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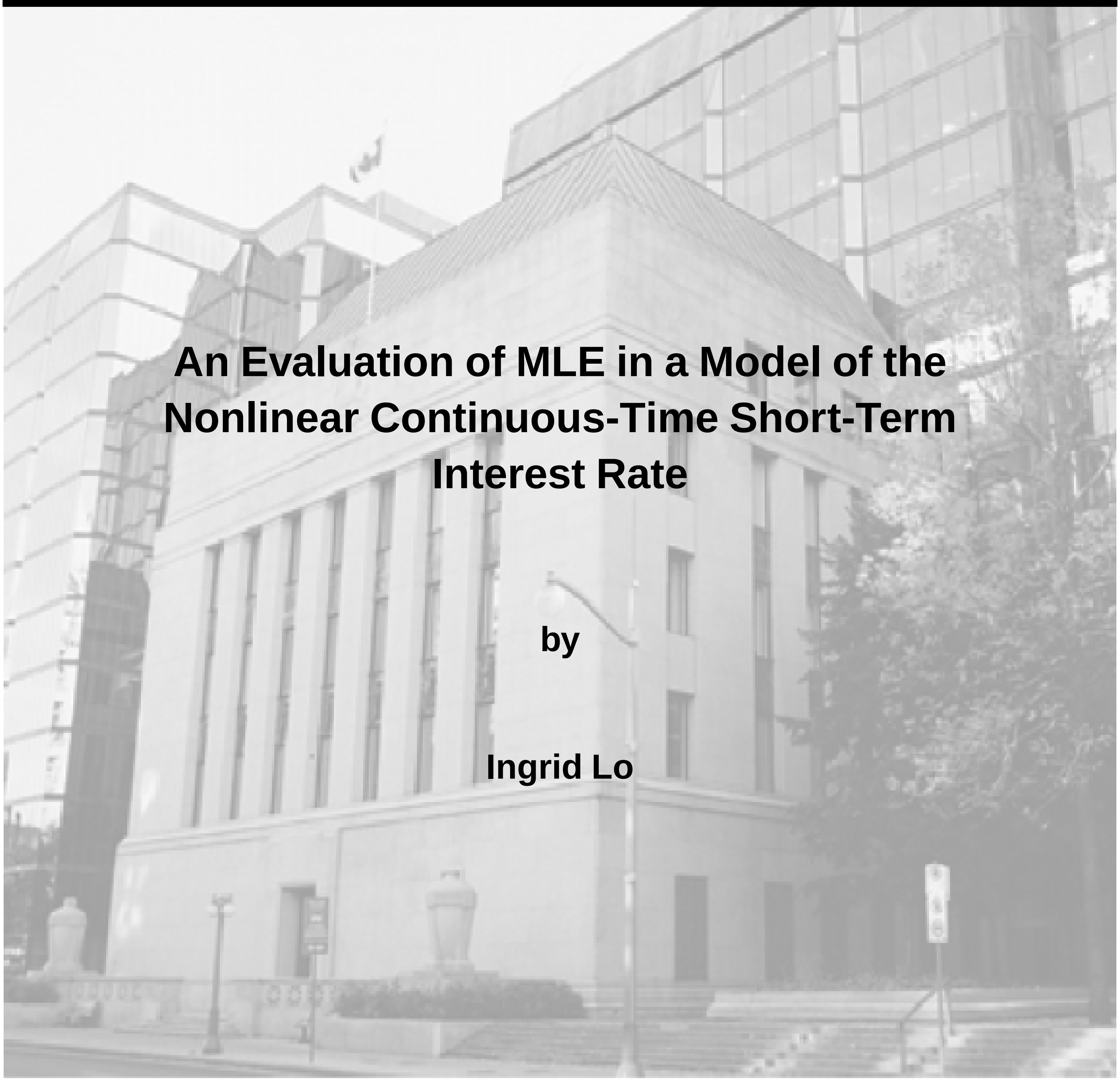

An Evaluation of MLE in a Model of the Nonlinear Continuous-Time Short-Term Interest Rate

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Ingrid Lo

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Ingrid Lo

Financial Markets Department
Bank of Canada
Ottawa, Ontario, Canada K1A 0G9
ingridlo@bankofcanada.ca

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Abstract

The author compares the performance of three Gaussian approximation methods—by Nowman (1997), Shoji and Ozaki (1998), and Yu and Phillips (2001)—in estimating a model of the nonlinear continuous-time short-term interest rate. She finds that the performance of Nowman's method is similar to that of Shoji and Ozaki's method, whereas the window width used in the Yu and Phillips method has a critical influence on parameter estimates. When a small window width is used, the Yu and Phillips method does not outperform the other two methods. Choosing a suitable window width can reduce estimation bias quite significantly, whereas too large a window width can worsen estimation bias and the fit of the model. An empirical study is implemented using Canadian and U.K. one-month interest rate data.

