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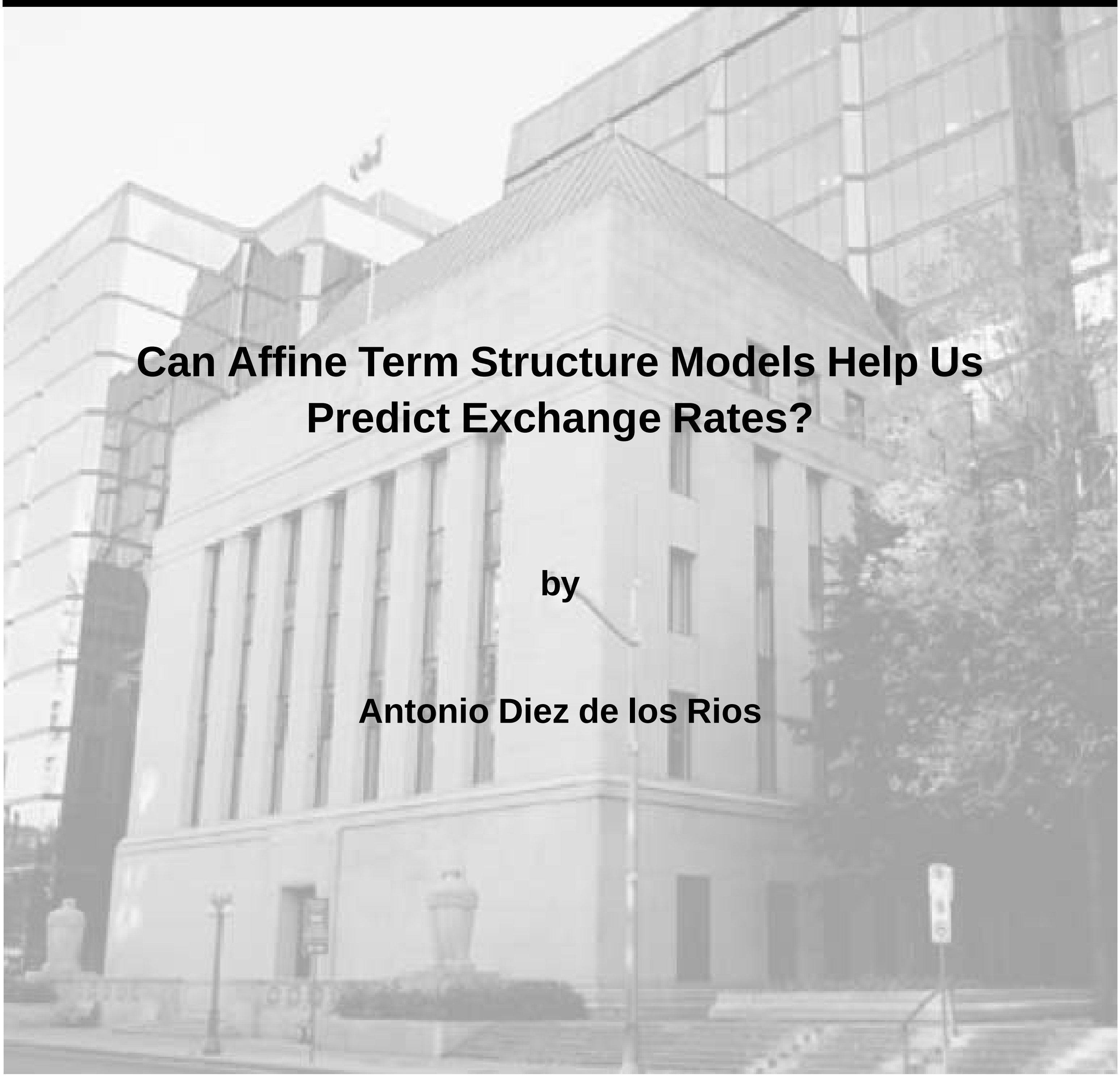

# Can Affine Term Structure Models Help Us Predict Exchange Rates?

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# Abstract

The author proposes an arbitrage-free model of the joint behaviour of interest and exchange rates whose exchange rate forecasts outperform those produced by a random-walk model, a vector autoregression on the forward premiums and the rate of depreciation, and the standard forward premium regression. In addition, the model is able to reproduce the forward premium puzzle.

*JEL classification: E43, F31, G12, G15*
*Bank classification: Exchange rates; Interest rates; Econometric and statistical methods*

# Résumé

L'auteur propose un modèle qui formalise le comportement conjoint des taux d'intérêt et des taux de change en l'absence de possibilités d'arbitrage. Son modèle permet de mieux prévoir l'évolution du taux de change qu'un modèle de marche aléatoire, un modèle vectoriel autorégressif (appliqué aux primes de terme et au taux de dépréciation) ou un modèle de régression classique du taux de change sur la prime de terme. Il a pour autre avantage d'expliquer l'énigme de la prime de terme.

*Classification JEL : E43, F31, G12, G15*
*Classification de la Banque : Taux de change; Taux d'intérêt; Méthodes économétriques et statistiques*

# 1. Introduction

The objective of this paper is to improve the ability to predict exchange rates. In particular, I present a model where restrictions derived from the assumption of no arbitrage are imposed on the joint behaviour of interest and exchange rates. In this model, prediction is based on the information embedded in interest rate differentials. An example of such information is the well-established fact that regressing ex post rates of depreciation of a given currency on a constant and the interest rate differential usually delivers a slope coefficient that is negative (see Hodrick 1987 and Engel 1996). This phenomenon is known as the "forward premium puzzle," and it implies that currencies where domestic interest rates are high relative to those in the foreign country tend to appreciate. In fact, Clarida and Taylor (1997), using a linear vector-error-correction model (VECM) framework for the term structure of forward premiums (interest rate differentials), are able to beat the long-standing and devastating result found by Meese and Rogoff (1983a,b) that standard empirical exchange rate models cannot outperform a simple random-walk forecast. Thus, interest rates across countries contain information that is useful to predict exchange rates.

However, the VECM framework is based solely on the time-series properties of interest and exchange rates. It does not take into account that, for example, a deposit denominated in foreign (domestic) currency is risky (due to exchange rate variability) for domestic (foreign) investors, and that therefore investors will demand compensation for bearing such risk. As a result, movements in interest and exchange rates must be related in such a way that they preclude the existence of arbitrage opportunities.

This paper investigates whether imposing such a set of restrictions in the estimation of the joint dynamics of interest and exchange rates marks an improvement over the Clarida and Taylor (1997) framework. In principle, imposing cross-equation restrictions will reduce the large number of parameters that characterize traditional time-series models and, consequently, it will reduce excessive parameter estimation uncertainty that may adversely affect its out-of-sample forecasting performance. This insight is confirmed by Duffee (2002) and Ang and Piazzesi (2003), who find that imposing no-arbitrage restrictions helps predict bond yields out-of-sample.

For the sake of tractability, the focus of this paper is on internationally affine term structure models; that is, models where not only interest rates (yields) are affine known functions of a set of state variables, but also the expected rate of depreciation (over any arbitrary period of time) satisfies this property. The main benefit of focusing on this class of models is that I avoid the use of Monte Carlo methods to compute the expected rate of depreciation. Although this is a perfectly valid approach (Dong 2005), the use of Monte Carlo methods in this set-up can be computationally costly, because the model is re-estimated at each point in time (of the out-of-sample period) in order to compute the corresponding dynamic forecasts. A first contribution of this paper is to provide conditions to obtain an expected rate of depreciation that is affine on the set of state variables. This paper shows that if the drift of the process that the log exchange rate follows is affine in a set of state variables and these state variables follow an affine diffusion, then the expected rate of depreciation (over any arbitrary period of time) will be affine. In addition, the expressions obtained for the expected rate of depreciation are exact and, therefore, they are not subject to discretization biases.

Two main families of affine dynamic term structure models are shown to satisfy these conditions. The first subgroup is the so-called completely affine term structure model introduced in Dai and Singleton (2000). It covers most of the work done on international term structure modelling: e.g., see Saa-Requejo (1993), Frachot (1996), Backus, Foresi, and Telmer (2001), Dewachter and Maes (2001), Hodrick and Vassalou (2002), and Ahn (2004). However, Backus, Foresi, and Telmer (2001) show that the specification of the prices of risk in these models constrains the relationship between interest rates and the risk premium in such a way that the ability to reproduce the forward premium puzzle, and, therefore, their ability to capture the properties of interest and exchange rates, is severely limited. The second group that falls within the internationally affine framework is the quadratic-Gaussian class of term structure models introduced in Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2002). I show in this paper that these models satisfy the conditions for an affine expected rate of depreciation once they are thought of as being "affine" in the original set of factors and their respective squares and cross-products. The multi-country models of the term structure in Leippold and Wu (2003) and Inci and Lu (2004) belong to this category. It is also worth mentioning that Gaussian essentially affine models can be viewed as a particular case of the quadratic case where interest rates are affine but the expected rate of depreciation is quadratic. Therefore, the models in Brennan and Xia (2004) and Dong (2005) also belong to the internationally affine class. Another model that does not fall between these two families but that generates affine interest rates and an affine expected rate of depreciation is the one in Graveline (2005).

Still, the main disadvantage of an internationally affine model is that tractability comes at the price of imposing more restrictions on top of the assumption of no arbitrage. The results obtained in this paper suggest, however, that this is not the case for two currencies: the

U.S. dollar – pounds sterling and the U.S. dollar – Canadian dollar. A two-factor Gaussian essentially affine model produces forecasts that are superior, on the basis of root-mean-square error (RMSE) and mean-absolute-error (MAE) criteria, to those produced by the random-walk model and the Clarida and Taylor (1997) approach. I find that imposing no-arbitrage restrictions reduces the RMSE in forecasting the spot U.S. dollar – pounds sterling rate by around 35 per cent at the one-year forecast horizon relative to the VECM approach, and by around 15 per cent for the case of the U.S. dollar – Canadian dollar. I also find that the gains (if any) from using a linear VECM model with respect to the use of a random-walk model are small. For example, the gain at the one-year horizon for the U.S. dollar – pounds sterling is only 2.4 per cent (versus the 40 per cent reported by Clarida and Taylor 1997). In addition, the model is able to reproduce the forward premium anomaly.

This paper is organized as follows. Section 2 describes the data. Section 3 introduces the concept of an internationally affine term structure model and discusses several specifications that fall within this framework. Section 4 presents the empirical exercise. Section 5 concludes.

## 2. Data

The data set comprises monthly observations over the period January 1976 to December 2004 of U.S. dollar – pounds sterling and U.S. dollar – Canadian dollar rates of depreciation, along with the corresponding American, British, and Canadian Eurocurrency interest rates of maturities one, three, six, and twelve months. These Eurocurrency deposits are essentially zero-coupon bonds whose payoffs at maturity are the principal plus the interest payment. Exchange rate (expressed as U.S. dollars per unit of foreign currency) and Eurocurrency interest rate data are obtained from Datastream. However, the estimations are carried out using data only over the period January 1976 to December 1997, in order to reserve the last seven years for an out-of-sample forecasting exercise.

Table 1 reports summary statistics for these variables. Following Bekaert and Hodrick (2001), all variables are measured in percentage points per year, and the monthly rates of depreciation are annualized multiplying by 1,200. Note that the rates of depreciation have lower means (in absolute value) than the ones corresponding to the interest rates, but, on the contrary, interest rates are less volatile. In addition, interest rates display a high level of autocorrelation, while the expected rates of depreciation do not. The rate of depreciation of the U.S. dollar with respect to the Canadian dollar is less volatile than the rate of depreciation of the U.S. dollar with respect to pounds sterling. The (average) spread between the one-year and the one-month interest rate is positive for the case of the United States, while negative for the case of the United Kingdom and Canada. Finally, the United Kingdom ranks first in terms of the highest (average) level of interest rates during the sample period. Canada and the United States rank second and third, respectively. These properties are consistent with previous studies such as Backus, Foresi, and Telmer (2001) and Bekaert and Hodrick (2001).

