates, and Exchange Rates in General Equilibrium

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ABSTRACT

Under mild assumptions, the data indicate that time-varying risk is the primary force driving nominal interest rate differentials on currency-denominated bonds. This finding is an immediate implication of the fact that exchange rates are roughly random walks. A general equilibrium monetary model with an endogenous source of risk variation—a variable degree of asset market segmentation—can produce key features of actual interest rates and exchange rates. In this model, the endogenous segmentation arises from a fixed cost for agents to exchange money for assets. As inflation varies, so does the benefit of asset market participation, and that changes the fraction of agents participating. These effects lead the risk premium to vary systematically with the level of inflation. This model produces variation in the risk premium even though the fundamental shocks have constant conditional variances.

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> Overall, the new view of finance amounts to a profound change. We have to get used to the fact that most returns and price variation comes from variation in *risk premia*. (Cochrane 2001, p. 451)

Cochrane's observation directs our attention to a critical counterfactual part of the standard general equilibrium monetary model: constant risk premia. Variation in risk over time is essential for understanding movements in asset prices; that has been widely documented. Yet the standard model does not generate time-varying risk premia. We develop a simple, general equilibrium monetary model that does. In our model, the asset market is segmented; at any time, only a fraction of the model's agents choose to participate in that market. Risk premia in our model vary over time because the degree of asset market segmentation varies endogenously in response to stochastic shocks.

We apply the model to interest rates and exchange rates because data on those variables provide some of the most compelling evidence that variation in risk premia is a prime mover behind variation in asset prices. In fact, a stylized view of the data on interest rates and exchange rates is that observed variations in the interest rate differential are accounted for entirely by variations in risk premia.

To make this view concrete, consider the risk, in nominal terms, faced by a U.S. investor choosing between bonds denominated in either dollars or euros. Clearly, for this investor, the dollar return on the euro bond is risky because next period's exchange rate is not known today. The risk premium compensates the investor who chooses to hold the euro bond for this exchange rate risk. Specifically, in logs, the *risk premium* p_t is equal to the expected log of the dollar return on a euro bond minus the log dollar return on a dollar bond,

$$p_t = i_t^* + E_t \log e_{t+1} - \log e_t - i_t,$$

where i_t^* and i_t are the logs of euro and dollar gross interest rates and e_t is the exchange rate between the currencies.[1] The difference in nominal interest rates across currencies can thus be divided into the expected change in the exchange rate between these currencies and a currency risk premium.

In standard equilibrium models of interest rates and exchange rates, since risk premia are constant, interest rate differentials move one-for-one with the expected change in the exchange rate. However, nearly the opposite seems to happen in the data. Indeed, one view of the data is that exchange rates are roughly random walks, so that the expected depreciation of a currency, $E_t \log e_{t+1} - \log e_t$, is roughly constant. (See, for example, the discussion in section 9.3.2 of Obstfeld and Rogoff 1996.) Under this view, the interest rate differential, $i_t^* - i_t$, is approximately equal to the risk premium p_t plus a constant. The observed variation in the interest rate differential is, thus, almost entirely accounted for by movement in the risk premium.

A more nuanced view of the data is that exchange rates are not exactly random walks; instead, when a currency's interest rate is high, that currency is expected to appreciate. This observation, documented by Fama (1984), Hodrick (1987), and Backus, Foresi, and Telmer (1995), among others, is widely referred to as the *forward premium anomaly.* The observation seems to contradict intuition, which predicts instead that investors will demand higher interest rates on currencies that are expected to fall, not rise, in value. To explain the data, then, theory requires large fluctuations in risk premia, larger even than those in the interest differentials.

We build a model that is consistent with that requirement. Our model is a two-country, pure exchange, cash-in-advance economy. The key difference between this model and the standard cash-in-advance model is that here agents must pay a fixed cost to transfer money between the goods market and the asset market. This fixed transfer cost is similar to that in the models of Baumol (1952) and Tobin (1956), and it differs across agents. In each period, agents with a fixed transfer cost below some cutoff level pay it and thus, at the margin, freely exchange money and bonds. Agents with a fixed transfer cost higher than the cutoff level choose not to pay it, so do not make these exchanges. This is the sense in which our model's asset market is segmented.

We show that this model can generate, qualitatively, the type of systematic variation in risk premia called for by the data on interest rates and exchange rates. Rather than build a quantitative model, we deliberately build a simple model in which the main mechanism can be clearly seen with pen and paper calculations. For example, throughout the body of this work, we abstract from trade in goods in order to focus on frictions in asset markets.[2] In Appendix A, we show how to extend the model to allow for trade in goods.

The model's mechanism through which asset market segmentation leads to variable risk premia is straightforward. Changes in the money growth rate change the inflation rate, which changes the net benefit of participating in the asset market. An increase in money growth, for example, increases the fraction of agents that participate in the asset market, reduces the effect of a given money injection on the marginal utility of any participating agent, and hence lowers the risk premium. We show that this type of variable risk premium can be the primary force driving interest rate differentials and that it can generate the forward premium anomaly.

Our model also has implications for the patterns of the forward premium observed across countries. One of these implications is that if inflation is permanently higher in one country, then asset market participation is too. With higher asset market participation, markets are less segmented; thus, the volatility of the risk premia should be smaller. The model thus predicts that countries with high enough inflation should not have a forward premium anomaly. This prediction is supported by Bansal and Dahlquist (2000), who study the forward premium in both developed and emerging economies.

Finally, our model has implications for the forward premium in individual countries over long horizons. We show that under fairly general conditions, asset market segmentation has no impact on long-term risk premia. Specifically, under these conditions, our model's implications for long-term risk premia are the same as those of a model with no segmentation. These risk premia are determined entirely by long-term inflation risk. We show that as long as the conditional distribution of long-term average inflation does not depend on the current state of the economy, long-term risk premia are constant. With constant risk premia, long-term expected depreciation rates move one-for-one with long-term interest differentials. In this sense, our model is consistent with the evidence of Alexius (2001) and Chinn and Meredith (2004), who show that in the data, long-term expected depreciation rates tend to move nearly one-for-one with long-term interest differentials.

The idea that segmented asset markets can generate large risk premia in certain asset prices is not new. (See, for example, Allen and Gale 1994, Basak and Cuoco 1998, and Alvarez and Jermann 2001.) Existing models, however, focus on generating constant risk premia, which for some applications is relevant. As we have argued, however, any attempt to account for the data on interest differentials and exchange rates requires risk premia that are not only large but also highly variable. Unlike other models, ours generates such premia. This success suggests that the friction in the model, asset market segmentation, may belong in a complete model of interest rates and exchange rates.

Our model is related to a huge literature on generating large and volatile risk premia in general equilibrium models. The work of Mehra and Prescott (1985) and Hansen and Jagannathan (1991) has established that in order to generate large risk premia, the general equilibrium model must produce extremely volatile pricing kernels. Also well-known is the fact that because of the data's rather small variations in aggregate consumption, a representative agent model with standard utility functions cannot generate large and variable risk premia. Therefore, attempts to account for foreign exchange risk premia in models of this type fail dramatically. (See Backus, Gregory, and Telmer 1993, Canova and Marrinan 1993, Bansal et al. 1995, Bekaert 1996, Engel 1996, and Obstfeld and Rogoff 2001.) Indeed, the only way such models could generate large and variable risk premia is by generating an implied series for aggregate consumption that both is many times more variable and has a variance that fluctuates much more than observed consumption.

Faced with these difficulties, researchers have split the study of risk in general equilibrium models into two branches. One branch investigates new classes of utility functions that make the marginal utility of consumption extremely sensitive to small variations in consumption. The work of Campbell and Cochrane (1999) typifies this branch. Bekaert (1996) examines the ability of a model along these lines to generate large and variable foreign exchange risk premia. The other research branch investigates limited participation models, in which the consumption of the marginal investor is not equal to aggregate consumption. The work of Alvarez and Jermann (2001) and Lustig and Van Nieuwerburgh (2005) typifies this branch. Our work here is firmly part of this second branch. In our model, the consumption of the marginal investor is quite variable even though aggregate consumption is essentially constant.

A body of empirical work supports the idea that limited participation in asset markets is quantitatively important in accounting for empirical failures of consumption-based asset-pricing models. Mankiw and Zeldes (1991) argue that the consumption of asset market participants, defined as stockholders, is more volatile and more highly correlated with the excess return on the stock market than the consumption of nonparticipants. Brav, Constantinides, and Geczy (2002) argue that if attention is restricted to the consumption of active market participants, then many standard asset-pricing puzzles, like the equity premium puzzle, can be partly accounted for in a consumption-based asset-pricing model with low and economically plausible values of the relative risk aversion coefficient. Vissing-Jorgensen (2002) provides similar evidence.

To keep our analysis here simple, we take an extreme view of the limited participation idea. In our model, aggregate consumption is (essentially) constant, so it plays no role in pricing risk. Instead, this risk is priced by the marginal investor, whose consumption is quite different from aggregate consumption. Lustig and Verdelhan (2005) present some interesting evidence that aggregate U.S. consumption growth may be useful for pricing exchange rate risk. In a more complicated version of our model, we could have both aggregate consumption and the consumption of the marginal investor playing a role in pricing exchange rate risk.

Backus, Foresi, and Telmer (1995) and Engel (1996) have emphasized that standard monetary models with standard utility functions have no chance of producing the forward premium anomaly because these models generate a constant risk premium as long as the underlying driving processes have constant conditional variances. Backus, Foresi, and Telmer argue that empirically this anomaly is not likely to be generated by primitive processes that have nonconstant conditional variances. (See also Hodrick 1989.) Instead, these researchers argue, what is needed is a model that generates nonconstant risk premia from driving processes that have constant conditional variances. Our model does exactly that.

Our work builds on that of Rotemberg (1985) and Alvarez and Atkeson (1997) and is most closely related to that of our earlier (2002) work. It is also related to the work of Grilli and Roubini (1992) and Schlagenhauf and Wrase (1995), who study the effects of money injections on exchange rates in two-country variants of the models of Lucas (1990) and Fuerst (1992) but do not address variations in the risk premium.

1. Risk, Interest Rates, and Exchange Rates in the Data

Here we document that fluctuations in interest differentials across bonds denominated in different currencies are large, and we develop our argument that these fluctuations are driven mainly by time-varying risk.

Backus, Foresi, and Telmer (2001) compute statistics on the difference between monthly euro currency interest rates denominated in U.S. dollars and the corresponding interest rates for the other G-7 currencies over the time period July 1974 through November 1994. The average of the standard deviations of these interest differentials is large: about one percentage point on an annualized basis. (To convert the monthly standard deviations expressed in percentage points in Table 1 to annualized percentage points, multiply them by 12.) Moreover, the interest differentials are quite persistent: at a monthly level, the average of their first-order autocorrelations is .83.

To see that these fluctuations in interest differentials are driven mainly by time-varying risk, start by defining the (log) risk premium for a euro-denominated bond as the expected log dollar return on a euro bond minus the log dollar return on a dollar bond. Let $\exp(i_t)$ and $\exp(i_t^*)$ be the nominal interest rates on the dollar and euro bonds and e_t be the price of euros (foreign currency) in units of dollars (home currency), or the exchange rate between the currencies, in a time period t. The dollar return on a euro bond, $\exp(i_t^*)e_{t+1}/e_t$, is obtained by converting a dollar in period t to $1/e_t$ euros, buying a euro bond paying interest $\exp(i_t^*)$, and then converting the resulting euros back to dollars in $t+1$ at the exchange rate e_{t+1}. The risk premium p_t is then defined as the difference between the expected log dollar return on a euro bond and the log return on a dollar bond:

$$(1) \qquad p_t = i_t^* + E_t \log e_{t+1} - \log e_t - i_t.$$

Clearly, the dollar return on the euro bond is risky because the future exchange rate e_{t+1} is not known in t. The risk premium compensates the holder of the euro bond for this exchange rate risk.

To see our argument in its simplest form, suppose that the exchange rate is a random walk, so that $E_t \log e_{t+1} - \log e_t$ is constant. Then (1) implies that

$$(2) \qquad i_t - i_t^* = -p_t + E_t \log e_{t+1} - \log e_t.$$

Here the interest differential is just the risk premium plus a constant. Hence, all of the movements in the interest differential are matched by corresponding movements in the risk premium:

$$\text{var}(p_t) = \text{var}(i_t - i_t^*).$$

In the data, however, exchange rates are only approximately random walks. In fact, one of the most puzzling features of the exchange rate data is the tendency for high interest rate currencies to appreciate, in that

$$\text{(3)} \qquad \text{cov}\left(i_t - i_t^*, \log e_{t+1} - \log e_t\right) \leq 0.$$

Notice that (3) is equivalent to

$$\text{(4)} \qquad \text{cov}\left(i_t - i_t^*, E_t \log e_{t+1} - \log e_t\right) \leq 0.$$

Thus, (3) implies that exchange rates are not random walks, because expected depreciation rates are correlated with interest differentials.

This tendency for high interest rate currencies to appreciate has been widely documented for the currencies of the major industrialized countries over the period of floating exchange rates. (For a recent discussion, see, for example, Backus, Foresi, and Telmer 2001.) The inequality (3) is referred to as the forward premium anomaly.[3] In the literature, this anomaly is documented by a regression of the change in the exchange rates on the interest differential of the form

$$\text{(5)} \qquad \log e_{t+1} - \log e_t = a + b(i_t - i_t^*) + u_{t+1}.$$

Such regressions typically yield estimates of b that are zero or negative. We refer to b as the *slope coefficient* in the Fama regression.

The estimated size of b is particularly puzzling because it implies that fluctuations in risk premia that are needed to account for fluctuations in interest differentials are even larger than those needed if exchange rates were random walks:

$$\text{(6)} \qquad \text{var}\left(p_t\right) \geq \text{var}\left(i_t - i_t^*\right).$$

To see that (4) implies (6), use (1) to rewrite (4) as $\text{var}(i_t - i_t^*) + \text{cov}(i_t - i_t^*, p_t) \leq 0$ or

$$\text{var}\,(i_t - i_t^*) \leq -\text{cov}\,(i_t - i_t^*, p_t) = -\text{corr}\,(i_t - i_t^*, p_t)\,\text{std}\,(i_t - i_t^*)\,\text{std}\,(p_t)\,.$$

Then, as does Fama (1984), divide by $\text{std}(i_t - i_t^*)$ and use the fact that a correlation is less than or equal to one in absolute value.

2. The Model Economy

Now we describe—first generally and then in detail—our general equilibrium monetary model with segmented markets that generates time-varying risk premia.

A. An Outline

We start by sketching out the basic structure of our model.

Consider a two-country, cash-in-advance economy with an infinite number of periods $t = 0, 1, 2, \ldots$. Call one country the *home* country and the other the *foreign* country. Each country has a government and a continuum of households of measure one. Households in the home country use the home currency, *dollars,* to purchase a home good. Households in the foreign country use the foreign currency, *euros,* to purchase a foreign good.

Trade in this economy in periods $t \geq 1$ occurs in three separate locations: an asset market and one goods market in each country. In the asset market, households trade the two currencies and dollar and euro bonds, which promise delivery of the relevant currency in the asset market in the next period, and the two countries' governments introduce their currencies via open market operations. In each goods market, households use the local currency to buy the local good subject to a cash-in-advance constraint and sell their endowment of the local good for local currency.

Each household must pay a real fixed cost γ for each transfer of cash between the asset market and a goods market. This fixed cost is constant over time for any specific household, but it varies across households in both countries according to a distribution with density $f(\gamma)$ and distribution $F(\gamma)$.[4] Households are indexed by their fixed cost γ. The fixed costs for households in each country are in units of the local good. We assume $F(0) > 0$, so that a positive mass of households has a zero fixed cost.

The only source of uncertainty in this economy is shocks to money growth in the two countries. The timing within each period $t \geq 1$ for a household in the home country is illustrated in Figure 1. We emphasize the physical separation of the markets by separating them in the figure. Households in the home country enter the period with the cash $P_{-1}y$ they obtained from selling their home good endowments in $t-1$, where P_{-1} is the price level and y is their endowment. Each government conducts an open market operation in the asset market, which determines the realizations of money growth rates μ and μ^* in the two countries and the current price levels in the two countries P and P^*.

The household then splits into a worker and a shopper. Each period the worker sells the household endowment y for cash Py and rejoins the shopper at the end of the period. The shopper takes the household's cash $P_{-1}y$ with real value $n = P_{-1}y/P$ and shops for goods. The shopper can choose to pay the fixed cost γ to transfer an amount of cash Px with real value x to or from the asset market. This fixed cost is paid in cash obtained in the asset market. If the shopper pays the fixed cost, then the cash-in-advance constraint is that consumption $c = n + x$; otherwise, this constraint is $c = n$.

The household also enters the period with bonds that are claims to cash in the asset market with payoffs contingent on the rates of money growth μ and μ^* in the current period. This cash can be either reinvested in the asset market or, if the fixed cost is paid, transferred to the goods market. With B denoting the current payoff of the state-contingent bonds purchased in the past, q the price of bonds, and $\int qB'$ the household's purchases of new bonds, the asset market constraint is $B = \int qB' + P(x+\gamma)$ if the fixed cost is paid and $B = \int qB'$ otherwise. At the beginning of period $t+1$, the household starts with cash Py in the goods market and a portfolio of contingent bonds B' in the asset market.

In equilibrium, households with a sufficiently low fixed cost pay it and transfer cash between the goods and asset markets while others do not. We refer to households that pay the fixed cost as *active* and those that do not as *inactive*. Inactive households simply consume their current real balances.

B. The Details

Now we flesh out this outline of the economy.

Throughout, we assume that the shopper's cash-in-advance constraint binds and that in the asset market, households hold their assets in interest-bearing securities rather than cash. It is easy to provide sufficient conditions for these assumptions to hold. Essentially, if the average inflation rate is high enough, then money held over from one period to another in a goods market loses much of its value, and households' cash-in-advance constraints bind.[5] If nominal interest rates are positive, then bonds dominate cash held in the asset market, and households hold their assets in interest-bearing securities rather than cash.

At the beginning of period 1, home households of type γ have M_0 units of home money (dollars), $\bar{B}_h(\gamma)$ units of the home government debt (bonds), and $\bar{B}_h^*$ units of the foreign government debt, which are claims on $\bar{B}_h(\gamma)$ dollars and $\bar{B}_h^*$ euros in the asset market in that period. Likewise, foreign households start period 1 with M_0^* euro holdings in the foreign goods market and start period 0 with $\bar{B}_f$ units of the home government debt and $\bar{B}_f^*(\gamma)$ units of the foreign government debt in the asset market.

Let M_t denote the stock of dollars in period t, and let $\mu_t = M_t/M_{t-1}$ denote the growth rate of this stock. Similarly, let μ_t^* be the growth rate of the stock of euros M_t^*. Let $s_t = (\mu_t, \mu_t^*)$ denote the aggregate event in period t. Then let $s^t = (s_1, \ldots, s_t)$ denote the *state,* consisting of the history of aggregate events through period t, and let $g(s^t)$ denote the density of the probability distribution over such histories.

The home government issues one-period dollar bonds contingent on the aggregate state s^t. In period t, given state s^t, the home government pays off outstanding bonds $B(s^t)$ in dollars and issues claims to dollars in the next asset market of the form $B(s^t, s_{t+1})$ at prices $q(s^t, s_{t+1})$. The home government budget constraint at s^t with $t \geq 1$ is

$$(7) \qquad B(s^t) = M(s^t) - M(s^{t-1}) + \int_{s_{t+1}} q(s^t, s_{t+1}) B(s^t, s_{t+1}) \, ds_{t+1}$$

with $M(s^0) = \bar{M}$ given, and in $t = 0$, the constraint is $\bar{B} = \int_{s_1} q(s^1) B(s^1) \, ds_1$. Likewise, the foreign government issues euro bonds denoted $B^*(s^t)$ with bond prices denoted $q^*(s^t, s_{t+1})$. The budget constraint for the foreign government is then analogous to the constraint above.

In the asset market in each period and state, home households trade a set of one-period dollar bonds and euro bonds that have payoffs next period contingent on the aggregate event s_{t+1}. Arbitrage between these bonds implies that

$$(8) \qquad q(s^t, s_{t+1}) = q^*(s^t, s_{t+1}) e(s^t)/e(s^{t+1}),$$

where $e(s^t)$ is the exchange rate for one euro in terms of dollars in state s^t. Thus, without loss of generality, we can assume that home households trade in home bonds and foreign households trade in foreign bonds. With these bonds there are complete markets within the asset market.

Consider now the problem of households of type γ in the home country. Let $P(s^t)$ denote the price level in dollars in the home goods market in period t. In each period $t \geq 1$, in the goods market, households of type γ start the period with dollar real balances $n(s^t, \gamma)$. They then choose transfers of real balances between the goods market and the asset market $x(s^t, \gamma)$, an indicator variable $z(s^t, \gamma)$ equal to zero if these transfers are zero and one if they are more than zero, and consumption of the home good $c(s^t, \gamma)$ subject to the cash-in-advance constraint and the transition law,

$$c(s^t, \gamma) = n(s^t, \gamma) + x(s^t, \gamma) z(s^t, \gamma) \tag{9}$$

$$n(s^{t+1}, \gamma) = \frac{P(s^t) y}{P(s^{t+1})}, \tag{10}$$

where in (9) in $t = 0$, the term $n(s^0, \gamma)$ is given by $M_0 / p(s^0)$. In the asset market in $t \geq 1$, home households begin with cash payments $B(s^t, \gamma)$ on their bonds. They purchase new bonds and make cash transfers to the goods market subject to the sequence of budget constraints

$$B(s^t, \gamma) = \int_{s_{t+1}} q(s^t, s_{t+1}) B(s^t, s_{t+1}, \gamma) \; ds_{t+1} + P(s^t) \left[x(s^t, \gamma) + \gamma \right] z(s^t, \gamma). \tag{11}$$

Assume that both consumption $c(s^t, \gamma)$ and real bond holdings $B(s^t, \gamma)/P(s^t)$ are uniformly bounded by some large constants.

The problem of the home household of type γ is to maximize utility

$$\sum_{t=1}^{\infty} \beta^t \int U\big(c(s^t, \gamma)\big) g(s^t) \; ds^t \tag{12}$$

subject to the constraints (9)–(11). Households in the foreign country solve the analogous problem, with $P^*(s^t)$ denoting the price level in the foreign country in euros. We require that $\int \bar{B}_h(\gamma) f(\gamma) \; d\gamma + \bar{B}_f = \bar{B}$ and $\bar{B}_h^* + \int \bar{B}_f^*(\gamma) f(\gamma) \; d\gamma = \bar{B}^*$.

Since each transfer of cash between the asset market and the home goods market consumes γ units of the home good, the total goods cost of carrying out all transfers between home households and the asset market in t is $\gamma \int z(s^t,\gamma) f(\gamma)\ d\gamma$, and likewise for the foreign households. The resource constraint in the home country is given by

$$\int \left[c(s^t,\gamma) + \gamma z(s^t,\gamma)\right] f(\gamma)\ d\gamma = y \tag{13}$$

for all t, s^t, with the analogous constraint in the foreign country. The fixed costs are paid for with cash obtained in the asset market. Thus, the home country money market–clearing condition in $t \geq 0$ is given by

$$\int \left(n(s^t,\gamma) + \left[x(s^t,\gamma) + \gamma\right] z(s^t,\gamma)\right) f(\gamma)\ d\gamma = M(s^t)/P(s^t) \tag{14}$$

for all s^t. The money market–clearing condition for the foreign country is analogous. We let c denote the sequences of functions $c(s^t,\gamma)$ and use similar notation for the other variables.

An *equilibrium* in this economy is a collection of bond and goods prices (q, q^*) and (P, P^*), together with bond holdings (B, B^*) and allocations for home and foreign households (c, x, z, n) and (c^*, x^*, z^*, n^*), such that for each γ, the bond holdings and the allocations solve the households' utility maximization problems, the governments' budget constraints hold, and the resource constraints and the money market–clearing conditions are satisfied.

3. Characterizing Equilibrium

Here, in our model economy, we solve for the equilibrium consumption and real balances of both active households (those that transfer cash between asset and goods markets) and inactive households (those that do not). We then characterize the link between the consumption of active households and asset prices. We focus on households in the home country; the analysis of households in the foreign country is similar.

A. Consumption and Real Balances of Active and Inactive Households

We start with a household's decision to pay the fixed cost. Under the assumption that the cash-in-advance constraint always binds, any household's decision to pay the fixed cost to trade in period t is static, since this decision affects only the household's current consumption and bond holdings and not the real balances it holds later in the goods market. Notice that the constraints (10), (13), and (14) imply that the price level is

$$P(s^t) = M(s^t)/y. \tag{15}$$

The inflation rate is $\pi_t = \mu_t$, and real money holdings are $n(s^t, \gamma) = y/\mu_t$. Hence, the consumption of inactive households is $c(s^t, \gamma) = y/\mu_t$. Let $c_A(s^t, \gamma)$ denote the consumption of an active household for a given s^t and γ.