Panel a of Table 2 presents the results of the estimation of the forward premium regressions for the U.S. dollar – pounds sterling and the U.S. dollar – Canadian dollar for the four different maturities available. These are ordinary least squares (OLS) regressions of the ex post rate of depreciation on a constant and the forward premium:

$$s_{t+h} - s_t = a + bp_t^{(h)} + u_{t+h}, \tag{1}$$

where $s_t$ is the logarithm of the spot exchange rate $S_t$ (i.e., dollars per pound); $p_t^{(h)} = f_t^{(h)} - s_t$ is the forward premium and $f_t$ is the logarithm of the forward rate $F_t^{(h)}$ contracted at time $t$ and that matures at $t + h$. The uncovered interest parity (UIP) states that, under risk neutrality, the nominal expected return to speculation in the forward foreign exchange market conditional on the available information must be equal to zero:

$$E_t \left[s_{t+h} - s_t\right] = f_t^{(h)} - s_t, \tag{2}$$

and, therefore, it implies that if I run a regression such as the one in equation (1), then the constant term should be found to be equal to zero while the slope is equal to one, that is, $a = 0$ and $b = 1$. Moreover, notice that this hypothesis implies that the (log) forward exchange rate is an unbiased predictor of the $h$-periods-ahead (log) spot exchange rate. This property has motivated another name for the uncovered interest parity: the "unbiasedness hypothesis." Most often, the uncovered interest parity is stated in terms of the interest rate differential between two countries. In particular, the covered interest parity states that the forward premium is equal to the interest rate differential between two countries: $f_t^{(h)} - s_t = r_t^{(h)} - r_t^{*(h)}$, where $r_t^{(h)}$and $r_t^{*(h)}$ are the $h$-period interest rates on a deposit denominated in domestic and foreign currency, respectively.

For the case of the pounds sterling, the data set implies a slope equal to -1.84 when considering a contract maturity of one month, -1.50 for three-month contracts, -1.36 for six-month contracts, and -0.82 for one-year contracts. As for the case of the Canadian dollar, the slope is -1.35, -0.83, -0.43, and -0.24 for one-, three-, six-, and twelve-month contracts, respectively. Moreover, it is possible to reject statistically the equality of these slopes to one on a maturity-by-maturity basis (Table 2 panel a), as well as using a joint test that the four coefficients are equal to one (Table 2 panel b).

As previously noted, this result is inconsistent with the uncovered interest parity, and it has been claimed that the main reason for this rejection lies in the fact that agents are not risk-neutral. This idea goes back to the influential work of Fama (1984), who shows that if certain conditions are met then the forward premium puzzle can be explained by the existence of rational (time-varying) risk premia in foreign exchange markets. To illustrate his argument, I start by the so-called Fama decomposition of the forward premium into an expected rate of depreciation and a risk premium component:

$$\underbrace{f_t - s_t}_{p_t^{(h)}} = \underbrace{E_t\left[s_{t+h} - s_t\right]}_{q_t^{(h)}} + \underbrace{f_t - E_t s_{t+1}}_{d_t^{(h)}}, \tag{3}$$

where $q_t^{(h)}$ is the expected rate of depreciation between time $t$ and time $t+h$, $p_t^{(h)}$ is the forward premium, and $d_t^{(h)}$ is the risk premium in Fama's terminology.

Using the law of iterated expectations and substituting this decomposition of the forward premium into the definition of the uncovered interest parity regression slope in equation (1), I obtain:

$$b(h) = \frac{Cov[q_t^{(h)}, p_t^{(h)}]}{Var[p_t^{(h)}]} = \frac{Var[q_t^{(h)}] + Cov[q_t^{(h)}, d_t^{(h)}]}{Var[p_t^{(h)}]},$$

where I write the slope as a function of the maturity $h$ to emphasize that there is a different slope for each value of $h$. Then, $b(h)$ can take negative values when the risk premium $d_t^{(h)}$ is time varying and satisfies the condition $Var[q_t^{(h)}] + Cov[q_t^{(h)}, d_t^{(h)}] < 0$. Fama (1984) translates this inequality into two conditions that have been extensively studied in the literature:

(i) negative covariance between $q_t^{(h)}$ and $d_t^{(h)}$,

(ii) greater variance of $d_t^{(h)}$ than $q_t^{(h)}$.

Therefore, a model of the joint behaviour of interest and exchange rates needs to satisfy these two conditions in order to be empirically plausible.

# 3. Internationally Affine Models

The analysis is similar to that in Backus, Foresi, and Telmer (2001), and Brandt and Santa-Clara (2002).[1] It is based on a two-country world where assets can be denominated in either domestic currency (i.e., "dollars") or foreign currency (i.e., "pounds"). As usual, starred $*$ variables are foreign counterparts of domestic variables; I use $(*)$ to denote domestic and foreign quantities at the same time and without distinction.

Initially, consider by a no-arbitrage argument the existence of a (strictly positive) discount factor (SDF), $M_t$, that prices any traded asset denominated in dollars through the following relationship[2]:

$$X_t = E_t\left[\frac{M_{t+h}}{M_t}X_{t+h}\right], \tag{4}$$

where $X_t$ is the value of a claim to a stochastic cash flow of $X_{t+h}$ dollars $h$ periods later. Equivalently, I can also divide both sides of this expression by $X_t$ to reformulate the previous expression as:

$$1 = E_t\left[\frac{M_{t+h}}{M_t}R_{t+h}\right], \tag{5}$$

where $R_{t+h} = X_{t+h}/X_t$ is just the gross $h$-periods return on the asset.

For example, this relationship can be used to price a zero-coupon bond that promises to pay one dollar $h$-periods ahead ($X_{t+h} = 1$). Let $P_t^{(h)}$ be the price of this bond. In this case, direct application of the pricing relationship in equation (4) gives that $P_t^{(h)}$ must equal the conditional expectation of the ratio of the future and actual value of the SDF:

$$P_t^{(h)} = E_t\left[\frac{M_{t+h}}{M_t}\right].$$

If, instead, I consider a position in an $h$-period contract in the forward foreign exchange market, which involves no payment at date $t$ while a payoff of $F_t^{(h)} - S_{t+h}$ at time $t+h$, I obtain:

$$0 = E_t\left[\frac{M_{t+h}}{M_t}\left(F_t^{(h)} - S_{t+h}\right)\right].$$

Alternatively, I might also need to price assets denominated in foreign currency such

[1] See also Guimarães (2006) for a more general setting, including jumps.

[2] The SDF is alternatively known as pricing kernel or state price density, and it is a concept related to the representative agent's nominal intertemporal marginal rate of substitution of consumption (see Cochrane 2001 for an extended discussion on the SDF).

as, for instance, a pound-denominated zero-coupon bond. Again, consider a no-arbitrage approach to postulate the existence of a foreign SDF, $M_t^*$, that prices any asset denominated in pounds through the following relationship:

$$1 = E_t \left[ \frac{M_{t+h}^*}{M_t^*} R_{t+h}^* \right], \tag{6}$$

where now $R_{t+h}^*$ is the gross $h$-periods return on an asset denominated in foreign currency.

However, any return denominated in pounds can be expressed in dollars once it is adjusted by the rate of change of the bilateral spot exchange rate, $S_{t+h}/S_t$. Thus, it must be the case that:

$$1 = E_t \left[ \frac{M_{t+h}}{M_t} \frac{S_{t+h}}{S_t} R_{t+h}^* \right].$$

In other words, the law of one price implies that any foreign asset must be correctly priced by both the domestic and the foreign SDFs:

$$E_t \left[ \frac{M_{t+h}}{M_t} \frac{S_{t+h}}{S_t} R_{t+h}^* \right] = E_t \left[ \frac{M_{t+h}^*}{M_t^*} R_{t+h}^* \right] = 1.$$

As noted by Backus, Foresi, and Telmer (2001) and Brandt and Santa-Clara (2002), this equation is trivially satisfied by a foreign SDF such that:

$$M_t^* = M_t S_t, \tag{7}$$

and, furthermore, if markets are complete then this specification of the foreign SDF is unique. Therefore, the exchange rate $S_t$ is uniquely determined by the ratio of the two pricing kernels. I can obtain the law of motion of the (log) exchange rate $s_t = \log S_t$ using Itô's lemma on the stochastic processes of $M_t$ and $M_t^*$.

Hence, assume the following dynamics of the domestic and foreign SDF:

$$\begin{aligned} \frac{dM_t}{M_t} &= -r(\mathbf{x}_t, t)dt - \mathbf{\Lambda}(\mathbf{x}_t, t)' d\mathbf{W}_t, \\ \frac{dM_t^*}{M_t^*} &= -r^*(\mathbf{x}_t, t)dt - \mathbf{\Lambda}^*(\mathbf{x}_t, t)' d\mathbf{W}_t, \end{aligned} \tag{8}$$

where $r_t$ and $r_t^*$ are the instantaneous domestic and foreign interest rates (also known as short rates); $\mathbf{W}_t$ is an $n$-dimensional vector of independent Brownian motions that describes the shocks in this economy; and $\mathbf{\Lambda}_t$ and $\mathbf{\Lambda}_t^*$ are two $n$-vectors that are usually called the market prices of risk, because they describe how the domestic and foreign SDFs respond to the shocks given by $\mathbf{W}_t$. In general, the short rates and the prices of risk are functions of time, $t$, and a Markovian $n$-dimensional vector, $\mathbf{x}_t$, that describes completely the state of the economy. The law of motion of these state variables, $\mathbf{x}_t$, is given by a diffusion such as:

$$d\mathbf{x}_t = \boldsymbol{\mu}_x(\mathbf{x}_t, t)dt + \boldsymbol{\sigma}_x(\mathbf{x}_t, t)d\mathbf{W}_t, \tag{9}$$

where $\boldsymbol{\mu}_x$ is an $n$-dimensional vector of drifts, and $\boldsymbol{\sigma}_x$ is an $n \times n$ state-dependent factor-volatility matrix.

Using Itô's lemma on (8) and subtracting, I get:

$$ds_t = \left[(r_t - r_t^*) + \frac{1}{2}(\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t^{*\prime}\boldsymbol{\Lambda}_t^*)\right] dt + (\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t^*)' d\mathbf{W}_t. \tag{10}$$

This equation ties the dynamic properties of the exchange rate to the specific parameterization of the drift (interest rates), the diffusion (price of risk) coefficients in (8), and the dynamic evolution of the set of state variables (because interest rates and the prices of risks are ultimately related to those). In particular, if I focus on a Euler discretization to the process of the exchange rate and take expectations, I find that:

$$q_t = E_t\left[s_{t+1} - s_t\right] \simeq (r_t - r_t^*) + \frac{1}{2}(\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t^{*\prime}\boldsymbol{\Lambda}_t^*). \tag{11}$$

Note that, since the covered interest parity implies that the first term of $q_t$ is equal to the forward premium $p_t = (r_t - r_t^*)$, this equation states Fama's decomposition where the risk premium is equal to $d_t = \frac{1}{2}(\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t - \boldsymbol{\Lambda}_t^{*\prime}\boldsymbol{\Lambda}_t^*)$. In other words, Fama's risk premium is proportional to the differential between the instantaneous variances of the SDFs across the two countries; and, more importantly, it is time varying because the prices of risks are usually functions of the state vector $\mathbf{x}_t$. Consequently, the uncovered interest parity does not necessarily hold in this no-arbitrage framework.[3]

Despite this result, (11) is only an approximation to the true expected rate of depreciation. First, it ignores the internal dynamics of the variables from moment $t$ to $t+1$. That is, if I approximate the expected rate of depreciation over one month using this equation, then

[3] One exception where the uncovered interest parity holds is the case when the SDF is conditionally homoscedastic ($\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t = \alpha$ and $\boldsymbol{\Lambda}_t^{*\prime}\boldsymbol{\Lambda}_t^* = \alpha^*$ where $\alpha$ and $\alpha^*$ are two positive constants). In this case, one is within the framework of Hansen and Hodrick (1983), who have shown that, with an additional constant term, the uncovered interest parity is consistent with a model of rational maximizing behaviour in which assets are priced by a no-arbitrage restriction. The intuition behind why this hypothesis still holds is that agents are risk-averse, but the risk premium is not time varying.

such an approach ignores that exchange rates move, say, "second-by-second" and that shocks accumulate during this time. Second, $r_t$ and $r_t^*$ are instantaneous interest rates and not the relevant interest rates on an $h$-period deposit denominated in domestic and foreign currency, $r_t^{(h)}$ and $r_t^{*(h)}$, respectively. Namely, it is hard to think of the instantaneous interest rates as being a good proxy of their one-year counterparts.