In this economy, inflation is distorting because it reduces the consumption of any household that chooses to be inactive. This effect induces some households to use real resources to pay the fixed cost, thereby reducing the total amount of resources available for consumption. This is the only distortion in the model. Because of this feature and our complete market assumptions, the competitive equilibrium allocations and asset prices can be found from the solution to the following planning problem for the home country, together with the analogous problem for the foreign country. Choose $z(s^t, \gamma) \in [0, 1]$, $c(s^t, \gamma) \geq 0$, and $c(s^t) \geq 0$ to solve

$$\max \sum_{t=1}^{\infty} \beta^t \int_{s^t} \int_{\gamma} U\left(c(s^t, \gamma)\right) f(\gamma) g(s^t)\; d\gamma ds^t$$

subject to the resource constraint (13) and

$$(16) \quad c(s^t, \gamma) = z(s^t, \gamma) c_A(s^t, \gamma) + [1 - z(s^t, \gamma)] y/\mu_t.$$

The constraint (16) captures the restriction that the consumption of households that do not pay the fixed cost is pinned down by their real money balances y/μ_t. Here the planning weight for households of type γ is simply the fraction of households of this type.

This planning problem can be decentralized with the appropriate settings of the initial endowments of home and foreign government debt $\bar{B}(\gamma)$ and $\bar{B}^*(\gamma)$. Asset prices are obtained from the multipliers on the resource constraints above.

Notice that the planning problem reduces to a sequence of static problems. We analyze first the consumption pattern for a fixed choice of z and then the optimal choice of z.

The first-order condition for c_A reduces to

$$(17) \quad \beta^t U'\left(c_A(s^t, \gamma)\right) g(s^t) = \lambda(s^t),$$

where $\lambda(s^t)$ is the multiplier on the resource constraint. This first-order condition clearly implies that all households that pay the fixed cost choose the same consumption level, which means that $c_A(s^t, \gamma)$ is independent of γ. Since this problem is static, this consumption level depends on only the current money growth shock μ_t. Hence, we denote this consumption as $c_A(\mu_t)$.

Now, since the solution to the planning problem depends on only current μ_t and γ, we drop its dependence on t. It should be clear that the optimal choice of z has a cutoff rule form: for each shock μ, there is some fixed cost level $\bar{\gamma}(\mu)$ at which the households with $\gamma \leq \bar{\gamma}(\mu)$ pay this fixed cost and consume $c_A(\mu)$, and all other households do not pay and consume instead γ/μ. For each μ, the planning problem thus reduces to choosing two numbers, $c_A(\mu)$ and $\bar{\gamma}(\mu)$, to solve

$$\max U\big(c_A(\mu)\big)F\big(\bar{\gamma}(\mu)\big) + U(y/\mu)\left[1 - F\big(\bar{\gamma}(\mu)\big)\right]$$

subject to

$$(18) \quad c_A(\mu)F\big(\bar{\gamma}(\mu)\big) + \int_0^{\bar{\gamma}(\mu)} \gamma f(\gamma)\, d\gamma + (y/\mu)\left[1 - F\big(\bar{\gamma}(\mu)\big)\right] = y.$$

The first-order conditions can be summarized by

$$(19) \quad U\big(c_A(\mu)\big) - U(y/\mu) - U'\big(c_A(\mu)\big)[c_A(\mu) + \bar{\gamma}(\mu) - (y/\mu)] = 0$$

and (18). In Appendix B, we show that the solution to these two equations, (18) and (19)—namely, $c_A(\mu)$ and $\bar{\gamma}(\mu)$—is unique. We then have the following proposition:

Proposition 1. The equilibrium consumption of households is given by

$$c(s^t, \gamma) = \begin{cases} y/\mu_t & \text{if } \gamma \leq \bar{\gamma}(\mu_t) \\ c_A(\mu_t) & \text{otherwise,} \end{cases}$$

where the functions $c_A(\mu)$ and $\bar{\gamma}(\mu)$ are the solutions to (18) and (19).

B. Active Household Consumption and Asset Prices

Now we turn to the active households and characterize the link between their consumption and asset prices.

In the decentralized economy corresponding to the planning problem, asset prices are given by the multipliers on the resource constraints for the planning problem. Here, from (17), these multipliers are equal to the marginal utility of active households.

Hence, the pricing kernel for dollar assets is

$$(20) \quad m(s^t, s_{t+1}) = \beta \frac{U'\big(c_A(\mu_{t+1})\big)}{U'\big(c_A(\mu_t)\big)} \frac{1}{\mu_{t+1}},$$

while the pricing kernel for euro assets is

$$(21) \quad m^*(s^t, s_{t+1}) = \beta \frac{U'\big(c_A^*(\mu_{t+1}^*)\big)}{U'\big(c_A^*(\mu_t^*)\big)} \frac{1}{\mu_{t+1}^*}.$$

These kernels are the state-contingent prices for dollars and euros normalized by the probabilities of the state.

These pricing kernels can price any dollar or euro asset. In particular, the pricing kernels immediately imply that any asset purchased in period t with a dollar return of R_{t+1} between periods t and $t+1$ satisfies the Euler equation

$$(22) \quad 1 = E_t m_{t+1} R_{t+1},$$

where, for simplicity here and in much of what follows, we drop the s^t notation. Likewise, every possible euro asset with rate of return R_{t+1}^* from t to $t+1$ satisfies the Euler equation

$$(23) \quad 1 = E_t m_{t+1}^* R_{t+1}^*.$$

Note that $\exp(i_t)$ is the dollar return on a dollar-denominated bond with interest rate i_t, and $\exp(i_t^*)$ is the expected euro return on a euro-denominated bond with interest rate i_t^*; these Euler equations thus imply that

$$(24) \quad i_t = -\log E_t m_{t+1} \text{ and } i_t^* = -\log E_t m_{t+1}^*.$$

The pricing kernels for dollars and euros have a natural relation: $m_{t+1}^* = m_{t+1} e_{t+1}/e_t$. This can be seen as follows. Every euro asset with euro rate of return R_{t+1}^* has a corresponding dollar asset with rate of return $R_{t+1} = R_{t+1}^* e_{t+1}/e_t$ formed when an investor converts dollars into euros in t, buys the euro asset, and converts the return back into dollars in $t+1$. Equilibrium requires that

$$(25) \quad 1 = E_t m_{t+1} R_{t+1} = E_t \left\{ \left[m_{t+1} \left(\frac{e_{t+1}}{e_t} \right) \right] R_{t+1}^* \right\}.$$

Since (25) holds for every euro return, $m_{t+1} e_{t+1}/e_t$ is an equilibrium pricing kernel for euro assets. Complete markets have only one euro pricing kernel, so

$$(26) \quad \log e_{t+1} - \log e_t = \log m_{t+1}^* - \log m_{t+1}.$$

Substituting (24) and (26) into our original expression for the risk premium, (1), gives that

$$p_t = E_t \log m^*_{t+1} - E_t \log m_{t+1} - (\log E_t m^*_{t+1} - \log E_t m_{t+1}). \quad (27)$$

Hence, the currency risk premium depends on the difference between the expected value of the log and the log of the expectation of the pricing kernel. Jensen's inequality implies that fluctuations in the risk premium are driven by fluctuations in the conditional variability of the pricing kernel.

Finally, note that given any period 0 exchange rate e_0, (26) together with the kernels gives the entire path of the nominal exchange rate e_t. It is easy to show that the period 0 nominal exchange rate e_0 is given by

$$e_0 = \left(\bar{B} - \bar{B}_h\right) / \bar{B}^*_h. \quad (28)$$

Clearly, this exchange rate exists and is positive as long as $\bar{B}_h < \bar{B}$ and $\bar{B}^*_h > 0$ or $\bar{B}_h > \bar{B}$ and $\bar{B}^*_h < 0$.

4. Linking Money Shocks and Active Households' Marginal Utility

In our model, the active households price assets in the sense that the pricing kernels (20) and (21) are determined by those households' marginal utilities. Thus, in order to characterize the link between money shocks and either exchange rates or interest rates, we need to determine how these marginal utilities respond to money shocks, or how $U'\big(c_A(\mu_t)\big)$ varies with μ_t.

A. The Theory

In the simplest monetary models (such as in Lucas 1982), all the agents are active every period, and changes in money growth have no impact on marginal utilities. Our model introduces two key innovations to those simple models. One is that here, because of the segmentation of asset markets, changes in money growth do have an impact on the consumption and, hence, marginal utility of active households. The other innovation is that, because the degree of market segmentation is endogenous, the size of this impact changes systematically with the level of money growth. In particular, as money growth increases, more households choose to be active in financial markets, and the degree of risk due to market segmentation falls. With these two innovations, our model can deliver large and variable currency risk premia even though the fundamental shocks have constant variance.

Mechanically, our model generates variable risk premia because $\log c_A(\mu)$ is increasing and concave in $\log \mu$. To see the link between risk premia and $\log c_A(\mu)$, define $\phi(\mu)$ to be the elasticity of the marginal utility of active households to a change in money growth. With constant relative risk aversion preferences of the form $U(c) = c^{1-\sigma}/(1-\sigma)$, this elasticity is given by

$$(29) \quad \phi(\mu) \equiv -\frac{d \log U'\big(c_A(\mu)\big)}{d \log \mu} = \sigma \frac{d \log c_A(\mu)}{d \log \mu}$$

where σ is the degree of relative risk aversion.

For later use, note that when $\log c_A(\mu)$ is increasing in $\log \mu$, $\phi(\mu) > 0$. The larger is $\phi(\mu)$, the more sensitive is the marginal utility of active households to money growth. Also note that when $\log c_A(\mu)$ is concave in $\log \mu$, $\phi(\mu)$ decreases in μ, so the marginal utility of active households is more sensitive to changes in money growth at low levels of money growth than at high levels. In this sense, the concavity implies that the variability of the pricing kernel decreases as money growth increases.

We now characterize features of our model's equilibrium in two propositions. In Proposition 2, we show that more households choose to become active as money growth and inflation increase. The result is intuitive: as inflation increases, so does the cost of not participating in the asset market, since the consumption of inactive households, y/μ, falls as money growth μ increases. In Proposition 3, we show that, at least for low values of money growth, $\log c_A(\mu)$ is increasing and concave in $\log \mu$.

Proposition 2. As μ increases, more households become active. In particular, $\bar{\gamma}'(\mu) > 0$ for $\mu > 1$, and $\bar{\gamma}'(1) = 0$.

Proof. Differentiating equations (18) and (19) with respect to μ and solving for $\bar{\gamma}'$ gives that

$$\bar{\gamma}'(\mu) = \frac{\left[U'(y/u) - U'(c_A)\right](y/\mu) - U''(c_A)\left[c_A + \bar{\gamma} - (y/\mu)\right]\frac{1-F}{F} y/\mu^2}{U'(c_A) - U''(c_A)\left[c_A + \bar{\gamma} - (y/\mu)\right] f/F},$$

where to simplify we have omitted the arguments in the functions F, f, c_A, and $\bar{\gamma}$. Note that $c_A(1) = y$ and $\bar{\gamma}(1) = 0$. Also note that (18) implies that if $\mu > 1$, then $c_A + \bar{\gamma} - (y/\mu) > 0$. To derive this result, rewrite (18) as

$$(30) \quad c_A(\mu) + \frac{\int_0^{\bar{\gamma}(\mu)} \gamma f(\gamma)\, d\gamma}{F\big(\bar{\gamma}(\mu)\big)} - y/\mu = \frac{y - y/\mu}{F\big(\bar{\gamma}(\mu)\big)},$$

use the inequality $\bar{\gamma}(\mu) \geq \left(\int_0^{\bar{\gamma}(\mu)} \gamma f(\gamma)\, d\gamma\right) / F(\bar{\gamma}(\mu))$, and note that the right side of (30) is strictly positive for $\mu > 1$. It follows from this result and (19) that $U'(y/\mu) - U'(c_A) > 0$ for $\mu > 1$. Finally, since U is strictly concave, $U''(c_A) < 0$; thus, $\bar{\gamma}' > 0$ for $\mu > 1$. Using similar results for $\mu = 1$, we get that $\gamma'(1) = 0$. *Q.E.D.*

Proposition 3. The log of the consumption of active households $c_A(\mu)$ is strictly increasing and strictly concave in $\log \mu$ around $\mu = 1$. In particular, $\phi(1) > 0$ and $\phi'(1) < 0$.

Proof. We first show that $\phi(1) = \sigma[1 - F(0)]/F(0)$, which is positive when $F(0) > 0$. To see this, differentiate (18) with respect to μ and $\bar{\gamma}$, and use, from Proposition 2, that $\bar{\gamma}'(1) = \bar{\gamma}(1) = 0$ in order to get that

$$c_A'(1) = y\frac{1 - F(0)}{F(0)}.$$

Using this expression for $c_A'(1)$ and using $c_A(1) = y$ in $\phi(1) = \sigma c_A'(1)/c_A(1)$ gives our intended result.

We next show that $\phi'(1) = -\phi(1)/F(0)$, which is negative because $\phi(1) > 0$ and $F(0) > 0$. To see this, first differentiate (29) to get that

$$\phi'(1) = \sigma\left[\frac{c_A''(1)}{c_A(1)} + \frac{c_A'(1)}{c_A(1)} - \left(\frac{c_A'(1)}{c_A(1)}\right)^2\right]. \tag{31}$$

Second, differentiate (18) with respect to μ and $\bar{\gamma}$, and use the result at $\mu = 1$, $\bar{\gamma}'(\mu) = \bar{\gamma}(\mu) = 0$, and $c_A(\mu) + \bar{\gamma}(\mu) - y/\mu = 0$ to get that

$$c_A''(1) = -2y\frac{1 - F(0)}{F(0)}.$$

Using these expressions for c_A' and c_A'' in (31) produces the desired result. *Q.E.D.*

In Proposition 2 we have shown that more households pay the fixed cost when money growth increases, and in Proposition 3 we have shown that locally the consumption of active households is increasing and concave in money growth. Note that the result in Proposition 3 does not depend on the finding of Proposition 2 that more households pay the fixed cost when money growth increases. This is because locally around $\mu = 1$, $\bar{\gamma}(\mu)$ does not vary with μ. As we show in our numerical example below, the $\log c_A(\mu)$ is much more concave when evaluated at a point $\bar{\mu} > 1$, where $\bar{\gamma}(\mu)$ does vary locally with μ.

B. A Numerical Example

Now we consider a simple numerical example that demonstrates the equilibrium features more broadly.

We assume that a time period is a month. We let $y = 1$ and $\sigma = 2$, and for fixed costs we let a fraction $F(0) = .125$ of the households have zero fixed costs and the remainder have fixed costs with a uniform distribution on $[0, b]$ with $b = .1$.

In Figure 2, we plot $\log c_A(\mu)$ against $\log \mu$ (annualized). This figure shows that the consumption of active households is increasing and concave in money growth in the relevant range. Because of this nonlinearity, even if the fundamental shocks—here, changes in money growth rates—have constant conditional variances, the resulting pricing kernels have time-varying conditional variances.

To capture the nonlinearity of $c_A(\mu)$ in a tractable way when computing the asset prices implied by our model, we take a second-order approximation to the marginal utility of active households of the form

$$(32) \quad \log U'\big(c_A(\mu_t)\big) = \log U'\big(c_A(\bar{\mu})\big) - \phi\hat{\mu}_t + \frac{1}{2}\eta\hat{\mu}_t^2,$$

where $\hat{\mu}_t = \log \mu_t - \log \bar{\mu}$,

$$(33) \quad \phi \equiv -\left.\frac{d \log U'\big(c_A(\mu)\big)}{d \log \mu}\right|_{\mu=\bar{\mu}} = \sigma \left.\frac{d \log c_A(\mu)}{d \log \mu}\right|_{\mu=\bar{\mu}}$$

$$\eta \equiv \left.\frac{d^2 \log U'\big(c_A(\mu)\big)}{(d \log \mu)^2}\right|_{\mu=\bar{\mu}} = -\sigma \left.\frac{d^2 \log c_A(\mu)}{(d \log \mu)^2}\right|_{\mu=\bar{\mu}}.$$

For our numerical example, $\phi = 10.9$ and $\eta = 1,007$ when $\bar{\mu}$ is equal to $\exp(5/1200)$, which is 5% at an annualized rate. Note that this high value of η is largely due to the endogeneity of segmentation, in that $\bar{\gamma}'(\mu) > 0$ so the number of active households increases with μ. To see that endogeneity is important for the value of η, consider an alternative economy where the number of active households is fixed at $F(0)$. In this case, ϕ and μ evaluated at the same $\bar{\mu}$ equal 13.5 and 105, respectively.

Motivated by our previous results, we assume that $\phi > 0$ and $\eta > 0$. With this parameterization, we have that the pricing kernel is given by

$$(34) \quad \log m_{t+1} = \log \beta/\bar{\mu} - (\phi + 1)\hat{\mu}_{t+1} + \frac{1}{2}\eta\hat{\mu}_{t+1}^2 + \phi\hat{\mu}_t - \frac{1}{2}\eta\hat{\mu}_t^2.$$

Throughout, we assume that the log of home money growth has normal innovations, or *shocks,* so that

$$(35) \quad \hat{\mu}_{t+1} = E_t\hat{\mu}_{t+1} + \varepsilon_{t+1}$$

and likewise for foreign money growth. Here ε_{t+1} and ε^*_{t+1} are the independent shocks across countries and are both normal with mean zero and variance σ^2_ε. For interest rates to be well-defined with our quadratic approximation, we need

$$(36) \quad \eta\sigma^2_\varepsilon < 1,$$

which we assume holds throughout.

5. Linking Money Growth and Risk Premia

Now we use our pricing kernel (34) to show how the risk premium varies systematically with changes in money growth. We show that the risk premium varies even if the shocks to money growth have constant conditional variances. In particular, we show that, locally, a persistent increase in money growth decreases the risk premium p_t. We also give conditions under which the variation in the risk premium is large.

Recall that the risk premium can be written in terms of the pricing kernels as in (27):

$$(37) \quad p_t = \log E_t m_{t+1} - E_t \log m_{t+1} - (\log E_t m^*_{t+1} - E_t \log m^*_{t+1}).$$

Note that if the pricing kernel m_{t+1} were a conditionally lognormal variable, then, as is well-known, $\log E_t m_{t+1} = E_t \log m_{t+1} + (1/2)\mathrm{var}_t(\log m_{t+1})$. In such a case, the risk premium p_t would equal half the difference of the conditional variances of the log kernels. Given our approximation (34), however, the pricing kernels are not conditionally lognormal; still, a similar relation between the risk premium and the conditional variances of the kernels holds, as we show in the next proposition (which is proved in Appendix C).

Proposition 4. Under (34), the risk premium is

$$(38) \quad p_t = \frac{1}{2}\frac{1}{(1-\eta\sigma^2_\varepsilon)}\left(\mathrm{var}_t \log m_{t+1} - \mathrm{var}_t \log m^*_{t+1}\right),$$

where

$$(39) \quad \mathrm{var}_t(\log m_{t+1}) = [-(1+\phi) + \eta E_t\hat{\mu}_{t+1}]^2\sigma^2_\varepsilon + \frac{3}{4}\eta^2\sigma^4_\varepsilon$$

and a symmetric formula holds for $\text{var}_t \left(\log m_{t+1}^* \right)$.

To see how the risk premium varies with money growth, we calculate the derivative of the risk premium and evaluate it at $\mu_t = \bar{\mu}$ to get that

$$\text{(40)} \quad \frac{dp_t}{d\hat{\mu}_t} = -\frac{\eta(\phi+1)\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2} \frac{dE_t\hat{\mu}_{t+1}}{d\hat{\mu}_t}.$$

Under (36), we know from (40) that the risk premium falls with home money growth if $\log c_A(\mu)$ is concave in $\log \mu$, so that $\eta > 0$, and if money growth is persistent, in that $dE_t\hat{\mu}_{t+1}/d\hat{\mu}_t$ is positive.

The basic idea behind why the risk premium decreases with the money growth rate has two parts. One is that, since money growth is persistent, a high money growth rate in period t leads households to forecast a higher money growth rate in period $t+1$. The other part is that, in any period, since η is positive, the marginal utility of active households is concave in the rate of money growth in that period. So as money growth increases, the sensitivity of marginal utility to fluctuations in money growth decreases. Thus, a high rate of money growth in period t leads households to predict that marginal utility in period $t+1$ will be less variable. Hence, the risk premium decreases with the money growth rate.

Next consider the variability of the risk premium. Expanding (39), we have that $\text{var}_t(\log m_{t+1})$ equals a constant plus

$$\frac{\eta\sigma_\varepsilon^2}{1-\eta\sigma_\varepsilon^2} \left[-(1+\phi)E_t\hat{\mu}_{t+1} + \frac{\eta}{2}(E_t\hat{\mu}_{t+1})^2 \right].$$

As long as $E_t\hat{\mu}_{t+1}$ is approximately normal, so that the covariance between $E_t\hat{\mu}_{t+1}$ and $(E_t\hat{\mu}_{t+1})^2$ is approximately zero, the variability of the risk premium is increasing in ϕ, η, and σ_ε^2.[6] The intuition for this result is the same as that for (40). As these parameters increase, the conditional variance of the pricing kernels changes more with a given change in the growth rates of money.

6. Generating the Forward Premium Anomaly

As we have described, the data exhibit a forward premium anomaly: high interest rate currencies are expected to appreciate. For our model to generate this forward premium anomaly, we must incorporate a shock that persistently moves the interest differential and the expected depreciation rate in opposite directions. Here we present sufficient conditions for a persistent shock to money growth to generate this pattern.

From the definition of the risk premium (1), we can write the interest differential as

$$(41) \quad i_t - i_t^* = E_t \log e_{t+1} - \log e_t - p_t.$$

As we have seen, a persistent increase in money growth leads the risk premium p_t to fall. When this increase in money growth also leads to an expected exchange rate appreciation smaller than the fall in the risk premium, then the interest differential increases, and our model generates the forward premium anomaly.

The simplest case to study is when exchange rates are random walks, for then an increase in money growth has no effect on the expected appreciation. Because the covariance between the interest differential and the expected change in the exchange rate is zero, the model generates, at least weakly, the forward premium anomaly.

The more general case is when a persistent increase in money growth leads to a moderate expected exchange rate appreciation. Recall that in standard models without market segmentation, the opposite occurs: a persistent increase in money growth leads to an expected depreciation. We discuss in some detail below how our model with segmentation delivers different implications for the effects of money growth on the exchange rate.

We begin our study of the general case with a discussion of the link between money growth, exchange rate depreciations, and interest differentials. Then we present a numerical example of the model's implications over time, followed by a brief discussion of the model's implications across countries.

A. The Link Between Money Growth, Exchange Rate Depreciations, and Interest Differentials

From (26) and (34), we can derive that the expected depreciation of the exchange rate is given by

$$(42) \quad E_t \log e_{t+1} - \log e_t =$$
$$-(\phi+1)E_t(\hat{\mu}^*_{t+1} - \hat{\mu}_{t+1}) + \frac{1}{2}\eta E_t(\hat{\mu}^{*2}_{t+1} - \hat{\mu}^2_{t+1}) + \phi(\hat{\mu}^*_t - \hat{\mu}_t) - \frac{1}{2}\eta(\hat{\mu}^{*2}_t - \hat{\mu}^2_t).$$

The interest differential (41) is then given by combining (38) and (42). We can use these formulas to establish the following proposition:

Proposition 5. If these inequalities are satisfied,

$$(43) \qquad 1 - \eta\sigma_\varepsilon^2 < \frac{1+\phi}{\phi}\frac{dE_t\hat{\mu}_{t+1}}{d\hat{\mu}_t} \le 1,$$

then for μ_t close to $\bar{\mu}$, a change in money growth leads the interest differential and the expected exchange rate depreciation to move in opposite directions. Specifically, an increase in home money growth $\hat{\mu}_t$ raises the interest differential, $i_t - i_t^*$, and lowers the expected depreciation, $E_t \log e_{t+1} - \log e_t$.

Proof. Differentiating (42), evaluating the resulting expressions at $\mu_t = \bar{\mu}$, and using $E_t\hat{\mu}_{t+1} = 0$ when $\mu_t = \bar{\mu}$ gives that

$$(44) \qquad \frac{d(E_t \log e_{t+1} - \log e_t)}{d\hat{\mu}_t} = (1+\phi)\frac{dE_t\hat{\mu}_{t+1}}{d\hat{\mu}_t} - \phi.$$

Adding (40) and (44) gives that

$$(45) \qquad \frac{d(i_t - i_t^*)}{d\hat{\mu}_t} = \frac{1+\phi}{1-\eta\sigma_\varepsilon^2}\frac{dE_t\hat{\mu}_{t+1}}{d\hat{\mu}_t} - \phi.$$

The first inequality in (43) implies that $d(i_t - i_t^*)/d\hat{\mu}$ is positive, so that an increase in money growth increases the interest differential. The second inequality in (43) implies that $d(E_t \log e_{t+1} - \log e_t)/d\hat{\mu}$ is negative, so that an increase in money growth leads to an expected exchange rate appreciation. *Q.E.D.*

Note that in our model, an increase in money growth leads to an expected appreciation of the nominal exchange rate e_t. To get some intuition for this feature, write this expected appreciation as the sum of the expected appreciation of the real exchange rate and the expected inflation differential:

$$(46) \qquad E_t \log e_{t+1} - \log e_t = (E_t \log v_{t+1} - \log v_t) + E_t[\log(P_{t+1}/P_t) - \log(P_{t+1}^*/P_t^*)],$$

where the real exchange rate $v_t = e_t P_t^*/P_t$. In a standard model, an increase in money growth leads to an expected nominal depreciation because the increased money growth increases expected inflation but has no effect on real exchange rates. In our model, an increase in money growth leads to an expected real depreciation that dominates the expected inflation effect.