Still, I need to compute the expected rate of depreciation for an arbitrary choice of $h$ (say, one, three, or six months, or one year) in order to predict the future exchange rate. A potential solution is to resort to Monte Carlo methods and, given a set of parameters and initial conditions, simulate paths of the exchange rate and obtain the expected rate of depreciation over an arbitrary sample period, $h$, computing the average change of the exchange rate across the simulated paths. However, this Monte Carlo approach can be computationally costly, especially if the model is re-estimated at each point of time, $t$, and then dynamic forecasts of the spot exchange rate are computed. Therefore, this paper follows a different avenue of research that is in the spirit of the literature on exponentially affine bond pricing: to restrict the specific functional forms of the short rates, prices of risk, and the drift and diffusion terms of the state variables so as to have a closed-form expression for the expected rate of depreciation ($h$-periods ahead).

In particular, I extend the class of affine term structure models to an *internationally affine* setting where not only the interest rate on an $h$-period deposit denominated in domestic and foreign currency ($r_t^{(h)}$and $r_t^{*(h)}$, respectively) is an affine (known) function of a set of state variables $\mathbf{x}_t$:

$$\begin{aligned} r_t^{(h)} &= A(h) + B(h)'\mathbf{x}_t, \\ r_t^{*(h)} &= A^*(h) + B^*(h)'\mathbf{x}_t, \end{aligned}$$

but also the expected rate of depreciation ($h$-periods ahead) is affine in the set of state variables $\mathbf{x}_t$:

$$q_t^{(h)} = E_t\left[s_{t+h} - s_t\right] = C(h) + D(h)'\mathbf{x}_t.$$

Nonetheless, the tractability obtained by using an internationally affine model must come at a price. In particular, I need to impose a set of restrictions on the model that will ultimately constrain the specific functional forms of $A(h)$, $B(h)$, $C(h)$, and $D(h)$ and the ability to predict the exchange rates. In particular, I require two sets of restrictions: (i) those needed to have interest rates in affine form and which can be found in Duffie and Kan

(1996) (see section 3.1 for examples), and (ii) those needed to obtain an affine expected rate of depreciation ($h$-periods ahead) and which can be found in the next proposition.

**Proposition 1** *If the drift of the process that the log exchange rate $s_t$ follows is affine in a set of state variables $\mathbf{x}_t$, that is,*

$$E_t ds_t = (\gamma_0 + \boldsymbol{\gamma}'\mathbf{x}_t)dt, \tag{12}$$

*with $\gamma_0 \in \mathbb{R}$ and $\boldsymbol{\gamma} \in \mathbb{R}^n$, and $\mathbf{x}_t$ follows an affine diffusion:*

$$d\mathbf{x}_t = \boldsymbol{\Phi}(\boldsymbol{\theta} - \mathbf{x}_t)dt + \boldsymbol{\Sigma}^{1/2}V(\mathbf{x}_t)^{1/2}d\mathbf{W}_t, \tag{13}$$

*where $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$ are $n \times n$ matrices, $\boldsymbol{\theta}$ is an $n$-vector, $V(\mathbf{x}_t)$ is a diagonal $n \times n$ matrix with $i$-th typical element $v_i(\mathbf{x}_t) = \alpha_i + \boldsymbol{\beta}_i'\mathbf{x}_t$, and $\mathbf{W}_t$ is an $n$-dimensional vector of independent Brownian motions; then, the expected rate of depreciation $h$-periods ahead is a (known) affine function of the state vector $\mathbf{x}_t$:*

$$q_t^{(h)} = E_t\left[s_{t+h} - s_t\right] = C(h) + D(h)'\mathbf{x}_t, \tag{14}$$

*where the coefficients $C(h) \in \mathbb{R}$ and $D(h) \in \mathbb{R}^n$ have the following expressions:*

$$C(h) = \gamma_0 h + \boldsymbol{\gamma}'\boldsymbol{\theta}h - \boldsymbol{\gamma}'\boldsymbol{\Phi}^{-1}\left[I - e^{-\boldsymbol{\Phi}h}\right]\boldsymbol{\theta},$$
$$D(h)' = \boldsymbol{\gamma}'\boldsymbol{\Phi}^{-1}\left[I - e^{-\boldsymbol{\Phi}h}\right].$$

**Proof.** See Appendix A. ■

The result in this proposition is novel because (to the best of my knowledge) the literature on multi-country affine models has focused almost entirely on Euler approximations to the expected rate of depreciation $h$-periods ahead and, therefore, their results are subject to the shortcomings mentioned earlier.[4] In particular, this proposition states that an affine expected rate of depreciation requires both the short rates ($r_t$ and $r_t^*$) and the instantaneous variances of the pricing kernels ($\boldsymbol{\Lambda}_t'\boldsymbol{\Lambda}_t$ and $\boldsymbol{\Lambda}_t^{*\prime}\boldsymbol{\Lambda}_t^*$) to be affine in $\mathbf{x}_t$ (which guarantees that the drift of the log exchange rate, $s_t$, is affine); and, at the same time, the process that $\mathbf{x}_t$ follows must

[4] For example, Hodrick and Vassalou (2002), Leippold and Wu (2003), and Ahn (2004) focus on Euler approximations of the law of motion of the (log) exchange rate, so their *formulae* regarding the expected rate of depreciation is valid only for arbitrary small $h$. One exception is Dewatcher and Maes (2001), who provide the expressions for the expected rate of depreciation for an arbitrary choice of $h$. However, their model is just a particular example of the general framework provided here.

be an affine diffusion. When I compare these conditions with those needed to obtain interest rates in affine form, I find that an internationally affine model imposes additional constraints with respect to the class of affine term structure models. For example, it is possible to obtain affine interest rates without having an instantaneous variance of the SDF that is affine in $\mathbf{x}_t$ (see Duffee 2002 and Cheridito, Filipovic, and Kimmel 2005) or without the condition that the state vector must follow an affine diffusion (see Duarte 2004). The next subsections investigate which models satisfy the internationally affine conditions and to what extent this represents a constraint to predicting exchange rates.

## 3.1 Affine models of currency pricing

Up to this point, I have been interested in finding those models that fall within the internationally affine framework; that is, those models where not only interest rates are affine (known) functions of a set of state variables, but the expected rate of depreciation also satisfies this property. Since one of the demanded characteristics is that the interest rates must be affine in a set of factors, and it is well known that the standard formulation of the affine term structure models shares this property, I start by establishing the properties of the exchange rate implied by this class of models.

To this end, I focus on a multi-country version of the Dai and Singleton (2000) standard formulation of these affine term structure models that nests most of the work on international term structure modelling.[5] These models can be considered as multivariate extensions of the Cox, Ingersoll, and Ross (1985) model, and they are characterized by an instantaneous interest rate (also known as short rate) that is an affine function of the set of state variables $\mathbf{x}_t$:

$$\begin{aligned} r_t &= \delta_0 + \boldsymbol{\delta}_1' \mathbf{x}_t, \\ r_t^* &= \delta_0^* + \boldsymbol{\delta}_1^{*\prime} \mathbf{x}_t, \end{aligned} \tag{15}$$

where $\delta_0$, $\delta_0^*$ are two scalars, and $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_1^*$ are two $n$-dimensional vectors. The dynamic evolution of these $n$ state variables is given by the following affine diffusion:

$$d\mathbf{x}_t = \boldsymbol{\Phi}(\boldsymbol{\theta} - \mathbf{x}_t)dt + \boldsymbol{\Sigma}^{1/2} V(\mathbf{x}_t)^{1/2} d\mathbf{W}_t, \tag{16}$$

where, again, $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$ are $n \times n$ matrices, $\boldsymbol{\theta}$ is an $n$-vector, and $V(\mathbf{x}_t)$ is a diagonal $n \times n$

[5] See Saa-Requejo (1993), Frachot (1996), Backus, Foresi, and Telmer (2001), Dewachter and Maes (2001), Hodrick and Vassalou (2002), and Ahn (2004).

matrix with $i$-th typical element $v_i(\mathbf{x}_t) = \alpha_i + \boldsymbol{\beta}_i'\mathbf{x}_t$. $\mathbf{W}_t$ is an $n$-dimensional vector of independent Brownian motions.[6] Finally, the model is completed by a specification of the domestic and foreign prices of risk such that:

$$\begin{aligned} \boldsymbol{\Lambda}_t &= V(\mathbf{x}_t)^{1/2}\boldsymbol{\lambda}, \\ \boldsymbol{\Lambda}_t^* &= V(\mathbf{x}_t)^{1/2}\boldsymbol{\lambda}^*. \end{aligned} \tag{17}$$

This standard formulation of the affine term structure models is also known as a "completely affine" specification (Duffee 2002), because it has an instantaneous variance of the SDFs, $\boldsymbol{\Lambda}_t^{(*)\prime}\boldsymbol{\Lambda}_t^{(*)}$, that is affine in the set of factors $\mathbf{x}_t$.

Under this parameterization, interest rates on $h$-period deposits denominated in domestic and foreign currencies satisfy:

$$r_t^{(h)} = A(h) + B(h)'\mathbf{x}_t,$$
$$r_t^{*(h)} = A^*(h) + B^*(h)'\mathbf{x}_t,$$

where the coefficients $A^{(*)}(h) \in \mathbb{R}$ and $B^{(*)}(h) \in \mathbb{R}^n$ solve two systems of ordinary differential equations whose details can be found in Duffie and Kan (1996), Dai and Singleton (2003), and Piazzesi (2003).

Substituting the expressions for the short rates and the prices of risk into the law of motion of the (log) exchange rate in equation (10) gives:

$$ds_t = (\gamma_0 + \boldsymbol{\gamma}'\mathbf{x}_t)dt + (\boldsymbol{\lambda} - \boldsymbol{\lambda}^*)'V(\mathbf{x}_t)^{1/2}d\mathbf{W}_t, \tag{18}$$

where $\gamma_0 = (\delta_0 - \delta_0^*) + \frac{1}{2}\sum_{i=1}^N(\lambda_i^2 - \lambda_i^{*2})\alpha_i$ and $\boldsymbol{\gamma} = (\boldsymbol{\delta}_1 - \boldsymbol{\delta}_1^*) + \frac{1}{2}\sum_{i=1}^N(\lambda_i^2 - \lambda_i^{*2})\boldsymbol{\beta}_i$. Therefore, Proposition 1 holds: the drift of the (log) exchange rate is affine in a set of state variables $\mathbf{x}_t$, and these state variables follow an affine diffusion. This property adds a new meaning to the term "completely affine specification."

However, it has been found that this "completely affine" specification of the prices of risk is empirically restrictive. For example, Duffee (2002) finds that this parameterization

[6] It is possible, for an arbitrary set of parameters, that the state variables $\mathbf{x}_t$ enter in a region where $v_i(\mathbf{x}_t) = \alpha_i + \boldsymbol{\beta}_i'\mathbf{x}_t$ is negative, which would imply that the state vector has a negative conditional variance. Dai and Singleton (2000) provide a set of restrictions on the parameters of the model that guarantees that the dynamics of $\mathbf{x}_t$ are well defined.

produces forecasts of future Treasury yields that are beaten by a random-walk specification[7]; Backus, Foresi, and Telmer (2001) point out that this model constrains the relationship between interest rates and the risk premium in such a way that the ability of the model to capture the forward premium puzzle, and, therefore, the ability to predict exchange rates, is severely limited. Therefore, I need to make more flexible assumptions on the form of the prices of risk. However, those models with more flexible specifications of the prices of risk, as the "essentially affine specification" in Duffee (2002) or the "extended affine specification" in Cheridito, Filipovic, and Kimmel (2005), do not necessarily have an instantaneous variance of the SDF that is affine in the state variables.[8] In particular, a similar point has been made by Guimarães (2006): "with these [Cheridito, Filipovic, and Kimmel (2005)] market prices of risk exchange rates will be a nonlinear (not even polynomial) function of latent state variables."

Still, there is hope in reproducing the forward premium puzzle. Dai and Singleton (2002) have been successful in explaining puzzles in a similar conceptual framework to the forward premium anomaly: the rejection of the expectations hypothesis of the term structure of interest rates.[9] In particular, they show that a Gaussian essentially affine model can generate flexible enough time-varying risk premia in holding-period bond returns so as to solve the failure of this traditional "expectation theory." Therefore, the question is whether this model can also reproduce the "forward premium puzzle."

In Dai and Singleton's (2002) model, the specification of the prices of risk is affine in a set of variables[10]:

$$\boldsymbol{\Lambda}_t^{(*)} = \boldsymbol{\lambda}_0^{(*)} + \boldsymbol{\lambda}_1^{(*)}\mathbf{x}_t,$$

where $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}_0^*$ are two $n$-dimensional vectors, and $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_1^*$ are two $n \times n$ matrices; the latent state variables follow a multivariate Orstein-Uhlenbeck (Gaussian) process. Again,

[7] Duffee (2002) claims that this is because (i) the price of risk variability comes only from $V(\mathbf{x}_t)^{1/2}$, and (ii) the sign of $\boldsymbol{\Lambda}_t^{(*)}$ cannot change because the elements of $V(\mathbf{x}_t)^{1/2}$ are restricted to be non-negative.