Using our pricing kernels (20) and (21), our expression for changes in exchange rates (26), and the expression for the home price level together with (15) and its foreign analog, we can write the right side of (46) in terms of the marginal utility of active households:

$$(47) \quad E_t[\log U'(c^*_{At+1})/U'(c_{At+1}) - \log U'(c^*_{At})/U'(c_{At})] + E_t[\log \mu_{t+1} - \log \mu^*_{t+1}],$$

where the first bracketed term corresponds to the change in the real exchange rate and the second to the expected inflation differential. Hence, we can decompose the effect of money growth changes on the expected change in the nominal exchange rate into two parts: a *market segmentation effect* and an *expected inflation effect*. The market segmentation effect measures the impact of an increase in money growth on the expected change in the real exchange rate through its impact on the marginal utilities in the first term in (47). This effect is not present in the standard general equilibrium model, which has no segmentation. The expected inflation effect, which is in the standard model, measures the impact of an increase in money growth on the expected inflation differential in the second term in (47).

Now consider the impact of a persistent increase in money growth on the expected change in the nominal exchange rate. The expected inflation effect is simply

$$(48) \quad d(E_t \log \mu_{t+1})/d \log \mu_t.$$

This effect is larger the more persistent is money growth. In the standard model, this is the only effect, so that an increase in money growth of one percentage point leads to an expected nominal depreciation of size $d(E_t \log \mu_{t+1})/d \log \mu_t$.

The size of the market segmentation effect depends on both the degree of market segmentation and the persistence of money growth. A persistent increase in the home money growth rate μ_t affects both the current real exchange rate

$$(49) \quad \log v_t = \log U'\big(c^*_A(\mu^*_t)\big)/U'\big(c_A(\mu_t)\big)$$

and, by increasing the expected money growth rate in $t+1$, the expected real exchange rate

$$(50) \quad E_t \log v_{t+1} = E_t \log U'\big(c^*_A(\mu^*_{t+1})\big)/U'\big(c_A(\mu_{t+1})\big).$$

To better understand the market segmentation effect, suppose first that money growth is not persistent but rather independently and identically distributed. Then changes in home money growth affect only the current real exchange rate. As can be seen from (49), an increase in money growth increases the consumption of the active home households in the current period, thus decreasing both the households' marginal utility and the current real exchange rate. Since the expected real exchange rate in $t+1$ is unchanged by the money growth shock, the real exchange rate is expected to appreciate from t to $t+1$; that is, $E_t \log v_{t+1} - \log v_t$ falls. This effect is larger the greater is the degree of market segmentation, as measured by $\phi(\mu_t)$.

Now suppose that money growth is persistent. Then changes in the money growth rate in period t also affect the expected money growth rate in $t+1$ and, thus, the expected real exchange rate in period $t+1$ as well. The effect of money growth on the expected real exchange rate depends on both the degree of market segmentation and the persistence of money growth, as measured by $d(E_t\hat{\mu}_{t+1})/d\hat{\mu}_t$. Using (47), we see that an increase in the home money growth rate $\hat{\mu}_t$ leads to an expected change in the real exchange rate of

$$\text{(51)} \qquad \frac{d}{d\hat{\mu}_t}(E_t \log v_{t+1} - \log v_t) = \phi \left[\frac{d(E_t\hat{\mu}_{t+1})}{d\hat{\mu}_t} - 1 \right],$$

where we have evaluated this derivative at $\mu_t = \bar{\mu}$. As long as money growth is mean-reverting, in that $d(E_t\hat{\mu}_{t+1})/d\hat{\mu}_t < 1$, an increase in money growth near the steady state leads to an expected real appreciation. Clearly, the magnitude of the expected real appreciation depends on both the degree of market segmentation, as measured by ϕ, and the degree of persistence in money growth, as measured by $d(E_t\hat{\mu}_{t+1})/d\hat{\mu}_t$.

Note that the market segmentation effect and the expected inflation effect have opposite signs. If the market segmentation effect dominates, then for values of μ_t close to $\bar{\mu}$, an increase in home money growth leads to an expected appreciation of the nominal exchange rate. This will occur when

$$\text{(52)} \qquad \frac{dE_t\hat{\mu}_{t+1}}{d\hat{\mu}_t} \leq \frac{\phi}{1+\phi},$$

which is equivalent to the second inequality in (43). For this condition to hold, markets must be sufficiently segmented relative to the persistence of money growth. If (52) holds as an equality, then the market segmentation effect exactly cancels the expected inflation effect, and the nominal exchange rate will be locally a random walk.

B. A Numerical Example of the Model's Time Series Implications

Now we use a simple numerical example to illustrate the type of interest rate and exchange rate behavior that our model can generate. We have constructed this example so that the exchange rate is a martingale. Hence, interest rates are driven entirely by movements in the risk premium, and the slope coefficient b in the Fama regression (5) is zero. The example has some qualitative properties that are similar to the data: interest differentials are persistent, and the exchange rate is an order of magnitude more volatile than interest differentials. However, we think of this example as simply illustrating some of the behavior our model can generate, not as being a definitive quantitative analysis of the properties of interest rates and exchange rates.

We choose the processes for the money growth rates in this example in order to ensure that the nominal exchange rate follows a random walk (actually, a martingale). Specifically, we choose these processes so that

$$E_t \log e_{t+1} - \log e_t = E_t(\log m^*_{t+1} - \log m_{t+1}) = 0.$$

Since the pricing kernel in each country is a function of only that country's money growth, we choose these processes so that for the home country $E_t \log m_{t+1} = \log \beta/\bar{\mu}$, where $\log m_{t+1}$ is given by (34); we do likewise for the foreign country. For both the home and foreign countries, we let these baseline processes be of the form

$$(53) \quad \hat{\mu}_{t+1} = g(\hat{\mu}_t) + \varepsilon_{t+1}, \ \hat{\mu}^*_{t+1} = g(\hat{\mu}^*_t) + \varepsilon^*_{t+1}.$$

Because (34) makes $\log m_{t+1}$ a quadratic function in μ_t and μ_{t+1}, the function $g(\cdot)$ that makes the exchange rate a martingale turns out to be quadratic in μ_t. To see this, notice that $g(\cdot)$ is obtained by substituting (50) into (40) and setting the expected depreciation rate to zero. The quadratic equation for $g(\cdot)$ has two solutions; we select the one that implies a mean-reverting process in the sense that $g'(\hat{\mu}_t) = d\left(E_t\mu_{t+1}\right)/d\hat{\mu}_t < 1$ when the derivatives are evaluated at $\hat{\mu}_t = 0$. We let ε_t and ε^*_t both be normal with mean zero and standard deviation σ_ε, and we let the correlation of ε_t and ε^*_t be ρ_ε.

We use the parameter values $\phi = 10$ and $\eta = 1,000$. We think of these values as round numbers that are motivated by those in our earlier example. Here, as before, we assume that a period in the model is a month, and we again let $\bar{\mu}$ correspond to an annualized inflation rate of 5%. We set $\sigma_\varepsilon = .0035$ and $\rho_\varepsilon = .5$. With these parameters, the resulting money growth process of the form (53) is similar to that of an AR(1) process with a serial correlation of .90. To demonstrate this similarity, in Figure 3 we plot 245 realizations of our baseline money growth process (53) and this AR(1) process based on the same driving shocks ε_t.

In Table 1 we report on some properties of exchange rates and interest rates implied by this example and provide, for comparison, some similar statistics from the data. The statistics in the model are computed as the means over 100,000 draws of length 245, whereas those in the data are averages of the statistics for seven European countries presented by Backus, Foresi, and Telmer (2001), each of which has 245 months of data. As the table demonstrates, in the data, changes in the exchange rate are an order of magnitude more volatile than interest differentials. Also, changes in the exchange rate have virtually no serial correlation, whereas interest differentials have a high serial correlation. At a qualitative level, our model successfully reproduces these features of the data.

In our model, by the construction of the function g, the slope coefficient b in the Fama regression would be zero in an infinite sample. We are also interested in what our model implies for this slope coefficient for samples of the length used in the data to estimate it. Figure 4 displays that. The figure is a histogram of 1,000 estimates of the slope coefficients of the Fama regression from simulated samples of length 245, which is the length used by Backus, Foresi, and Telmer (2001). As is evident, our model is consistent with having a wide variety of slope coefficients in small samples, including very negative ones. In addition, note that the mean value for the slope coefficient across the 100,000 draws is -3.69, which is substantially lower than its population value of zero. This indicates the presence of significant small-sample bias.

C. Some of the Model's Cross-Country Implications

So far we have focused on the time series implications of our model. Now we discuss some of its cross-country implications.

A key mechanism at work in our market segmentation model is that as the money growth rate rises, so does the inflation rate; thus, gains from participating in the asset market rise with money growth. As these gains rise, more households choose to be active, and the amount of risk in the economy falls. In an economy with a high enough mean inflation rate, then, risk in the asset market is sufficiently low that the forward premium anomaly disappears. Our model thus implies that the market segmentation effect is smaller in countries with higher inflation rates.

More precisely, if the distribution of fixed costs is bounded and the risk aversion parameter σ is greater than one, then clearly, beyond some sufficiently high inflation rate, all households are active and consumption is constant. Then our model reduces to a standard one similar to that of Lucas (1982), with constant risk premia and no forward premium anomaly.

Some evidence for this cross-country implication has been found by Bansal and Dahlquist (2000). They study a data set for 28 developed and emerging countries and find that the forward premium anomaly is mostly present in the developed countries and mostly absent in the emerging countries. In regressions for their entire data set, Bansal and Dahlquist find that countries with a higher inflation rate tend to have a smaller forward premium anomaly.

7. Long-Term Risk Premia

So far we have discussed our model's ability to account for the forward premium anomaly: in the data a Fama regression of short-term changes in exchange rates on short-term interest differentials tends to produce a zero or negative slope coefficient (at least for low inflation countries). Our model accounts for this anomaly by generating a sufficient amount of time-varying risk over the short term. We now ask whether our model can also account for the evidence from Fama regressions that run long-term changes in exchange rates on long-term interest differentials. As Alexius (2001) and Chinn and Meredith (2004) show, the slope coefficient in a Fama regression tends to be close to one in long-horizon regressions. We show that our model can account for these regressions.

We begin by showing that under fairly general conditions, our model's implications for long-term risk premia are the same as those of a model with no asset market segmentation. In this sense, all the risk that arises in pricing long-term nominal bonds does so solely because of the risk of inflation over the long term. This feature holds because the (real) risk coming from market segmentation does not grow with the time horizon and so matters little in pricing long-term bonds. We then show that if we impose stricter conditions on the long-term behavior of the money growth process, then the long-term risk premia are constant over time. Under these conditions, our model implies that long-term expected depreciation rates move one-for-one with long-term interest differentials. Thus, our model implies that the slope coefficient in a Fama regression of long-term changes in exchange rates on long-term interest differentials is equal to one. In this sense, our model is consistent with the evidence of Alexius (2001) and Chinn and Meredith (2004).

A. Definitions

We begin with some definitions. Define the k-period dollar nominal rate $i_{t,k}$ as

$$i_{t,k} = -\frac{1}{k} \log Q_{t,k},$$

where $Q_{t,k}$ is the price of a zero coupon nominal bond at t paying one dollar at $t+k$. Clearly,

$$Q_{t,k} = E_t \left[\beta^k \frac{U'\big(c_A(\mu_{t+k})\big)}{U'\big(c_A(\mu)\big)} \frac{P_t}{P_{t+k}} \right].$$

Define the k-period euro nominal rate $i^*_{t,k}$ and the price $Q^*_{t,k}$ for a k-period zero coupon euro bond in a similar way. The exchange rate change between t and $t+k$ is then given by

$$\frac{e_{t+k}}{e_t} = \left(\beta^k \frac{U'\big(c_A(\mu^*_{t+k})\big)}{U'\big(c_A(\mu^*_t)\big)} \frac{P^*_t}{P^*_{t+k}} \right) \Big/ \left(\beta^k \frac{U'\big(c_A(\mu_{t+k})\big)}{U'\big(c_A(\mu_t)\big)} \frac{P_t}{P_{t+k}} \right).$$

The long-term risk premium is given by

$$p_{t,k} = E_t \left[\frac{1}{k} \log \frac{e_{t+k}}{e_t} \right] - \left(i_{t,k} - i^*_{t,k} \right),$$

which can be written as

$$p_{t,k} = b_{t,k} - b^*_{t,k},$$

where $b_{t,k}$ represents

$$\text{(54)} \qquad \frac{1}{k} \Bigg(\log E_t \left[U'\big(c_A(\mu_{t+k})\big) \exp(-\sum_{s=1}^{k} \log \mu_{t+s}) \right] - E_t \left[\log U'\big(c_A(\mu_{t+k})\big) \right] + E_t \left[\sum_{s=1}^{k} \log \mu_{t+s} \right] \Bigg)$$

and $b^*_{t,k}$ is analogous.