[8] One exception is Graveline (2005), whose model is based on the "extended affine specification" in Cheridito, Filipovic, and Kimmel (2005). However, he restricts these prices of risk in such a way that the non-linear terms in equation (10) cancel out, which delivers an affine diffusion for the exchange rate.

[9] See Bekaert and Hodrick (2001) for this relationship between the expectation hypothesis of the term structure of interest rates and the uncovered interest parity (also known as expectation hypothesis of the foreign exchange market).

[10] As claimed by Dai and Singleton (2002): "Since LPY *(the expectations hypothesis of the term structrure)* refers to the properties of the *(conditional)* first moments of yields, and the family of Gaussian models (family $A_0(3)$) gives the most flexibility to the structure of factor correlations and conditional means, one might conjecture a priori that these models would perform at least as well as other affine models." Therefore, since the uncovered interest parity also refers to the properties on the (conditional) first moments, but now of exchange and interest rates, I can apply the same reasoning to my case.

this model falls within the essentially affine specification in Duffee (2002) and it does not generate an affine expected rate of depreciation, because the variance of the SDF, $\mathbf{\Lambda}_t^{(*)\prime}\mathbf{\Lambda}_t^{(*)}$, is quadratic in $\mathbf{x}_t$:

$$\mathbf{\Lambda}_t^{(*)\prime}\mathbf{\Lambda}_t^{(*)} = \boldsymbol{\lambda}_0^{(*)\prime}\boldsymbol{\lambda}_0^{(*)} + 2\boldsymbol{\lambda}_0^{(*)\prime}\boldsymbol{\lambda}_1^{(*)}\mathbf{x}_t + \mathbf{x}_t'\boldsymbol{\lambda}_1^{(*)\prime}\boldsymbol{\lambda}_1^{(*)}\mathbf{x}_t.$$

However, quadratic models can be viewed as being "affine" in the original set of factors and their respective squares and cross-products.[11] This idea is exploited in the next subsection.

## 3.2 Quadratic models of currency pricing

These term structure models are characterized by an instantaneous interest rate that is a quadratic function of the set of state variables $\mathbf{x}_t$:

$$\begin{aligned} r_t &= \delta_0 + \boldsymbol{\delta}_1'\mathbf{x}_t + \mathbf{x}_t'\boldsymbol{\delta}_2\mathbf{x}_t, \\ r_t^* &= \delta_0^* + \boldsymbol{\delta}_1^{*\prime}\mathbf{x}_t + \mathbf{x}_t'\boldsymbol{\delta}_2^*\mathbf{x}_t, \end{aligned} \tag{19}$$

where $\delta_0$, $\delta_0^*$ are two scalars, $\boldsymbol{\delta}_1$, $\boldsymbol{\delta}_1^*$ are two $n$-dimensional vectors, and $\boldsymbol{\delta}_2$, $\boldsymbol{\delta}_2^*$ are two symmetric $n \times n$ matrices. The state variables follow a multivariate Orstein-Uhlenbeck (Gaussian) process:

$$d\mathbf{x}_t = \mathbf{\Phi}(\boldsymbol{\theta} - \mathbf{x}_t)dt + \mathbf{\Sigma}^{1/2}d\mathbf{W}_t, \tag{20}$$

where $\mathbf{\Phi}$ and $\mathbf{\Sigma}$ are $n \times n$ matrices, $\boldsymbol{\theta}$ is an $n$-vector; and $\mathbf{W}_t$ is an $n$-dimensional vector of independent Brownian motions. As a difference with the prices of risk in a completely affine framework, the price of risk is a linear function of the state variables $\mathbf{x}_t$:

$$\begin{aligned} \mathbf{\Lambda}_t &= \boldsymbol{\lambda}_0 + \boldsymbol{\lambda}_1\mathbf{x}_t, \\ \mathbf{\Lambda}_t^* &= \boldsymbol{\lambda}_0^* + \boldsymbol{\lambda}_1^*\mathbf{x}_t, \end{aligned} \tag{21}$$

where $\boldsymbol{\lambda}_0$ and $\boldsymbol{\lambda}_0^*$ are two $n$-dimensional vectors, and $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_1^*$ are two $n \times n$ matrices. Moreover, note that the Gaussian essentially affine specification in Dai and Singleton (2002), which has both short rates and the prices of risk being affine in a set of Gaussian state variables, is nested by this quadratic formulation when $\boldsymbol{\delta}_2 = \boldsymbol{\delta}_2^* = 0$. I will return to this model shortly.

Once again, substituting these expressions for the short rates and the prices of risk into

[11] A similar argument has been given in Cheng and Scaillet (2002), Dai and Singleton (2003b), and Gouriéroux and Sufana (2003) within the one-country set-up.

the law of motion of the (log) exchange rate in equation (10) gives:

$$ds_t = (\gamma_0 + \boldsymbol{\gamma}_1'\mathbf{x}_t + \mathbf{x}_t'\boldsymbol{\gamma}_2\mathbf{x}_t)dt + [(\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_0^*) + (\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_1^*)\mathbf{x}_t]'\, d\mathbf{W}_t, \tag{22}$$

where $\gamma_0 = (\delta_0 - \delta_0^*) + \frac{1}{2}(\boldsymbol{\lambda}_0'\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_0^{*\prime}\boldsymbol{\lambda}_0^*)$, $\boldsymbol{\gamma}_1 = (\boldsymbol{\delta}_1 - \boldsymbol{\delta}_1^*) + (\boldsymbol{\lambda}_1'\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_1^{*\prime}\boldsymbol{\lambda}_0^*)$, and $\boldsymbol{\gamma}_2 = \boldsymbol{\delta}_2 - \boldsymbol{\delta}_2^* + \frac{1}{2}(\boldsymbol{\lambda}_1'\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_1^{*\prime}\boldsymbol{\lambda}_1^*)$, which makes the drift of the process that the (log) exchange rate follows quadratic in the set of state variables. Similarly, Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2002) show that in this framework interest rates have a quadratic form:

$$\begin{aligned} r_t^{(h)} &= A(h) + B_1(h)'\mathbf{x}_t + \mathbf{x}_t'B_2(h)\mathbf{x}_t, \\ r_t^{*(h)} &= A^*(h) + B_1^*(h)'\mathbf{x}_t + \mathbf{x}_t'B_2^*(h)\mathbf{x}_t, \end{aligned}$$

where the coefficients $A^{(*)}(h) \in \mathbb{R}$, $B_1^{(*)}(h) \in \mathbb{R}^n$, and $B_2^{(*)}(h) \in \mathbb{R}^{n\times n}$ solve two systems of ordinary differential equations.

Therefore, this model does not fall within the internationally affine framework. However, it is still possible to view any quadratic model as being affine in the original set of variables and their respective squares and cross-products. This point can be illustrated with an example. First, assume that short rates in both countries are quadratic in a global factor $x_t$ that follows Gaussian process:

$$\begin{aligned} r_t^{(*)} &= \delta_0^{(*)} + \delta_1^{(*)}x_t + \delta_2^{(*)}x_t^2, \\ dx_t &= \phi(\theta - x_t)dt + \sigma dW_t, \end{aligned}$$

and assume that the prices of risk are affine in this global factor:

$$\Lambda_t^{(*)} = \lambda_0^{(*)} + \lambda_1^{(*)}x_t.$$

Note that this set of assumptions implies that the $h$-period domestic and foreign interest rates will satisfy a quadratic relationship:

$$\begin{aligned} r_t^{(*)(h)} &= a^{(*)}(h) + b_1^{(*)}(h)x_t + b_2^{(*)}(h)x_t^2, \\ r_t^{(*)(h)} &= a^{(*)}(h) + B^{(*)}(h)'\widetilde{\mathbf{x}}_t, \end{aligned}$$

which can be rewritten using a new variable, $z_t = x_t^2$, that captures the square of the global factor, and where $B^{(*)}(h) = \left[b_1^{(*)}(h), b_2^{(*)}(h)\right]'$ and $\widetilde{\mathbf{x}}_t = [x_t, z_t]'$. That is, interest rates are affine in the original global factor $x_t$ and its square $z_t = x_t^2$. Therefore, this quadratic model can be viewed as an affine model in the new set of factors given by $\widetilde{\mathbf{x}}_t$.

Similarly, it can be shown that the expected rate of depreciation is also affine in $\widetilde{\mathbf{x}}_t$. Recall that Proposition 1 requires, first, the drift of the (log) exchange rate process to be affine in this new set of state variables and, second, the process that these variables follow must be itself an affine diffusion. The first condition is satisfied because short rates and the instantaneous variance of the SDFs are quadratic in $x_t$ so they can be expressed in terms of $\widetilde{\mathbf{x}}_t = [x_t, z_t]$. Second, it can be shown that, if Itô's lemma is applied on $z_t = x_t^2$, then the joint process for $x_t$ and $z_t$ is an affine diffusion.[12] Therefore, this model satisfies the two conditions for an expected rate of depreciation.

In brief, the interest rates and the expected rate of depreciation are affine in a new set of state variables. Consequently, this one-factor quadratic model can be interpreted as a two-factor internationally affine model. Furthermore, the generalization of this result to the case where there is more than one factor is straightforward and the details can be found in Appendix B. In the general case, $\mathbf{z}_t$ must include the squares of the original set of state variables $\mathbf{x}_t$ and their cross-products. A compact way to do so is to apply the matrix *vech* operator, which stacks the elements on and below the main diagonal of a square matrix, to the matrix given by $\mathbf{x}_t\mathbf{x}_t'$. As a result, quadratic models can be viewed as part of the internationally affine framework, because they provide interest rates and an expected rate of depreciation that are affine in the augmented set of state variables given by $\widetilde{\mathbf{x}}_t = [\mathbf{x}_t', \mathbf{z}_t']'$ with $\mathbf{z}_t = vech(\mathbf{x}_t\mathbf{x}_t')$.

### *3.2.1 Gaussian essentially affine models*

As mentioned earlier, the Gaussian subfamily of the essentially affine models introduced in Duffee (2002) provides a sufficiently flexible risk premia to explain the puzzles of the expectations hypothesis of the term structure of interest rates. Therefore, it can potentially help to address the forward premium anomaly. These models are characterized by instantaneous

[12] In particular, the law of motion of the augmented set of factors $\widetilde{\mathbf{x}}_t = [x_t, z_t]$ satisfies:

$$d\begin{bmatrix} x_t \\ z_t \end{bmatrix} = \begin{bmatrix} \phi & 0 \\ -2\phi\theta & 2\phi \end{bmatrix}\left[\begin{pmatrix} \theta \\ \sigma^2/2\phi + \theta^2 \end{pmatrix} - \begin{pmatrix} x_t \\ z_t \end{pmatrix}\right]dt + \begin{bmatrix} \sigma \\ 2\sigma\sqrt{z_t} \end{bmatrix} dW_t.$$

interest rates and prices of risk that are affine in a set of state variables $\mathbf{x}_t$:

$$\begin{aligned} r_t^{(*)} &= \delta_0^{(*)} + \boldsymbol{\delta}_1^{(*)\prime}\mathbf{x}_t, \\ \boldsymbol{\Lambda}_t^{(*)} &= \boldsymbol{\lambda}_0^{(*)} + \boldsymbol{\lambda}_1^{(*)}\mathbf{x}_t, \end{aligned}$$

where the state vector $\mathbf{x}_t$ follows a multivariate Orstein-Uhlenbeck (Gaussian) process:

$$d\mathbf{x}_t = \boldsymbol{\Phi}(\boldsymbol{\theta} - \mathbf{x}_t)dt + \boldsymbol{\Sigma}^{1/2}d\mathbf{W}_t.$$

Duffee (2002) shows that this model generates interest rates that are affine in the factors $\mathbf{x}_t$. However, notice as before that the instantaneous variance of the SDF, $\boldsymbol{\Lambda}_t^{(*)\prime}\boldsymbol{\Lambda}_t^{(*)}$, is quadratic in the set of state variables $\mathbf{x}_t$. Still, this model can be viewed as a particular case of the quadratic specification with $\boldsymbol{\delta}_2^{(*)} = \mathbf{0}_{n\times n}$ in equation (19), and, therefore, the expected rate of depreciation will be affine in the original set of state variables and their respective squares and cross-products. An example of such a Gaussian essentially affine model of currency pricing can be found in Brennan and Xia (2004) and Dong (2005).

Table 3 summarizes the theoretical findings. First, the completely affine framework implies that interest rates and the expected rate of depreciation are linear (known) functions of a set of state variables. Second, the quadratic framework implies interest rates and an expected rate of depreciation that are both linear in the augmented set of factors that includes the original set of factors and their squares and cross-products. Finally, the Gaussian essentially affine model is in middle ground, because the interest rates are linear, while the expected rate of depreciation is quadratic (or linear in the augmented set of factors).

# 4. Empirical Model

## 4.1 A two-factor Gaussian essentially affine model

In this section, I focus on the estimation of a Gaussian essentially affine model. Several reasons justify the choice of this particular model. First, a multivariate Gaussian process gives the highest degree of flexibility to the structure of correlations and conditional means of the state vector. Therefore, this model is expected to perform at least as well as the other members of the family of completely affine models, whose general structure is restricted by the requirement that the conditional variance must always be positive (Dai and Singleton 2000). Moreover, since there is no state variable driving the conditional variance in this proposed model, there is no need to worry about these variables entering some non-admissible space where the volatilities are negative. Second, the specification of the prices of risk is the same as in the quadratic models, so a similar degree of flexibility is expected in reproducing the forward premium puzzle. Besides, and in contrast with these quadratic models, the Gaussian essentially affine model generates a one-to-one mapping from interest rates to the state vector $\mathbf{x}_t$, so the estimation exercise is easier and can be done by quasi-maximum likelihood.

Moreover, since the data set contains only interest rates with maturities up to one year, I focus on a two-factor model where these two state variables correspond with the instantaneous interest rates in each of the countries. In terms of our general framework this is translated into: $\mathbf{x}_t = [r_t, r_t^*]$, $\delta_0 = \delta_0^* = 0$, $\boldsymbol{\delta} = (1,0)'$, and $\boldsymbol{\delta}^* = (0,1)'$, where the joint process for the short rates is a multivariate Orstein-Uhlenbeck process:

$$d\begin{pmatrix} r_t \\ r_t^* \end{pmatrix} = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \left[ \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} - \begin{pmatrix} r_t \\ r_t^* \end{pmatrix} \right] dt + \begin{pmatrix} \sigma_{11} & 0 \\ \sigma_{21} & \sigma_{22} \end{pmatrix} d\mathbf{W}_t, \tag{23}$$

and the prices of risks are assumed to be affine (known) functions of these state variables:

$$\begin{aligned} \boldsymbol{\Lambda}_t &= \begin{pmatrix} \lambda_{01} \\ \lambda_{02} \end{pmatrix} + \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \mathbf{x}_t, \\ \boldsymbol{\Lambda}_t^* &= \begin{pmatrix} \lambda_{01}^* \\ \lambda_{02}^* \end{pmatrix} + \begin{pmatrix} \lambda_{11}^* & \lambda_{12}^* \\ \lambda_{21}^* & \lambda_{22}^* \end{pmatrix} \mathbf{x}_t. \end{aligned} \tag{24}$$

## 4.2 Estimation

Under the above assumptions, and given the results presented in the previous section, both domestic and foreign interest rates are affine functions of the set of state variables given by $\mathbf{x}_t = [r_t, r_t^*]'$:

$$\begin{aligned} r_t^{(h)} &= A(h) + B(h)'\mathbf{x}_t, \\ r_t^{*(h)} &= A^*(h) + B^*(h)'\mathbf{x}_t, \end{aligned} \tag{25}$$

and the expected rate of depreciation is quadratic in the same set of state variables, or, if preferred, linear in the augmented set of factor given by $\widetilde{\mathbf{x}}_t = [\mathbf{x}_t', \mathbf{z}_t']'$ with $\mathbf{z}_t = vech(\mathbf{x}_t\mathbf{x}_t')$:

$$q_t^{(h)} = C(h) + \widetilde{D}(h)'\widetilde{\mathbf{x}}_t. \tag{26}$$

In the (hypothetical) absence of exchange rate data, the estimation of this model can be done by maximum likelihood (ML) exploiting the fact that the conditional distribution of the state variables is Gaussian. For example, and following the usual convention in the literature (Dai and Singleton 2002; Duffee 2002), I assume that some of the interest rates are observed without measurement error, while the interest rates on the remaining maturities are assumed to be measured with serially uncorrelated, zero-mean errors. In particular, I assume that domestic and foreign one-month interest rates do not contain any source of measurement error, which allows me to recover the state variable $\mathbf{x}_t = [r_t, r_t^*]'$ by inversion of the one-to-one mapping given in equation (25) evaluated at $h = 1$[13]:

$$\mathbf{x}_t = H_1^{-1}(\mathbf{r}_t^{(1)} - H_0), \tag{27}$$

where $\mathbf{r}_t^{(1)} = [r_t^{(1)}, r_t^{*(1)\prime}]$, $H_0 = [A(1), A^*(1)]'$, and $H_1 = [B(1), B^*(1)]'$.

Given the value of the state vector $\mathbf{x}_t$ obtained in equation (27), the model-implied interest rates for the remaining maturities (three, six, and twelve months) can be computed. Denote by $\mathbf{r}_t^{(-1)}$ a vector that contains the observed domestic and foreign interest rates on these remaining maturities, and denote their implied counterparts by $\widehat{\mathbf{r}}_t^{(-1)}$. Then, the measurement error is $\boldsymbol{\epsilon}_t = \mathbf{r}_t^{(-1)} - \widehat{\mathbf{r}}_t^{(-1)}$, and let it be *i.i.d.* zero-mean normally distributed with density given by $f_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_t)$.

The log likelihood has, then, two parts. The first one is the contribution of the interest rates that are observed without measurement error. This can be computed using the fact that the conditional distribution of the state variables $\mathbf{x}_t$ is a first-order vector autoregression with Gaussian innovations:

$$f_{\mathbf{x}}(\mathbf{x}_{t+1}|\,\mathbf{x}_t) \sim N\left[\mu_t, \boldsymbol{\Omega}\right],$$

where

$$\mu_t = (I - e^{-\boldsymbol{\Phi}})\boldsymbol{\theta} + e^{-\boldsymbol{\Phi}}\mathbf{x}_t,$$
$$vec\left[\boldsymbol{\Omega}\right] = \left[\boldsymbol{\Phi} \otimes I + I \otimes \boldsymbol{\Phi}\right]^{-1}\left[I \otimes I - e^{-\boldsymbol{\Phi}} \otimes e^{-\boldsymbol{\Phi}}\right] vec(\boldsymbol{\Sigma}).$$

Then, the conditional density of $\mathbf{r}_t^{(1)}$ can be obtained by a change of variable:

$$f_{\mathbf{r}^{(1)}}\left(\mathbf{r}_{t+1}^{(1)}\middle|\,\mathbf{r}_t^{(1)}\right) = \frac{1}{|H_1|} f_{\mathbf{x}}\left(\mathbf{x}_{t+1}|\,\mathbf{x}_t\right).$$

[13] Time is measured in months.

Second, I have assumed that $\boldsymbol{\epsilon}_t$ is normal *i.i.d.* Thus, the log likelihood of an observation at time $t$ is

$$l_t(\Theta) = \log f_{\mathbf{r}^{(1)}}\left(\mathbf{r}_{t+1}^{(1)}\middle|\,\mathbf{r}_t^{(1)}\right) + \log f_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_t),$$

where $\Theta$ is a vector that contains the parameters of the model. The log likelihood of the whole sample is constructed as the usual sum of these log densities over the sample $L_t(\Theta|\,\mathbf{r}_1^{(1)}) = \sum_t l_t(\Theta)$ where, for simplicity, I have conditioned on the first observation of the one-month interest rates.

Still, this approach does not use the information that exchange rates contain on the ratio of the SDFs (see equation (7)). Therefore, and to exploit such information, notice that the assumption of rational expectations in foreign exchange markets allows me to write:

$$\Delta s_{t+1} = E_t\left[\Delta s_{t+1}\right] + v_{t+1}, \tag{28}$$

where $v_{t+1}$ is a rational-expectation forecasting error with zero mean and uncorrelated with any variable in the time $t$ information set. Traditionally, the empirical literature in international finance has combined this last equation with the uncovered interest parity to obtain a testable implication of this theory (see section 2). Instead, note that the assumption on the absence of measurement errors in the one-month interest rates implies that the "backed out" state variables $\mathbf{x}_t$, and any function of those, such as $\mathbf{z}_t = vech(\mathbf{x}_t\mathbf{x}_t')$, belong to the time $t$ information set. Therefore, I can combine equation (28) with (26) to obtain:

$$\Delta s_{t+1} = C(1) + \widetilde{D}(1)'\widetilde{\mathbf{x}}_t + v_{t+1}, \tag{29}$$

where, again, $\widetilde{\mathbf{x}}_t = [\mathbf{x}_t', \mathbf{z}_t']'$. This equation can form the basis of an estimation by quasi-maximum likelihood (QML) if I assume that $v_t$ is $N(0, \sigma_v^2)$ and independent of $W_t$ and the measurement errors $\epsilon_t$. The parameter $\sigma_v^2$ can be interpreted as a general characterization of the mean squared error of the (restricted) projection of the rate of depreciation on the factors and their squares. In this case, the log likelihood has a third component that captures the contribution of the exchange rates:

$$l_t(\Theta) = \log f_{\mathbf{r}^{(1)}}\left(\mathbf{r}_t^{(1)}\middle|\,\mathbf{r}_{t-1}^{(1)}\right) + \log f_{\boldsymbol{\epsilon}}(\boldsymbol{\epsilon}_t) + \log f_v(v_t),$$

and the estimated parameter $\widehat{\Theta}$ can be obtained by maximizing the log likelihood of the whole sample $L_t(\Theta|\,\mathbf{r}_1^{(1)}) = \sum_t l_t(\Theta)$.

Although somewhat extreme, the assumption on the independence of the error term $v_t$

can be justified from either estimation simplicity and/or from the empirical observation that a predominant portion of the exchange rate movement is independent of the interest rate movements of either country. For example, Lothian and Wu (2003) point out that the forward premium regressions usually show very low $R^2$ statistics. This feature can be accounted for in my model once I allow the pricing kernels to be driven by an additional source of risk that is orthogonal to the forces driving the short rates.[14] Still, the expressions for the expected rate of depreciation in equation (14) are valid because this new source of risk is orthogonal to the interest rates. More important is that the QML estimation approach can be viewed as a generalized method of moments (GMM) estimation based on the scores of the quasi-likelihood function. Given that the estimation of the model is based on the right conditional moments, this approach is expected to deliver consistent (albeit less efficient) estimates of the parameters of the model.[15]

## 4.3 Results

Tables 4 and 5 present the QML estimates of the two-factor Gaussian essentially affine model, along with robust estimates of the corresponding standard errors. Table 4 contains the results of the estimation exercise for the U.S. dollar – pounds sterling, while Table 5 contains those of the U.S. dollar – Canadian dollar. Since the objective of this study is to investigate the ability of affine models in the out-of-sample prediction of exchange rates, I follow Dai and Singleton (2002) and Duffee (2002) to re-estimate the model after setting to zero those coefficients with the largest relative standard errors ($\phi_{12}$, $\sigma_{21}$, $\lambda_{11}$, $\lambda^*_{02}$ for the pair U.S. dollar – pounds sterling, and $\phi_{12}$, $\sigma_{21}$, $\lambda_{11}$, $\lambda_{12}$, $\lambda_{01}$, $\lambda^*_{21}$, $\lambda^*_{22}$, $\lambda^*_{02}$ for the pair U.S. dollar – Canadian dollar). This approach will help to reduce the large number of parameters that the model contains and therefore it will reduce excessive parameter estimation uncertainty that may adversely affect the out-of-sample forecast performance of this model. For the sake of saving space, I report the results for only these restricted models.

The results for the U.S. dollar – pounds sterling (Table 4) indicate that the process for

[14]Brandt and Santa-Clara (2002) attribute this additional source of risk to market incompleteness; Dewachter and Maes (2001) or Leippold and Wu (2003) assume that bond returns do not necessarily span the returns in the foreign exchange market, interpreting this orthogonal risk factor as a factor outside the bond market (e.g., stocks).