B. General Conditions and Implications

From (54), we see that the long-term risk premium depends on the variability of long-term inflation and the variability of the level of the marginal utility of active households.

Under two mild regularity conditions, in the long term, the accumulated effect of inflation dominates the component of risk associated with the marginal utility $U'(c_A)$; that is, the long-term premium is independent of the variability of the marginal utility of active households.

One regularity condition is that long-term average inflation (per period) has a finite conditional expectation, in the sense that for all possible histories,

$$\text{(55)} \quad \lim_{k\to\infty} \frac{1}{k} \log E_t \left[\frac{P_t}{P_{t+k}}\right] = \lim_{k\to\infty} \frac{1}{k} \log E_t \left[\exp\left(-\sum_{s=1}^{k} \log \mu_{t+s}\right)\right] < \infty.$$

The other regularity condition is that the consumption of active households is uniformly bounded away from zero, in the sense that

$$\text{(56)} \quad c_A(\mu) \geq \underline{c} > 0 \text{ for all } \mu.$$

Note that the consumption of active households is bounded above by $\bar{c} = y/F(0)$ with $F(0) > 0$.

Now we establish that under these two mild regularity conditions, asset market segmentation does not affect the risk premia in the long term.

Proposition 6. Under (55) and (56), the long-term risk premium in our model is the same as that in a version of our model with no fixed costs and, hence, no asset market segmentation.

Proof. To prove this proposition, we show that in the limit, the expression (54) reduces to

$$\text{(57)} \quad \lim_{k\to\infty} \frac{1}{k} \left(\log E_t \left[\exp\left(-\sum_{s=1}^{k} \log \mu_{t+s}\right)\right] + E_t \left[\sum_{s=1}^{k} \log \mu_{t+s}\right]\right)$$

and, hence, does not depend on the marginal utility of active households at all. Since the consumption of active agents is bounded above and below, so is the marginal utility; thus, $U'(\bar{c}) \leq U'\big(c_A(\mu)\big) \leq U'(\underline{c})$. In the limit, the first bracketed term in (54) satisfies these inequalities:

$$
\begin{aligned}
& \lim_{k\to\infty} \frac{1}{k}\left(\log U'(\bar{c}) + \log E_t\left[\exp\left(-\sum_{s=1}^{k}\log\mu_{t+s}\right)\right]\right) \\
\leq\; & \lim_{k\to\infty} \frac{1}{k}\log E_t\left[U'\big(c_A(\mu_{t+k})\big)\exp\left(-\sum_{s=1}^{k}\log\mu_{t+s}\right)\right] \\
\leq\; & \lim_{k\to\infty} \frac{1}{k}\left(\log U'(\underline{c}) + \log E_t\left[\exp\left(-\sum_{s=1}^{k}\log\mu_{t+s}\right)\right]\right).
\end{aligned}
$$

Since the terms $U'(\bar{c})$ and $U'(\underline{c})$ are constants and (55) holds, it follows that

$$
\begin{aligned}
(58)\qquad & \lim_{k\to\infty} \frac{1}{k}\log E_t\left[U'\big(c_A(\mu_{t+k})\big)\exp\left(-\sum_{s=1}^{k}\log\mu_{t+s}\right)\right] \\
=\; & \lim_{k\to\infty} \frac{1}{k}\log E_t\left[\exp\left(-\sum_{s=1}^{k}\log\mu_{t+s}\right)\right].
\end{aligned}
$$

Clearly, the same bounding argument implies that $\lim_{k\to\infty}(1/k)\log E_t\big[U'\big(c_A(\mu_{t+k})\big)\big] = 0$. *Q.E.D.*

C. Stricter Conditions and Implications

We now turn to the question of whether the long-term risk premium is constant. From (57), we know that the long-term risk premium depends on

$$
(59)\qquad \lim_{k\to\infty} \frac{1}{k}\left(\log E_t\left[\frac{P_t}{P_{t+k}}\right] - E_t\left[\log\frac{P_t}{P_{t+k}}\right]\right).
$$

Hence, a sufficient condition for the long-term risk premium to be constant is that the conditional distribution of long-term average inflation is independent of the conditioning information in period t.

For example, if μ_t is lognormal, then the standard formula for the mean of a lognormal implies that (59) reduces to

$$
(60)\qquad \lim_{k\to\infty} \frac{1}{2k}\mathrm{var}_t\left(-\sum_{s=1}^{k}\log\mu_{t+s}\right).
$$

The expression in (60) will be independent of conditioning information under relatively weak conditions. One such condition is that $\log\mu_t$ follows a covariance stationary process that has a finite spectral density at frequency zero.

For another example, suppose that $\log \mu_t$ follows a Markov chain. If the Markov chain has strictly positive transition probabilities, then the long-term risk premium is constant. Moreover, that premium can be computed, as we show in the next proposition.

Consider an n state Markov chain, where the transition probabilities are $\pi_{ij} = \Pr\{\log \mu_{t+1} = \log \mu_j \mid \log \mu_t = \log \mu_i\}$. We prove the following proposition in Appendix D:

Proposition 7. If the transition probabilities π_{ij} are all positive, then the long-term risk premium, defined as $\lim p_{t,k}$ as $k \to \infty$, is a constant given by

$$\lim_{k\to\infty} p_{t,k} = [\delta - E(\log \mu)] - [\delta^* - E(\log \mu^*)],$$

where δ is the log of the largest positive (dominant) eigenvalue of the matrix with entries

$$A_{ij} = \mu_j \pi_{ij}$$

for $i, j = 1, 2, \ . \ . \ . \ , n$, and $E(\log \mu)$ is the unconditional expectation of the log of the money supply, and the foreign country variables are similarly defined.

As discussed by Hansen and Scheinkman (2005), the quantity δ measures the importance of long-term risk in pricing risky assets. This quantity has the dimension of the yield on a long-term return. The intuition for why these long-term premia are constant is that if all the transition probabilities are positive, then there is sufficient mixing so that households' expectations of long-term average inflation rates do not depend on the initial state.

8. Conclusion

We have constructed a simple, general equilibrium monetary model with endogenously segmented asset markets and shown that this sort of friction is a potentially important part of a complete model of interest rates and exchange rates. The fundamental problem behind this exercise is to develop a model in which exchange rates are roughly a random walk (so that expected changes in exchange rates are roughly constant) while interest differentials are highly variable and persistent. In such a model, by definition, time-varying risk must be the prime mover of interest differentials. Our main contribution here is to highlight the economic forces in a model with endogenous market segmentation that allow the model to produce these features of interest rates and exchange rates. We argue that such asset market segmentation may belong in a complete model of interest rates and exchange rates.

Appendix A: An Extension with Trade in Goods

In the work above, we have kept the model simple by abstracting from the possibility of trade in goods. Here we sketch out a version of the model with trade in final goods that works similarly to the original model.

Let there be two goods h and f, referred to as *home* and *foreign* goods. Households in the home country have endowments y_h and y_f of these goods, while households in the foreign country have endowments y_h^* and y_f^*. Home households have an additively separable period utility function over these goods

$$\alpha U(c_h) + (1-\alpha) U(c_f),$$

where $\alpha \in (0,1]$ and where (c_h, c_f) denotes the consumption of the h and f goods by the home household. Foreign households have a similar period utility function

$$\alpha U(c_f^*) + (1-\alpha) U(c_h^*),$$

where (c_h^*, c_f^*) denotes the consumption of the h and f goods. When $\alpha \geq 1/2$, preferences exhibit a type of home bias: home country households consume relatively more home goods, and foreign households consume relatively more foreign goods.

Home goods must be purchased with home currency and foreign goods with foreign currency. Specifically, households in each country have one cash-in-advance constraint for purchases on home goods and one for purchases of foreign goods. Home households have one fixed cost γ that applies to each transfer of home currency between the home goods market and the asset market, and a separate fixed cost γ^* that applies to each transfer of foreign currency between the foreign goods market and the asset market. Home households are indexed by (γ, γ^*), which we assume have joint distribution given by $F(\gamma)F^*(\gamma^*)$. Foreign households are indexed by a symmetric distribution of costs: $F(\gamma)$ for transfers between foreign goods markets and the asset market and $F^*(\gamma^*)$ for transfers between home goods markets and the asset market.

In the model, households now have more options of participation in the goods and asset markets. They can transfer only home currency, only foreign currency, both currencies, or none at all. For these different patterns of transfer, the home households will pay $\gamma, \gamma^*, \gamma + \gamma^*$, and 0, respectively, while the foreign households will pay $\gamma^*, \gamma, \gamma^* + \gamma$, and 0, respectively. It can be shown that the equilibrium allocations of home goods solve the following planning problem that is the obvious generalization of the one in the simple model:

$$\max_{c_{hA}, c^*_{hA}, \bar{\gamma}_h, \bar{\gamma}^*_h} \alpha U(c_{hA}) F(\bar{\gamma}_h) + \alpha U(y_h/\mu)[1 - F(\bar{\gamma}_h)]$$

$$+ (1 - \alpha) U(c^*_{hA}) F^*(\bar{\gamma}^*_h) + (1 - \alpha) U(y^*_h/\mu)[1 - F^*(\bar{\gamma}^*_h)]$$

subject to

$$c_{hA} \; F(\bar{\gamma}_h) + \int_0^{\bar{\gamma}_h} \gamma f(\gamma) d\gamma + [1 - F(\bar{\gamma}_h)] y_h/\mu$$

$$+ c^*_{hA} \; F^*(\bar{\gamma}^*_h) + \int_0^{\bar{\gamma}^*_h} \gamma f^*(\gamma) d\gamma + [1 - F^*(\bar{\gamma}^*_h)] y^*_h/\mu = y_h + y^*_h.$$

Here we denote the consumption of home goods by the home and foreign households by c_{hA} and c^*_{hA}. We also denote the cutoff values for transferring home currency to the home goods market by the home and foreign households by $\bar{\gamma}_h$ and $\bar{\gamma}^*_h$. The equilibrium allocations of foreign goods solve a similar problem.

The solution to the problem for home goods is similar to the earlier problem in which goods are not tradable. The link between money injections and households' marginal utilities is also similar. The key distinctions between the model with and without tradables are as follows. Here all active households equate their marginal utilities; hence, the consumption of home goods of home and foreign active households moves together. If a home household does not make a transfer of home currency, then the home consumption of home goods is $c_h = y_h/\mu$,. If a foreign household does not make a transfer of home currency, then its consumption of the home good is $c^*_h = y^*_h/\mu$. Hence, the value of making a transfer of home currency for a home household differs from that of making one for a foreign household. Likewise, the cost of making such a transfer is drawn from $F(\gamma)$ for a home household and from $F^*(\gamma^*)$ for a foreign household. Because of these differences in the value and costs of making transfers, in general, the home households have a cutoff function for transfers of home currency $\bar{\gamma}_h(\mu)$ which differs from the cutoff that foreign households have for transfers of home currency $\bar{\gamma}^*_h(\mu)$. A similar distinction holds with respect to foreign currency transfers.

Consider now a utility function of the form $U(c) = c^{1-\sigma}/(1-\sigma)$. It is easy to show that the optimal allocations $\{c_{hA}(\mu),\ c^*_{hA}(\mu),\ \bar{\gamma}_h(\mu),\ \bar{\gamma}^*_h(\mu)\}$ are increasing functions of μ and that the consumption of home goods of the home and foreign active households are proportional, $c^*_{hA}(\mu) = \omega\ c_{hA}(\mu)$, where $\omega = [(1-\alpha)/\alpha]^{\frac{1}{\sigma}}$.

Next we present a proposition in which the determination of the active households' consumption and cutoff function is identical to that in the model with nontradable goods.

Proposition 8. Assume that endowments satisfy

$$(61) \quad y^*_h / y_h = \omega$$

and that the upper bound of the support for γ, denoted by $\gamma_{\max}$, satisfies $F^*(\omega\gamma_{\max}) = 1$. Then $\bar{\gamma}_h(\mu)$ and $\bar{\gamma}^*_h(\mu)$ satisfy $\bar{\gamma}^*_h(\mu) = \omega\ \bar{\gamma}_h(\mu)$, and the values of $c_{hA}(\mu)$ and $\bar{\gamma}_h(\mu)$ are identical to those in an economy with no tradable goods, an aggregate endowment of $y_h + y^*_h$, and a distribution of costs given by

$$\tilde{F}(\gamma) = \frac{F(\gamma) + \omega F^*(\omega\gamma)}{1+\omega}.$$

The proof of this proposition follows from verifying that the candidate solution satisfies the first-order conditions of the problem stated above. The assumption (61) includes the case of completely symmetric countries, $\alpha = 1/2$ and $y_h = y^*_h$, but it is more general. In particular, this assumption allows for a type of nationalistic bias preference of $\alpha \geq 1/2$ and specialization in the endowments in the sense of $y_h > y^*_h$. The nationalistic bias implies that $c_h \geq c_f$ at $\mu = \mu^*$. This assumption implies that for $\mu = 1$, exports are zero, since $c_h = y_h$. For $\mu > 1$, however, there typically will be trade in equilibrium, provided that F and F^* differ.