[15]Moreover, the scores of this Gaussian model can be computed algebraically (see Harvey 1989, 140–42 for a related example). Another advantage of this model is that it is possible to show that both the covariance matrix of the measurement errors and the $\sigma_v^2$ (that is, the mean squared error of the (restricted) projection of the rate of depreciation) can be concentrated out of the likelihood function. In particular, the estimates of these two objects are $\widehat{\boldsymbol{\Omega}}_\epsilon = T^{-1}\sum \widehat{\boldsymbol{\epsilon}}_t \widehat{\boldsymbol{\epsilon}}_t'$ and $\widehat{\sigma}_v^2 = T^{-1}\sum \widehat{v}_t^2$, respectively. Finally, the QML approach allows the standard errors of the parameters to be computed using the standard GMM *formulae* (see Hamilton 1994, 428–29).

the short rates is mean reverting. However, this mean reversion is slow (both elements in the diagonal of $\mathbf{\Phi}$ are positive but close to zero). The British short rate seems to revert to a higher level than its American counterpart and, in addition, it is more volatile. Since the process that short rates follow is Gaussian, it is possible for them to take on negative values with positive probability. Still, I find that, with estimated parameters, the probability of a negative short rate is small: 2.59 per cent and 0.30 per cent for the case of the United States and United Kingdom, respectively. The implied yield curve for the United States is upward sloping with an implied long-term yield of 17.75 per cent. On the contrary, the implied yield curve for the United Kingdom is downward sloping with long-term yields reaching as high as 9.54 per cent. These seem to be reasonable numbers. First, Backus, Foresi, and Telmer (2001) report that their estimates implied long-term yields reaching as high as 80 per cent. Second, note that the estimation is done without interest rate data on maturities greater than one year, which are the ones that can potentially help to anchor these long-term yields. In addition, this result is consistent with an (average) spread between the one-year and the one-month interest rate that is positive for the case of the United States while negative for the case of the United Kingdom (Table 1).

Notice that almost all the elements of the matrices $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^*$ in the prices of risk are statistically different from zero. Therefore, and in line with the work of Dai and Singleton (2002), extending the specification of the prices of risk seems to be an important factor for the estimation of the model. In addition, this model is able to reproduce the forward premium puzzle. Table 6 presents the term structure of forward premium regression slopes implied by the model. These are computed using the closed-form *formulae* derived in Appendix C and by treating the estimates of the two-factor model as truth. The sample OLS estimates of these slopes are reproduced here for the sake of comparison. The model-implied slopes are negative and reasonably close to their sample counterparts. However, the model tends to generate uncovered interest parity slopes that are more negative than the ones I estimated using standard OLS techniques. Therefore, the results produced by this model for the case of the U.S. dollar – pounds sterling do not seem to be unreasonable.

The results for the U.S. dollar – Canadian dollar are presented in Table 5. Again, the process for the short rates is mean reverting, and the Canadian short term seems to revert to a higher level than its American counterpart. However, if compared with the estimates obtained for the U.S. dollar – pounds sterling, the long-run mean for the short rates (given by $\boldsymbol{\theta}$) is unusually high. In fact, both implied yield curves for the United States and Canada are downward sloping with implied long-term yields being unreasonably low: 1.03 per cent and 3.84 per cent for the United States and Canada, respectively. Conversely, the probability of having a negative short rate is almost zero regardless of the choice of the country.

Still, the model is able to reproduce the forward premium anomaly. Again, almost all the elements of the matrices $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^*$ are statistically different from zero and the model-implied slopes are negative (Table 6, panel b). The model underestimates (in absolute terms) the slope for the case of the one-month and three-month contracts, while the implied slope is more negative than the OLS counterparts for the six-month and twelve-month contracts. Therefore, and contrary to the results for the pair U.S. dollar – pounds sterling, here it seems that the good fit of the exchange rate is done at the expense of the interest rates.

## 4.4 Out-of-sample forecasting

In this section, an out-of-sample forecasting experiment is conducted over the period January 1998 – December 2004 to evaluate the performance of the Gaussian essentially affine term structure model estimated and described in the previous section. These forecasts are computed according to the recursive procedure employed in Clarida and Taylor (1997) and Clarida et al. (2003): at each date $t$, the model is re-estimated using data up to and including time $t$ and then dynamic forecasts of the spot exchange rate up to $t+12$ are obtained. These forecasts are computed using equation (14).

The first column in Table 7 presents the results of the accuracy of these forecasts using the root-mean-square error (RMSE) and mean-absolute error (MAE) criteria. Panel a contains the results of the forecasting exercise for the U.S. dollar – pounds sterling, while panel b contains those of the U.S. dollar – Canadian dollar. This table also compares these forecasts with those generated by three alternative benchmarks: a random walk (RW), a vector autoregression on the forward premia and the rate of depreciation (VAR), and the forward premium regression (OLS). The comparison of my forecasts with those produced by the random-walk model is motivated by the fact that the random-walk model is considered the usual metric in which to evaluate exchange rate forecasts since the original work of Meese and Rogoff (1983a, b). However, Clarida and Taylor (1997) show that if one uses a linear vector-error-correction (VECM) model in the spot and forward exchange rates, it is possible to obtain out-of-sample forecasts of spot exchange rates that beat the random-walk model. Therefore, I include as a second benchmark the forecasts obtained by the use of a vector autoregression (VAR) on the forward premia and the rate of depreciation,[16] where the number of lags in the VAR is

[16] This approach is equivalent to the VECM in Clarida and Taylor (1997) if I impose that the spot and the forward exchange rates are cointegrated with a known cointegration vector (1, -1). See Mark (2001, 51).

chosen to be equal to $p = 2$ for the United Kingdom, and $p = 1$ for Canada, as suggested by the Bayesian information criteria (BIC).[17] Finally, and for completeness, I also include the forecast produced by a standard ordinary least squares regression of the rate of depreciation onto a constant and the lagged forward premium.

Following Clarida and Taylor (1997), I report the level of the RMSE and the MAE for the affine term structure model, while for the alternative forecasts the results are expressed as the ratio of the RMSE or the MAE to that obtained by the alternative method. For example, the level of the RMSE of the affine forecast for the U.S. dollar – pounds sterling rate one year ahead is 0.0496, while the ratio of this to the forecast obtained using a random-walk forecast is 0.637. That means a 36.3 per cent reduction in RMSE by using the forecasts produced by the affine term structure model as opposed to the random walk.

The results for the U.S. dollar – pounds sterling (Table 7 panel a) indicate that the affine term structure model produces the best out-of-sample forecasts among the four competing models. The linear vector autoregression and the random-walk model rank second and third, respectively. On the other side of the spectrum, the forward premium regression forecasts fail to outperform the random walk at all four horizons. In addition, the improvement of the affine term structure model forecasts with respect to those obtained from alternative models grows with the forecast horizon. For example, the improvement in RMSE with respect to the random walk at the one-month horizon is 2.2 per cent, at the three-month horizon is 9.7 per cent, at the six-month horizon is 18.2 per cent and, finally, at the twelve-month horizon is 36.3 per cent.

That the VAR forecasts are also able to beat the random walk is consistent with the results found by Clarida and Taylor (1997). However, these gains are smaller than those reported by the authors. For example, the improvement of the forecasts in RMSE at the one-year horizon produced by the VAR compared with those of a random walk is only 2.4 per cent.

Similar to the findings for the U.S. dollar – pounds sterling exchange rate, the affine term structure model produces the best out-of-sample forecasts of the U.S. dollar – Canadian dollar among the four competing models (Table 7, panel b). However, the VAR model fails to outperform the random-walk forecasts at all horizons. Thus, the random-walk model ranks second and the VAR and forward premium forecasts rank third and fourth, respectively.

[17] The results provided below are qualitatively similar to those obtained with other choices of the number of lags.

Again, the improvement of the affine term structure model forecasts with respect to those obtained from alternative models grows with the forecast horizon. In particular, at the one-month horizon, the improvement in RMSE with respect to the random walk is 1.6 per cent, at the three-month horizon is 4.0 per cent, at the six-month horizon is 6.2 per cent, and at the twelve-month horizon is 9.4 per cent. Because this gain is smaller than the one reported for the U.S. dollar – pounds sterling exchange rate, note that this model is successful in beating the random-walk model while the linear VAR is not. In particular, at the one-month horizon, the improvement in RMSE with respect to the VAR is 2.5 per cent, at the three-month horizon is 5.4 per cent, at the six-month horizon is 8.0 per cent, and at the twelve-month horizon is 13.1 per cent.

These results extend those of Clarida and Taylor (1997), who, using a linear VECM model in the spot and forward exchange rates, are able to obtain out-of-sample forecasts of spot exchange rates that beat the random-walk model. The results presented in this paper also extend those in Ang and Piazzesi (2003), who, imposing the cross-equation restrictions from no-arbitrage, are able to beat the random walk when they forecast bond yields, again, out-of-sample. Therefore, imposing the cross-equation restrictions from no-arbitrage can help in extracting information contained in the term structure of forward exchange premia that is useful to forecast exchange rates.

# 5. Conclusion

This paper provides an arbitrage-free empirical model that produces exchange forecasts that are superior to those produced by a purely time-series method such as a random-walk model or a vector autoregression on the forward premiums and the rate of depreciation. The intuition behind this success is that imposing no-arbitrage restrictions in the estimation of the joint dynamics of interest and exchange rates reduces the large number of parameters that characterize traditional time-series models. Consequently, it also reduces excessive parameter estimation uncertainty that may adversely affect the out-of-sample forecasting performance of a purely time-series model.

Several questions are left for further research. The first one is the role of non-linearities. Clarida et al. (2003) show that there is strong evidence of the presence of non-linearities in the joint behaviour of interest and exchange rates that can be used to outperform the forecasts obtained by using linear methods. Therefore, imposing no-arbitrage restrictions in non-linear models can potentially improve upon the results presented in this paper. A second possible extension is the use of an arbitrage-free joint model of interest rates, exchange rates, and macro variables to extract the information that interest rates and macro variables contain about the future evolution of exchange rates. Along these lines, Dong (2005) presents a structural VAR identified by the assumption of the absence of arbitrage, where the macro variables correspond to the output gap and inflation, and where the correlation between the model-implied rate of depreciation and the data is over 60 per cent. However, Dong (2005) does not conduct an out-of-sample prediction exercise. Finally, another possible extension is to include commodity prices into the set of macro variables that potentially can predict exchange rate movements. Since the Canadian dollar is usually considered as a commodity currency (see Amano and Van Norden 1998; Chen and Rogoff 2003),[18] this exercise can be especially helpful to improve upon the results presented here.

[18] For a review of the work on exchange rate modelling done at the Bank of Canada, see Bailliu and King (2005).

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Table 1: Summary Statistics for All Variables

| | Mean | Std. deviation | Autocorr. |
|---|---|---|---|
| **I. Depreciation Rate** $s_{t+1} - s_t$ | | | |
| Pounds sterling | -0.940 | 40.073 | 0.080 |
| Canadian dollar | -1.557 | 15.761 | -0.059 |
| | | | |
| **II. Interest Rates** $r_t$ | | | |
| *U.S.* | | | |
| 1 month | 7.966 | 3.558 | 0.969 |
| 3 months | 8.046 | 3.488 | 0.973 |
| 6 months | 8.103 | 3.372 | 0.974 |
| 12 months | 8.073 | 3.020 | 0.977 |
| | | | |
| *U.K.* | | | |
| 1 month | 10.571 | 3.437 | 0.959 |
| 3 months | 10.526 | 3.294 | 0.961 |
| 6 months | 10.388 | 3.082 | 0.963 |
| 12 months | 10.100 | 2.715 | 0.965 |
| | | | |
| *Canada* | | | |
| 1 month | 9.142 | 3.628 | 0.978 |
| 3 months | 9.199 | 3.513 | 0.979 |
| 6 months | 9.183 | 3.319 | 0.978 |
| 12 months | 9.069 | 2.987 | 0.978 |

Data are monthly and the sample is January 1976 to December 1997 (252 observations). All variables are measured in percentage points per year, and monthly rates of depreciation are annualized by multiplying by 1,200.

Table 2: Forward Premium Regressions

Panel a: Estimates and Individual Wald Tests

| | $a^{(h)}$ | $b^{(h)}$ | $H_0 : b^{(h)} = 1$ |
|---|---|---|---|
| **I. Pounds sterling** | | | |
| 1 month | -5.760 | -1.840 | 8.996 |
| | (3.044) | (0.947) | [0.003] |
| 3 months | -4.700 | -1.505 | 7.215 |
| | (2.937) | (0.933) | [0.007] |
| 6 months | -4.218 | -1.361 | 7.573 |
| | (2.647) | (0.858) | [0.006] |
| 12 months | -3.101 | -0.817 | 6.358 |
| | (2.249) | (0.721) | [0.012] |
| **II. Canadian dollar** | | | |
| 1 month | -3.172 | -1.351 | 32.337 |
| | (0.904) | (0.414) | [0.000] |
| 3 months | -2.526 | -0.827 | 21.941 |
| | (0.818) | (0.390) | [0.000] |
| 6 months | -1.927 | -0.425 | 16.349 |
| | (0.691) | (0.352) | [0.000] |
| 12 months | -1.680 | -0.240 | 8.856 |
| | (0.622) | (0.417) | [0.003] |

Panel b: Wald Test of the Joint Equality of the Four Forward Premium Regression Slopes

| $H_0 : b^{(h)} = 1$ | $\forall h = 1, 3, 6, 12$ |
|---|---|
| Pounds sterling | 12.990 |
| | [0.011] |
| Canadian dollar | 36.821 |
| | [0.000] |

Data are monthly and the sample is January 1976 to December 1997 (252 observations). Forward premium regressions are of the form $s_{t+h} - s_t = a^{(h)} + b^{(h)} p_t^{(h)} + u_{t+h}$, where $p_t^{(h)}$ is the interest rate differential $r_t^{(h)} - r_t^{*(h)}$ (also known as the forward premium). This equation is estimated by GMM, and Newey-West standard errors are presented in parentheses. The last column $H_0 : b^{(h)} = 1$ in panel a presents the value of the Wald test of the null hypothesis that the slope coefficient is equal to one. In large samples, this test is distributed as a $\chi^2$ with one degree of freedom. Panel b presents an equivalent Wald test of the null hypothesis that all four slopes coefficients are equal to one. In large samples, this test is distributed as a $\chi^2$ with four degrees of freedom. *P*-values are presented in brackets.

Table 3: Summary of the Properties of *Internationally Affine* Models

| | Interest rates $r_t^{(h)}$, $r_t^{*(h)}$ | | |
|---|---|---|---|
| | | *Linear* | *Quadratic* |
| **Expected rate of depreciation** $q_t^{(h)}$ | *Linear* | "Completely" affine DTSM | |
| | *Quadratic* | Gaussian "essentially" affine DTSM | Quadratic DTSM |

Table 4: Estimates of the Two-Factor Essentially Affine Model: U.S. dollar - Pounds sterling

| | Index number ($i$) | |
|---|---|---|
| Parameter | 1 | 2 |
| $\phi_{1i}$ | 0.0238 | 0 |
| | (0.0079) | |
| $\phi_{2i}$ | -0.0785 | 0.0935 |
| | (0.0324) | (0.0266) |
| $\theta_i$ | 0.6745 | 0.9006 |
| | (0.1977) | (0.1832) |
| $\sigma_{1i}$ | 0.0756 | 0 |
| | (0.0082) | |
| $\sigma_{2i}$ | 0 | 0.0862 |
| | | (0.0065) |
| $\lambda_{1i}$ | 0 | 0.1261 |
| | | (0.0527) |
| $\lambda_{2i}$ | -2.0673 | -0.7399 |
| | (0.8182) | (0.6636) |
| $\lambda_{0i}$ | -0.2412 | 5.9025 |
| | (0.0637) | (0.9747) |
| $\lambda^*_{1i}$ | 1.1885 | 0.6846 |
| | (0.6829) | (0.3947) |
| $\lambda^*_{2i}$ | 1.3813 | -0.4585 |
| | (0.6970) | (0.3224) |
| $\lambda^*_{0i}$ | -5.8778 | 0 |
| | (0.9960) | |

This table presents quasi-maximum likelihood (QML) estimates of the two-factor Gaussian essentially affine model defined in equations (22) and (23). These estimates are based on monthly observations of the rate of depreciation of the U.S. dollar - pounds sterling and 1-, 3-, 6-, and 12-month Eurocurrency interest rates in the United States and United Kingdom. The sample period is January 1976 to December 1997 (252 observations). American variables correspond with the index number 1 and British variables correspond with the number 2. Robust standard errors are provided in parentheses.

Table 5: Estimates of the Two-Factor Essentially Affine Model: U.S. dollar - Canadian dollar

| | Index number ($i$) | |
|---|---|---|
| Parameter | 1 | 2 |
| $\phi_{1i}$ | 0.0458 | 0 |
| | (0.0037) | |
| $\phi_{2i}$ | -0.1995 | 0.1999 |
| | (0.0283) | (0.0238) |
| $\theta_i$ | 1.0570 | 1.1795 |
| | (0.0589) | (0.0733) |
| $\sigma_{1i}$ | 0.0639 | 0 |
| | (0.0063) | |
| $\sigma_{2i}$ | 0 | 0.0637 |
| | | (0.0043) |
| $\lambda_{1i}$ | 0 | 0 |
| $\lambda_{2i}$ | 16.2445 | -12.2519 |
| | (3.8478) | (2.8206) |
| $\lambda_{0i}$ | 0 | -1.9329 |
| | | (1.1279) |
| $\lambda^*_{1i}$ | 16.2852 | -12.2643 |
| | (3.8170) | (2.8048) |
| $\lambda^*_{2i}$ | 0 | 0 |
| $\lambda^*_{0i}$ | -1.8498 | 0 |
| | (1.1153) | |

This table presents quasi-maximum likelihood (QML) estimates of the two-factor Gaussian essentially affine model defined in equations (22) and (23). These estimates are based on monthly observations of the rate of depreciation of the U.S. dollar - Canadian dollar and 1-, 3-, 6-, and 12-month Eurocurrency interest rates in the United States and Canada. The sample period is January 1976 to December 1997 (252 observations). American variables correspond with the index number 1 and Canadian variables correspond with the number 2. Robust standard errors are provided in parentheses.

Table 6: Implied Forward Premium Regression Slopes

Panel a: U.S. dollar - pounds sterling

| | 1-month | 3-month | 6-month | 12-month |
|---|---|---|---|---|
| OLS | -1.840 | -1.505 | -1.361 | -0.817 |
| Implied | -2.001 | -1.945 | -1.878 | -1.788 |

Panel b: U.S. dollar - Canadian dollar

| | 1-month | 3-month | 6-month | 12-month |
|---|---|---|---|---|
| OLS | -1.351 | -0.827 | -0.425 | -0.240 |
| Implied | -0.578 | -0.536 | -0.481 | -0.411 |

This table presents the term structure of forward premium regression slopes implied by the two-factor Gaussian essentially affine model defined in equations (22) and (23). These are computed using the closed-form formulae derived in the appendix and by treating the estimates displayed in Tables 4 and 5 as truth. The sample OLS estimates of these slopes are reproduced again for the sake of comparison.

Table 7: Comparison of Out-of-sample Forecasting Performance

Panel a: U.S. dollar - pounds sterling

| | Affine (level) | VAR(2) (ratio) | RW (ratio) | OLS (ratio) |
|---|---|---|---|---|
| Root mean square error (*RMSE*) | | | | |
| 1-month horizon | 0.0205 | 0.975 | 0.978 | 0.980 |
| 3-month horizon | 0.0300 | 0.925 | 0.903 | 0.909 |
| 6-month horizon | 0.0403 | 0.845 | 0.819 | 0.820 |
| 12-month horizon | 0.0496 | 0.653 | 0.637 | 0.584 |
| | | | | |
| Mean absolute error (*MAE*) | | | | |
| 1-month horizon | 0.0167 | 0.971 | 0.959 | 0.975 |
| 3-month horizon | 0.0255 | 0.999 | 0.952 | 0.988 |
| 6-month horizon | 0.0312 | 0.830 | 0.828 | 0.818 |
| 12-month horizon | 0.0414 | 0.666 | 0.632 | 0.601 |

Panel b: U.S. dollar - Canadian dollar

| | Affine (level) | VAR(1) (ratio) | RW (ratio) | OLS (ratio) |
|---|---|---|---|---|
| Root mean square error (*RMSE*) | | | | |
| 1-month horizon | 0.0189 | 0.975 | 0.984 | 0.977 |
| 3-month horizon | 0.0343 | 0.946 | 0.960 | 0.944 |
| 6-month horizon | 0.0480 | 0.920 | 0.938 | 0.914 |
| 12-month horizon | 0.0626 | 0.847 | 0.907 | 0.813 |
| | | | | |
| Mean absolute error (*MAE*) | | | | |
| 1-month horizon | 0.0150 | 0.959 | 0.967 | 0.965 |
| 3-month horizon | 0.0277 | 0.976 | 0.995 | 0.973 |
| 6-month horizon | 0.0363 | 0.932 | 0.933 | 0.935 |
| 12-month horizon | 0.0467 | 0.869 | 0.887 | 0.824 |

This table presents the results of the out-of-sample forecasting exercise during the last seven years of the sample (January 1998 - December 2004). For the two-factor Gaussian essentially affine model, the RMSE or the MAE is expressed in levels. For the alternative forecasts, the RMSE or the MAE is expressed as the inverse of its ratio to the corresponding figure for the affine model. Therefore, a figure less than one indicates superior relative performance by the VECM.

# Appendix A: Proof of Proposition 1

In this appendix, I show that if the drift of the log exchange rate, $s_t$, is linear in a set of state variables $\mathbf{x}_t$, and $\mathbf{x}_t$ follows an affine diffusion, then the expected rate of depreciation is a (known) linear function of the state vector $\mathbf{x}_t$. However, to show this point I need one previous result.

**Lemma 2** *If the process $\mathbf{x}_t$ follows the affine diffusion given by (13), then*

$$E_t\left[\int_t^{t+h}\mathbf{x}_\tau d\tau\right]=\boldsymbol{\theta}h+\boldsymbol{\Phi}^{-1}\left[I-e^{-\boldsymbol{\Phi}h}\right]\left[\mathbf{x}_t-\boldsymbol{\theta}\right]. \tag{A1}$$

**Proof.** First note that (Fackler 2000) when $\mathbf{x}_t$ follows an affine diffusion:

$$E_t\mathbf{x}_{t+h}=\boldsymbol{\theta}+e^{-\boldsymbol{\Phi}h}(\mathbf{x}_t-\boldsymbol{\theta}).$$

Second, take expectations with respect to the integral form of (13):

$$E_t\left[\int_t^{t+h}d\mathbf{x}_\tau\right]=\boldsymbol{\Phi}\boldsymbol{\theta}+\boldsymbol{\Phi}E_t\left[\int_t^{t+h}\mathbf{x}_\tau d\tau\right].$$

Finally, notice that $E_t\left[\int_t^{t+h}d\mathbf{x}_\tau\right]=E_t\mathbf{x}_{t+h}-\mathbf{x}_t$ and solve this last equation for $E_t\left[\int_t^{t+h}\mathbf{x}_\tau d\tau\right]$ to obtain (A1) ■

Note that the variable inside the conditional expectation $\mathbf{y}_t^{(h)}=\int_t^{t+h}\mathbf{x}_\tau d\tau$ is a flow variable. In particular, a lot of attention has been paid to obtaining the process that a set of discretely sampled data follows when these observations (whether stock or flow, or a combination of both) have been generated by an underlying continuous-time model (Bergstrom 1984). However, this literature has relied on the assumption that this underlying continuous-time process is a multivariate version of the Orstein-Uhlenbeck process. Here, the distributional assumption is relaxed to allow for an affine diffusion at the cost of restricting the predictions only to the conditional expectation of the flow variable (instead of the whole distribution of $\mathbf{y}_t^{(h)}$). Nonetheless, this result is enough to prove the linearity of the expected rate of depreciation, because the expected rate of depreciation satisfies

$$E_t\left[s_{t+h}-s_t\right]=E_t\left[\int_t^{t+h}ds_\tau\right]=\gamma_0 h+\boldsymbol{\gamma}'E_t\left[\int_t^{t+h}\mathbf{x}_\tau d\tau\right],$$

and once I substitute (A1) into this last expression I obtain the desired result.