When assumption (61) is not satisfied, $\bar{\gamma}_h$ and $\bar{\gamma}^*_h$ move together with μ, but they are not necessarily proportional; hence, the expression of $c_{hA}(\mu)$ does not reduce exactly to that of the model with no tradable goods. Nevertheless, the expressions for c_{hA} and $\bar{\gamma}_h$ are similar to those obtained in that model. To see why, consider the extreme case in which $y^*_h = 0$, so that the foreign country has no endowment of the home good. In this case, under appropriate conditions, all foreign households engage in transfers of home currency, so that $\bar{\gamma}^*_h(\mu) = \gamma^*_{\max}$. The resulting expressions for $c_{hA}(\mu)$ and $\bar{\gamma}_h(\mu)$ correspond to those for the model with no tradable goods, the cost functions $\tilde{F}(\gamma) = [F(\gamma) + \omega] / (1 + \omega)$ and $\tilde{F}(0) = \omega \ / (1 + \omega)$, the consumption of inactive home households y_h/μ, and the aggregate endowment $[y_h - \int \gamma^* dF^*(\gamma^*)] / (1 + \omega)$. *Q.E.D.*

Appendix B: Proof of Uniqueness

Here we show that equations (18) and (19) have at most one solution for any given μ.

To see this result, solve for $\bar{\gamma}$ as a function of c_A from (19) and suppress explicit dependence of μ to get

$$\bar{\gamma}(c_A) = \frac{U(c_A) - U(y/\mu)}{U'(c_A)} - [c_A - (y/\mu)] \,.$$

Note that

$$\text{(62)} \quad \frac{d\bar{\gamma}(c_A)}{dc_A} = -\frac{U''(c_A)}{[U'(c_A)]^2}[U(c_A) - U(y/\mu)].$$

Use (19) to see that $d\bar{\gamma}(c_A)/dc_A$ is positive when $c_A + \bar{\gamma} - (y/\mu) > 0$ and negative when $c_A + \bar{\gamma} - (y/\mu) < 0$. Substituting $\bar{\gamma}(c_A)$ into (18) and differentiating the left side of the resulting expression with respect to c_A gives

$$\text{(63)} \quad F\big(\bar{\gamma}(c_A)\big) + [c_A + \bar{\gamma}(c_A) - y/\mu] \, \frac{d\bar{\gamma}(c_A)}{dc_A}.$$

Using (62), we see that (63) is strictly positive; hence, the equations have at most one solution. *Q.E.D.*

Appendix C: Proof of Proposition 4

To prove Proposition 4, we derive two equations, (38) and (39).

To derive (38), start with (37), the risk premium defined in terms of the pricing kernels. Compute $E_t \log m_{t+1}$ from (34). To compute $\log E_t m_{t+1}$, we must compute

$$\log E_t \exp\left(\left[-(\phi + 1) + \eta E_t \hat{\mu}_{t+1}\right] \varepsilon_{t+1} + \frac{\eta}{2} \varepsilon_{t+1}^2\right).$$

To do that, use the result that if x is normally distributed with mean zero and variance σ^2 and satisfies $1 - 2b\sigma^2 > 0$, then

$$\text{(64)} \quad E \exp\left(ax + bx^2\right) = \exp\left(\frac{1}{2} \frac{a^2\sigma^2}{(1 - 2b\sigma^2)}\right) \left(\frac{1}{1 - 2\sigma^2 b}\right)^{1/2}.$$

To derive (64), note that

$$E\exp\left(ax+bx^2\right)=\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(ax+bx^2\right)\exp\left(-\frac{x^2}{2\sigma^2}\right)dx=$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left[2\sigma^2ax+\left(2\sigma^2b-1\right)x^2\right]\right)dx=$$

$$\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(\frac{1}{2\sigma^2}\left[-\left(1-2\sigma^2b\right)x^2+2\sigma^2ax-\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)+\left(\frac{\sigma^4a^2}{1-2\sigma^2b}\right)\right]\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{(1-2b\sigma^2)}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{1}{2\sigma^2}\left[\left(1-2\sigma^2b\right)^{1/2}x-\frac{\sigma^2a}{(1-2\sigma^2b)^{1/2}}\right]^2\right)dx=$$

$$\exp\left(\frac{1}{2}\frac{a^2\sigma^2}{(1-2b\sigma^2)}\right)\frac{1}{\sigma\sqrt{2\pi}}\int\exp\left(-\frac{(1-2\sigma^2b)}{2\sigma^2}\left[x-\frac{\sigma^2a}{(1-2\sigma^2b)}\right]^2\right)dx,$$

which equals (64).

We can derive (39) using (34) together with the standard results that $E_t\varepsilon_{t+1}^4=3\sigma_\varepsilon^4$ and $E_t\varepsilon_{t+1}^3=0$. *Q.E.D.*

Appendix D: Proof of Proposition 7

Here we prove Proposition 7.

Note that $E[\mu_{t+1}|\mu_t=\mu_j]$ and $E[\Pi_{s=1}^k\mu_{t+s}|\mu_t=\mu_j]$ are the jth elements of the vectors $A\overrightarrow{1}$ and $A^k\overrightarrow{1}$, where $\overrightarrow{1}$ is a column vector of n ones. Let $\lambda_1>\lambda_2\geq\ldots\geq\lambda_n$ be the eigenvalues of A; let $f_1,f_2,\ldots,f_n$ be the corresponding eigenvectors; and let f_{ij} be the jth element of the ith eigenvector. Since A is a strictly positive matrix, by the Perron-Frobenius theorem, $\lambda_1>0$, $|\lambda_1|>|\lambda_i|$ for $i>1$, and $f_{1j}>0$ for all j.

Since the eigenvectors of A form an orthogonal basis for R^n, we have that $\overrightarrow{1}=\sum_{i=1}^n\alpha_if_i$, where $\alpha_i=\ <f_i,\overrightarrow{1}>/<f_i,f_i>$. Since $f_1>0$, $\alpha_1>0$. Then

$$\lim_{k\to\infty}\frac{1}{k}\log E[\Pi_{s=1}^k\mu_{t+s}|\mu_t=\mu_j]=\lim_{k\to\infty}\frac{1}{k}\log\left(\sum_{i=1}^n\alpha_i\lambda_i^kf_{ij}\right).$$

We can rewrite this expression and then use the fact that λ_1 is the dominant eigenvalue to compute this limit:

$$\lim_{k\to\infty}\frac{1}{k}\log\left(\lambda_1^k\sum_{i=1}^n\alpha_i\left(\frac{\lambda_i}{\lambda_1}\right)^kf_{ij}\right)=\lim_{k\to\infty}\frac{1}{k}\log\left(\alpha_1\lambda_1^kf_{1j}\right)=\log\lambda_1.$$

Note that this limit is independent of the current state $\mu_t=\mu_j$. Hence,

$$\lim_{k\to\infty}\frac{1}{k}b_{t,k}=\log\lambda_1-E\log\mu,$$

and a similar expression holds for the foreign country variables.

Notes

[1]Technically, p_t is simply the log of the excess return on foreign currency bonds. In general, this excess return could arise for many reasons, including differences in taxes, liquidity services, or transaction costs across bonds. We take the view here that the fluctuations in this excess return are driven primarily by risk; hence, we refer to the excess return as the *risk premium*.

[2]Our economy can be interpreted as the limit of a sequence of economies in which the amount of goods traded goes to zero.

[3]The forward premium anomaly can also be stated in terms of forward exchange rates. The forward exchange rate f_t is the price specified in a contract in period t in which the buyer has the obligation to transfer f_t dollars in $t+1$ in exchange for one euro. The forward premium is then the forward rate relative to the spot rate f_t/e_t. Arbitrage implies that $\log f_t - \log e_t = i_t - i_t^*$. Thus, (3) can be restated as $\text{cov}(\log f_t - \log e_t, \log e_{t+1} - \log e_t) < 0$. The forward premium and the expected change in exchange rates, therefore, tend to move in opposite directions. This observation contradicts the hypothesis that the forward rate is a good predictor of the future exchange rate.

[4]Variants of this model can be considered in which the fixed cost for each household varies randomly over time. As will be clear from what follows, for the appropriate set of sufficient conditions, the cash-in-advance constraints would always bind in those variants, and the equilibrium would be identical to that which we discussed.

[5]Although this condition is intuitive, the problem's nonconvexity requires that its proof be more than just a verification of the relevant first-order condition. For the formal treatment of a similar problem, see the appendix in our earlier (2002) work.

[6]Technically, a sufficient condition for the variability of the risk premium, $\text{var}(p_t)$, to be increasing in ϕ, η, and σ_ε^2 is that $\text{cov}\big(E_t\hat{\mu}_{t+1}, (E_t\hat{\mu}_{t+1})^2\big) \leq 0$. This inequality holds with equality if the distribution of $E_t\hat{\mu}_{t+1}$ is symmetric, as, for example, is true with normally distributed variables.

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Table 1
Data and Model Properties of Exchange Rates and Interest Rates
Monthly

Statistic/Variable	Value: Data	Value: Model
Standard Deviations (in percent)		
Exchange Rates $(\log e_{t+1} - \log e_t)$	3.0	4.8
Interest Rates $(i_t - i_t^*)$	.3	.1
Autocorrelations		
Exchange Rates $(\log e_{t+1} - \log e_t)$	.04	0
Interest Rates $(i_t - i_t^*)$	.83	.92

Note: The data values are based on Backus, Foresi, and Telmer's (2001) monthly values on U.S. and euro currencies from July 1974 through November 1994. Both the data and the model are monthly. The standard deviations are then multiplied by 100 to express them in percentage points.

Figure 1 **Timing in the Two Markets for a Household in the Home Country**

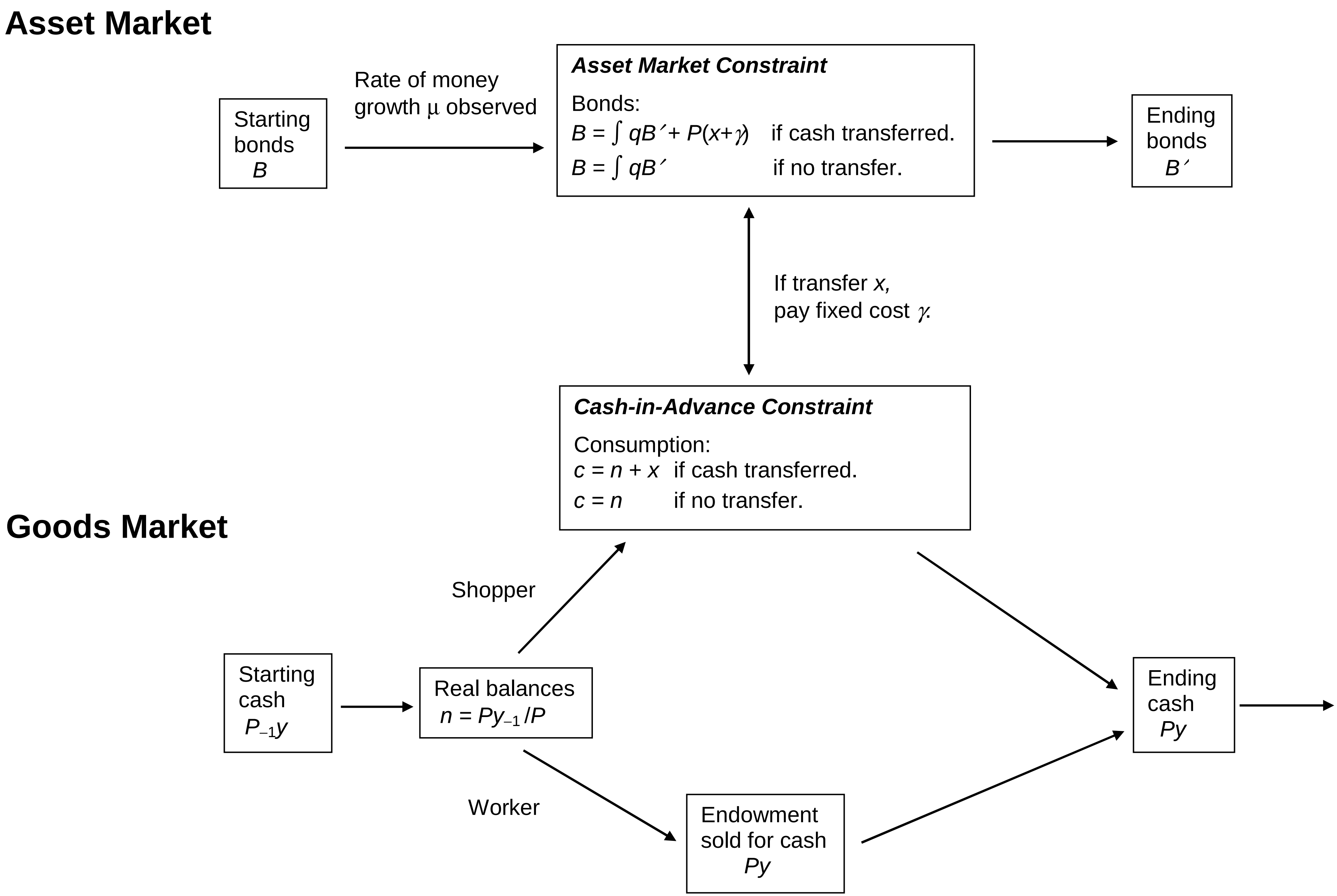


Figure 2 **The Log of the Consumption of Active Households**

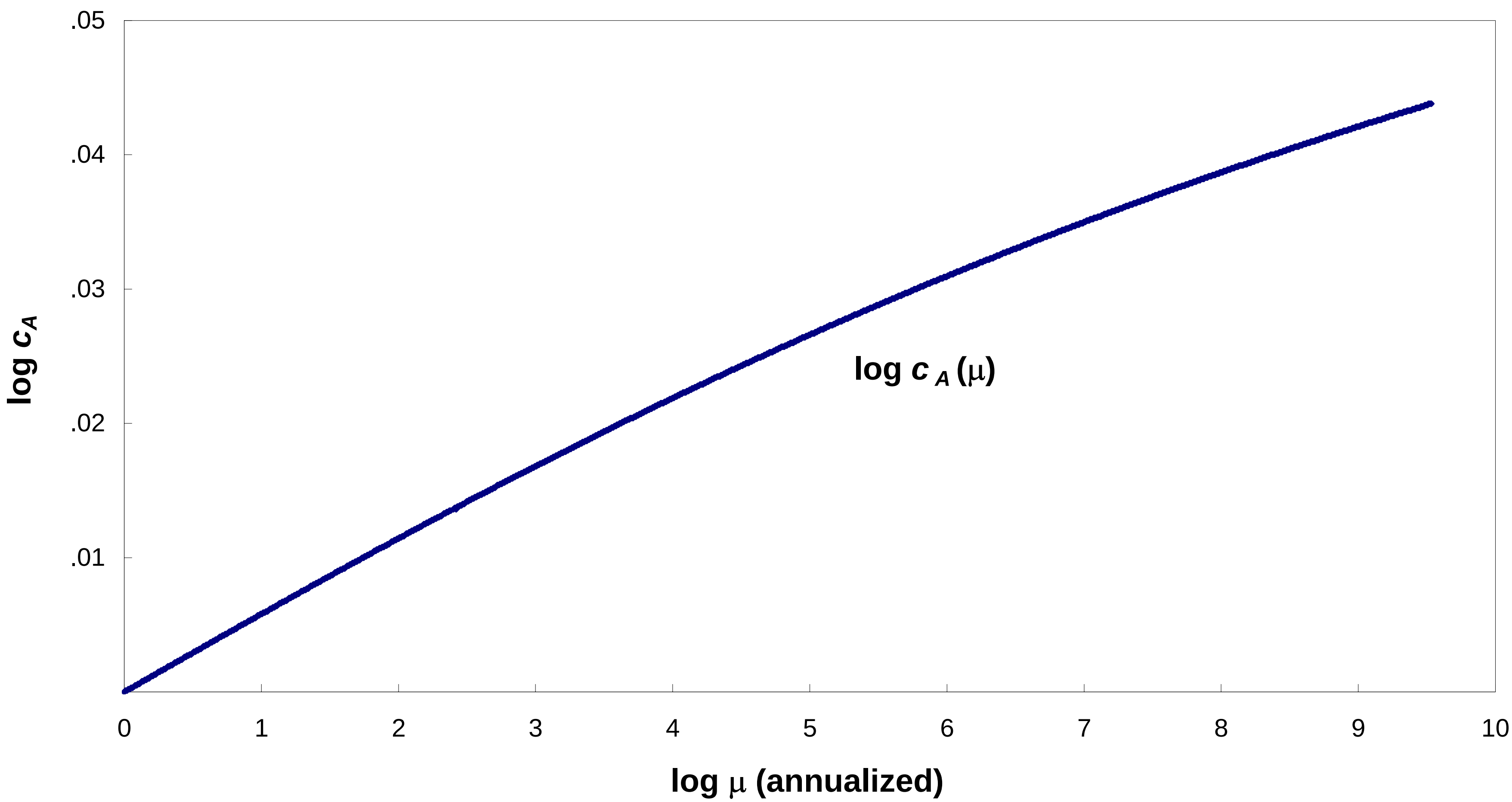


Figure 3 **Realizations of Money Growth Using Our Baseline Process and an AR(1) Process**

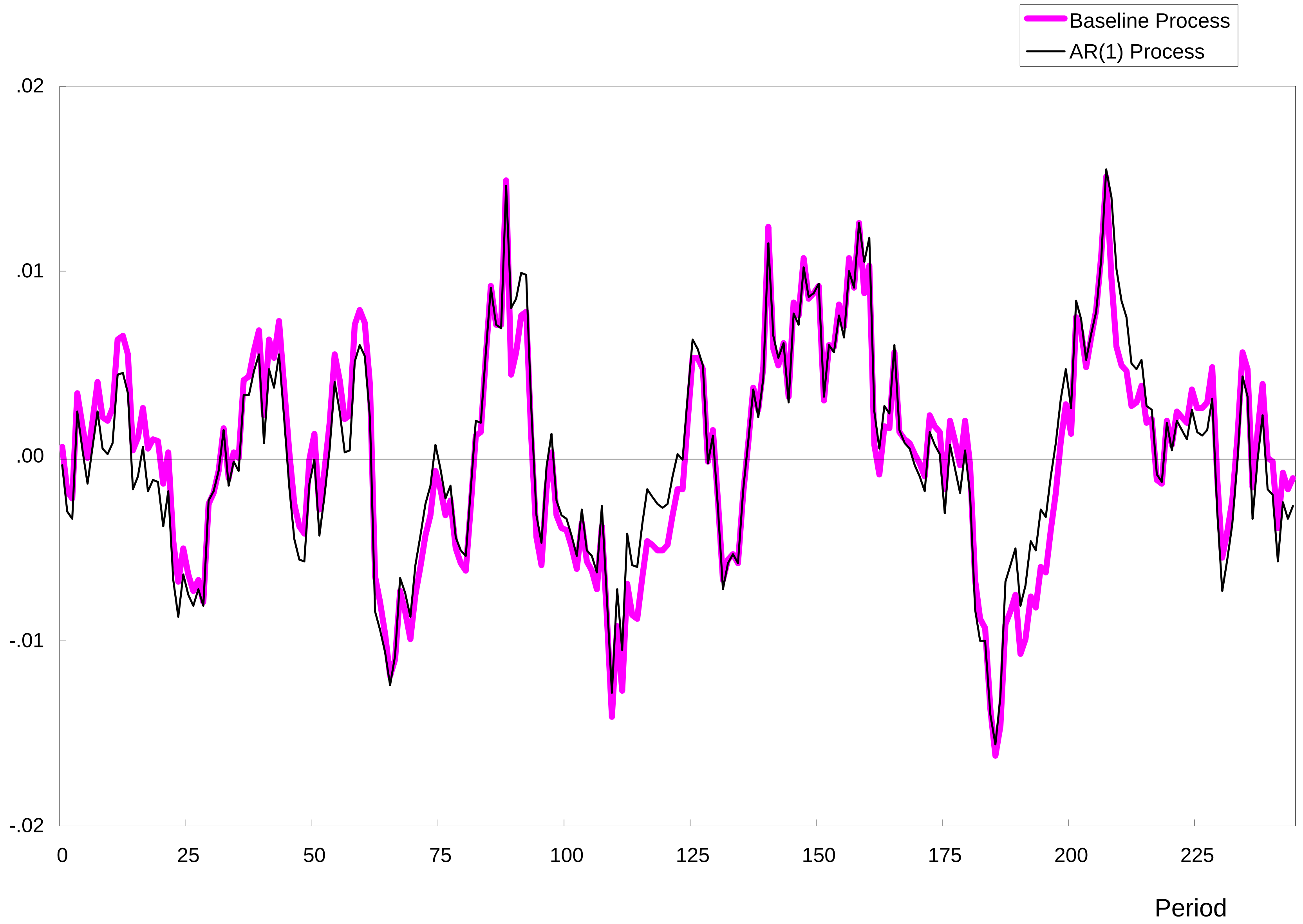
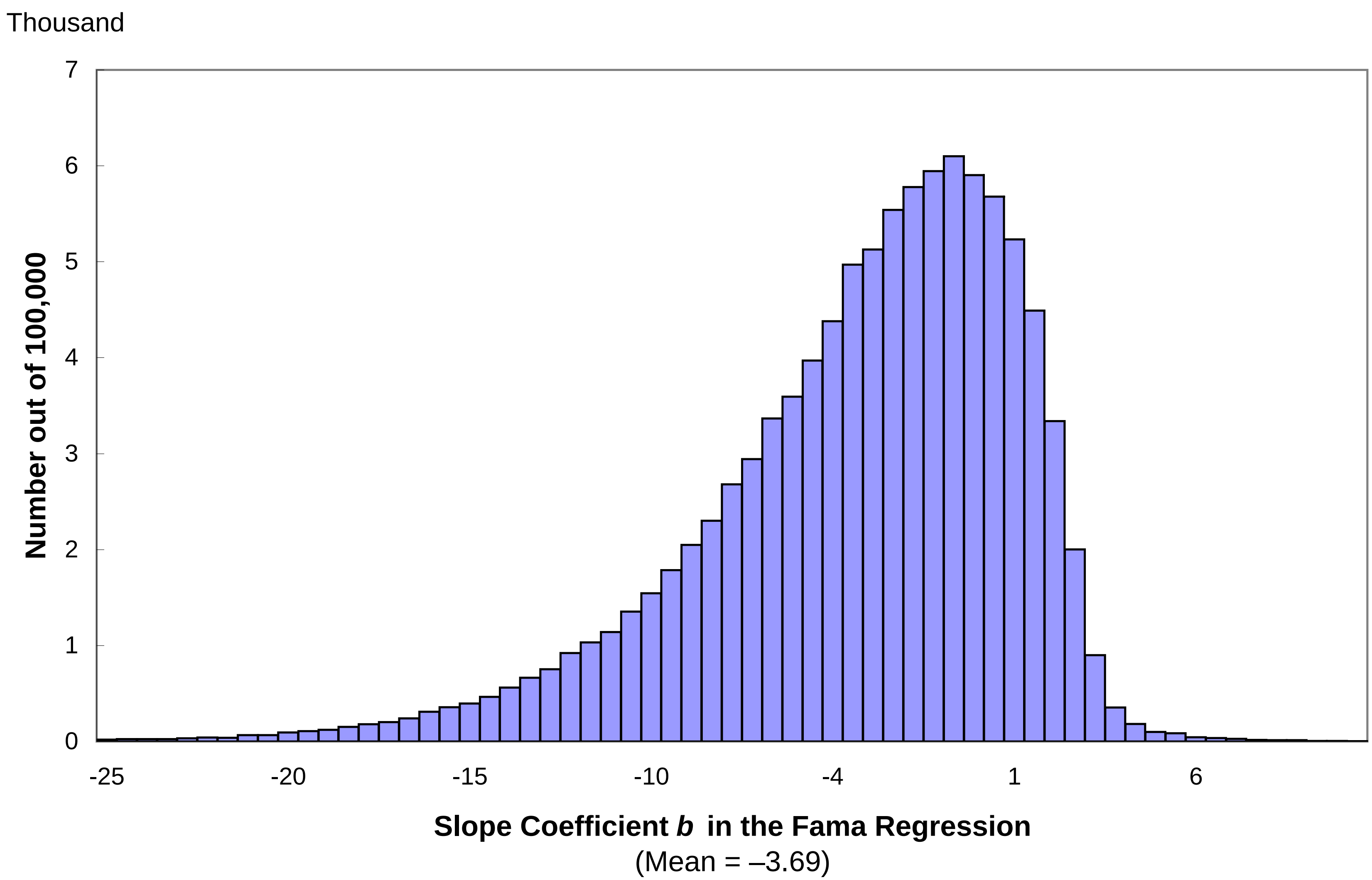


Figure 4 **What the Model Implies for the Slope Coefficient in the Fama Regression**



Note: Each simulation is of length 245 using the parameters in Table 1.