# Appendix B: From Quadratic to Affine in an Augmented Set of State Variables

Quadratic models of the term structure generate interest rates that are quadratic (known) functions of a set of state variables denoted by $\mathbf{x}_t = [x_{1t}, x_{2t}, \ldots, x_{nt}]'$:

$$r_t^{(*)(h)} = A^{(*)}(h) + B^{(*)}(h)'\mathbf{x}_t + \mathbf{x}_t' C^{(*)}(h)\mathbf{x}_t,$$

where the coefficients $A^{(*)}(h) \in \mathbb{R}$, $B^{(*)}(h) \in \mathbb{R}^n$, and $C^{(*)}(h) \in \mathbb{R}^{n\times n}$ solve two systems of ordinary differential equations. As claimed in the main text, this expression can be expressed as an affine function of the original set of state variables and a new set of state variables $\mathbf{z}_t = vech\left[\mathbf{x}_t\mathbf{x}_t'\right]$, that includes the squares of the original ones as well as the corresponding cross-products. To realize why this is true, first note that for a given $n \times n$ matrix $\mathbf{\Gamma}$ it can be shown that

$$\mathbf{x}_t'\mathbf{\Gamma}\mathbf{x}_t = tr(\mathbf{x}_t'\mathbf{\Gamma}\mathbf{x}_t) = tr(\mathbf{\Gamma}\mathbf{x}_t\mathbf{x}_t').$$

Then, use the fact that $tr(\mathbf{\Gamma}\mathbf{x}_t\mathbf{x}_t') = vec(\mathbf{\Gamma})'vec(\mathbf{x}_t\mathbf{x}_t')$; and notice that $\mathbf{x}_t\mathbf{x}_t'$ is an $n \times n$ symmetric matrix so that, $vec(\mathbf{x}_t\mathbf{x}_t') = \mathbf{D}_n vech(\mathbf{x}_t\mathbf{x}_t')$, where $\mathbf{D}_n$ is the duplication matrix and whose details can be found in Lütkepohl (1993, 464-5). This makes:

$$\mathbf{x}_t'\mathbf{\Gamma}\mathbf{x}_t = vec(\mathbf{\Gamma})'\mathbf{D}_n vech(\mathbf{x}_t\mathbf{x}_t') = vec(\mathbf{\Gamma})'\mathbf{D}_n\mathbf{z}_t. \tag{B1}$$

If I specialize this result to the case when $\mathbf{\Gamma} = B_2^{(*)}(h)$, it delivers interest rates that are (known) affine in the augmented set of state variables given by $\widetilde{\mathbf{x}}_t = [\mathbf{x}_t', \mathbf{z}_t']'$ with $\mathbf{z}_t = vech(\mathbf{x}_t\mathbf{x}_t')$:

$$r_t^{(*)(h)} = A^{(*)}(h) + B_1^{(*)}(h)'\mathbf{x}_t + \left[vec\left(B_2^{(*)}(h)\right)\right]'\mathbf{D}_n\mathbf{z}_t.$$

By a similar reasoning, it can be shown that the expected rate of depreciation is also affine in this augmented set of factors. This property requires the drift of the (log) exchange rate process to be affine in this new set of state variables, and the process that these variables follow must be itself an affine diffusion. First, remember that the law of motion of the (log) exchange rate is:

$$ds_t = (\gamma_0 + \boldsymbol{\gamma}_1'\mathbf{x}_t + \mathbf{x}_t'\gamma_2\mathbf{x}_t)dt + \left[(\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_0^*) + (\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_1^*)\mathbf{x}_t\right]' d\mathbf{W}_t, \tag{B2}$$

where $\gamma_0 = (\delta_0 - \delta_0^*) + \frac{1}{2}(\boldsymbol{\lambda}_0'\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_0^{*\prime}\boldsymbol{\lambda}_0^*)$, $\boldsymbol{\gamma}_1 = (\boldsymbol{\delta}_1 - \boldsymbol{\delta}_1^*) + (\boldsymbol{\lambda}_1'\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_1^{*\prime}\boldsymbol{\lambda}_0^*)$, and $\boldsymbol{\gamma}_2 = \boldsymbol{\delta}_2 - \boldsymbol{\delta}_2^* + \frac{1}{2}(\boldsymbol{\lambda}_1'\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_1^{*\prime}\boldsymbol{\lambda}_1^*)$. Then, we use again equation (B1) with $\boldsymbol{\Gamma} = \boldsymbol{\gamma}_2$ to obtain:

$$E_t ds_t = \left[\gamma_0 + \boldsymbol{\gamma}_1'\mathbf{x}_t + [vec(\boldsymbol{\gamma}_2)]'\,\mathbf{D}_n\mathbf{z}_t\right] dt, \tag{B3}$$
$$E_t ds_t = (\gamma_0 + \widetilde{\boldsymbol{\gamma}}'\widetilde{\mathbf{x}}_t)dt,$$

with $\widetilde{\mathbf{x}}_t = [\mathbf{x}_t', \mathbf{z}_t']'$ and $\widetilde{\boldsymbol{\gamma}} = \left[\boldsymbol{\gamma}_1', [vec(\boldsymbol{\gamma}_2)]'\,\mathbf{D}_n\right]'$.

Second, it can be shown that if I apply Itô's lemma on $\mathbf{z}_t = vech\,[\mathbf{x}_t\mathbf{x}_t']$ then the joint process for $\mathbf{x}_t$ and $\mathbf{z}_t$ is an affine diffusion (see Appendix B in Cheng and Scaillet 2002). In particular, the law of motion of the augmented set of factors $\widetilde{\mathbf{x}}_t$ satisfies:

$$d\begin{bmatrix}\mathbf{x}_t \\ \mathbf{z}_t\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Phi} & 0 \\ \boldsymbol{\Phi}_{\mathbf{zx}} & \boldsymbol{\Phi}_{\mathbf{zz}}\end{bmatrix}\left[\begin{pmatrix}\boldsymbol{\theta} \\ \boldsymbol{\theta}_{\mathbf{z}}\end{pmatrix} - \begin{pmatrix}\mathbf{x}_t \\ \mathbf{z}_t\end{pmatrix}\right]dt + \begin{bmatrix}\boldsymbol{\Sigma}^{1/2} \\ \boldsymbol{\Sigma}_{\mathbf{z}}(\mathbf{x}_t)^{1/2}\end{bmatrix}d\mathbf{W}_t, \tag{B4}$$
$$d\widetilde{\mathbf{x}}_t = \widetilde{\boldsymbol{\Phi}}(\widetilde{\boldsymbol{\theta}} - \widetilde{\mathbf{x}}_t)dt + \widetilde{\boldsymbol{\Sigma}}(\mathbf{x}_t)^{1/2}d\mathbf{W}_t,$$

where the drift is linear with:

$$\boldsymbol{\Phi}_{\mathbf{zz}} = 2\mathbf{D}_n^+(\boldsymbol{\Phi} \otimes \mathbf{I}_n)\mathbf{D}_n,$$
$$\boldsymbol{\Phi}_{\mathbf{zx}} = -2\mathbf{D}_n^+(\boldsymbol{\Phi}\boldsymbol{\theta} \otimes \mathbf{I}_n),$$
$$\boldsymbol{\theta}_{\mathbf{z}} = \boldsymbol{\Phi}_{\mathbf{zz}}^{-1}\left(vech(\boldsymbol{\Sigma}) - \boldsymbol{\Phi}_{\mathbf{zx}}\boldsymbol{\theta}\right),$$

where $\mathbf{D}_n^+$ is the Moore-Penrose inverse of matrix $\mathbf{D}_n$: $\mathbf{D}_n^+ = (\mathbf{D}_n'\mathbf{D}_n)^{-1}\mathbf{D}_n'$. In addition, the diffusion term satisfies:

$$\boldsymbol{\Sigma}_{\mathbf{z}}(\mathbf{x}_t)^{1/2} = 2\mathbf{D}_n^+(\boldsymbol{\Sigma}^{1/2} \otimes \mathbf{x}_t),$$

which implies a volatility matrix $\widetilde{\boldsymbol{\Sigma}}$ whose elements are affine in $\mathbf{x}_t$ and $\mathbf{x}_t\mathbf{x}_t'$ (and, therefore, affine in $\mathbf{x}_t$ and $\mathbf{z}_t$):

$$\widetilde{\boldsymbol{\Sigma}} = \begin{pmatrix}\boldsymbol{\Sigma} & 2(\boldsymbol{\Sigma}' \otimes \mathbf{x}_t')\mathbf{D}_n^+ \\ 2\mathbf{D}_n^+(\boldsymbol{\Sigma} \otimes \mathbf{x}_t) & 4\mathbf{D}_n^+(\boldsymbol{\Sigma} \otimes \mathbf{x}_t\mathbf{x}_t')\mathbf{D}_n^{+\prime}\end{pmatrix}.$$

# Appendix C: Closed-Form Expressions of the Implied Uncovered Interest Parity Regression Slope

For expositional purposes, collect the expected rate of depreciation and the forward premia in a vector, $\mathbf{y}_t^{(h)} = [q_t^{(h)}, p_t^{(h)}]$. In addition, denote $\mathbf{\Omega}(h) = Var[\mathbf{y}_t^{(h)}]$ as the unconditional variance of $\mathbf{y}_t^{(h)}$.

The definition of the (population) uncovered interest parity regression slope in equation (1) is

$$b(h) = \frac{Cov[s_{t+h} - s_t, p_t^{(h)}]}{Var[p_t^{(h)}]} = \frac{Cov[q_t^{(h)}, p_t^{(h)}]}{Var[p_t^{(h)}]},$$

where the second equality comes from the law of iterated expectations. For convenience, rewrite this expression as:

$$b(h) = \frac{e_1' \mathbf{\Omega}(h) e_2}{e_2' \mathbf{\Omega}(h) e_2},$$

where $e_1' = (1,0)$ and $e_2' = (0,1)$. The numerator of this equation is the covariance between the expected rate of depreciation and the forward premia, while the denominator is the variance of the forward premia.

However, given that $\mathbf{y}_t^{(h)}$ is a linear function of the states $\mathbf{x}_t$:

$$\mathbf{y}_t^{(h)} = \mathbf{\Psi}_0(h) + \mathbf{\Psi}_1(h)' \mathbf{x}_t,$$

with

$$\mathbf{\Psi}_0(h) = \begin{bmatrix} C(h) \\ A(h) - A^*(h) \end{bmatrix},$$

$$\mathbf{\Psi}_1(h) = \begin{bmatrix} D(h) \\ B(h) - B^*(h) \end{bmatrix},$$

it is straightforward to realize that the unconditional variance of $\mathbf{y}_t^{(h)}$ is related to the unconditional variance of the factors in the following way:

$$\mathbf{\Omega}(h) = Var[\mathbf{z}_t^{(h)}] = \mathbf{\Psi}_1(h)' Var[\mathbf{x}_t] \mathbf{\Psi}_1(h).$$

Therefore, computing the implied uncovered interest parity regression slope amounts to computing the unconditional variance of the factor $\mathbf{x}_t$. If the state-vector $\mathbf{x}_t$ follows an affine diffusion (as in the case of the affine models of currency pricing), I can use the explicit *formulae* for the unconditional variance provided in Fackler (2000). Specializing his results to this case, I obtain:

$$vec\left[Var(\mathbf{x}_t)\right] =$$
$$\left[\mathbf{\Phi} \otimes I + I \otimes \mathbf{\Phi}\right]^{-1} \left[\mathbf{\Sigma}^{1/2} \otimes \mathbf{\Sigma}^{1/2}\right] vec\left[diag\left[\mathbf{a} + \mathbf{B}\boldsymbol{\theta}\right]\right],$$

being

$$\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} \beta_1' \\ \beta_2' \\ \vdots \\ \beta_N' \end{bmatrix}.$$

For the quadratic models, I can exploit the fact that the unconditional distribution of the original state-vector $\mathbf{x}_t$ is Gaussian with mean $\boldsymbol{\theta}$ and variance such that $vec\left[Var(\mathbf{x}_t)\right] = \left[\mathbf{\Phi} \otimes I + I \otimes \mathbf{\Phi}\right]^{-1} vec(\mathbf{\Sigma})$. Still, I need the covariances between $\mathbf{x}_t$ and $\mathbf{z}_t = vech\left[\mathbf{x}_t\mathbf{x}_t'\right]$. The elements of this matrix can be computed using the fact that, if three variables $x$, $y$, and $z$ are jointly normally distributed, then:

$$Cov(xy, z) = \mu_x Cov(y, z) + \mu_y Cov(x, z).$$

Finally, the variance-covariance matrix of $\mathbf{z}_t$ can be computed using the fact that, if four variables $x$, $y$, $u$, and $v$ are jointly normally distributed, then:

$$Cov(xy, uv) = \mu_x\mu_u Cov(y, v) + \mu_y\mu_u Cov(x, v) +$$
$$\mu_x\mu_v Cov(y, u) + \mu_y\mu_v Cov(x, u) +$$
$$Cov(y, u)Cov(x, v) + Cov(x, u)Cov(y, v).$$

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