JEL classification: C1, E4
Bank classification: Interest rates; Econometric and statistical methods

Résumé

L'auteure compare l'efficacité de trois méthodes d'approximation gaussienne — proposées par Nowman (1997), Shoji et Ozaki (1998) et Yu et Phillips (2001) — pour l'estimation d'un modèle en temps continu non linéaire du taux d'intérêt à court terme. Elle constate que la méthode de Nowman est aussi efficace que celle de Shoji et Ozaki, mais que la largeur de la fenêtre retenue dans le cas de la méthode de Yu et Phillips a un effet déterminant sur la valeur estimée des paramètres. Lorsque la fenêtre utilisée est trop étroite, la méthode de Yu et Phillips n'est pas supérieure aux deux autres. Le choix d'une fenêtre de largeur appropriée peut réduire le biais d'estimation de façon importante, alors que celui d'une fenêtre trop large peut entraîner une détérioration de l'ajustement statistique du modèle et accentuer le biais d'estimation. L'analyse empirique de l'auteure met à contribution les données relatives aux taux d'intérêt à un mois canadien et britannique.

Classification JEL : C1, E4
Classification de la Banque : Taux d'intérêt; Méthodes économétriques et statistiques

1. Introduction

Use of the continuous-time framework in modelling interest rates has been popular since the seminal work of Merton (1973). Much of the literature utilizes nonlinear stochastic differential equations in formulating interest rate models. Examples include Dothan (1978), Brennan and Schwartz (1979), Cox, Ingersoll, and Ross (1985) (hereafter, CIR), and Chan et al. (1992) (hereafter, CKLS). Two common features of these models are (i) the volatility of the interest rate is dependent, linearly or nonlinearly, on the level of the interest rate, and (ii) the drift is linear in the interest rate. The exact distribution of the interest rate for discrete time intervals in this class of models is often complicated or even unknown, so it is a challenge for econometric estimation. The discrete-time solution exists only for the Ornstein-Uhlenbeck process, the Brownian motion, and the CIR process. Except for these cases, maximum-likelihood estimation (MLE) cannot be undertaken because the transition density in this class has no closed functional form.

This paper compares three Gaussian approximation methods proposed by Nowman (1997), Shoji and Ozaki (1998), and Yu and Phillips (2001). Both Nowman's method and Shoji and Ozaki's method transform the stochastic differential equation into one in which a closed-form solution exists. Nowman (1997) assumes that the volatility of the interest rate in the current period is proportional to the interest rate in the previous period. Thus, the approximated process becomes an Ornstein-Uhlenbeck process. Shoji and Ozaki (1998) first transform the nonlinear diffusion process into a process with a unit diffusion coefficient. Then the transformed drift, which is nonlinear, is approximated by a locally linear function. Thus, their process also becomes an Ornstein-Uhlenbeck process and the transition density function remains Gaussian. Yu and Phillips (2001) utilize the continuous martingale property: after a suitable time change, a non-Gaussian diffusion process becomes an exact Gaussian process.

Since all three methods are only approximations of the true data-generating process, the following question arises: which method gives the best approximation of the true process? Shoji and Ozaki compare their approximation method with the Euler method. Yu and Phillips compare theirs with Nowman's method. The literature lacks a comprehensive study of how well each of these Gaussian approximations work. This paper aims to fill this void by comparing the performance of these approximation methods. In this study, two models are investigated: the CIR model and the nonlinear diffusion process (i.e, the CKLS model). Three measures of performance are used: the empirical distribution of the parameters, the average likelihood of the model, and the mean squared error (MSE) of the model. The performance of Shoji and Ozaki's method is found to be similar to Nowman's method. Another contribution of this paper is that different values of the window width in the Yu and Phillips method are examined. The window width plays a critical role in the transformation of the original process into an exact Gaussian process, but it is an exogenous parameter. It is found that (i) with too small a window width, the Yu and Phillips method does not outperform the other two methods, (ii) choosing a suitable window width can reduce estimation bias quite significantly, and (iii) too large a window width can worsen the estimation bias and the fit of the model.

Many other empirical estimation methods have been proposed recently to deal with the continuous-time interest rate models. They include Lo (1988), which solves for the PDF numerically; Pedersen (1995); and Santa-Clara (1995), which uses a simulated MLE method and simulates a large number of sample paths, and then computes the conditional density by numerical integration. These methods do not allow for a closed-form solution to be maximized. Other likelihood-based methods include Ait-Sahalia (1999) and Jiang and Knight (2001), which use Hermite/Graham-Charlier expansions on the conditional density; Singleton (2001), which applies Fourier inversion of the characteristic function; and Jiang and Knight (2002), which minimizes the integrated MSE between empirical and theoretical characteristic functions. For simplicity, this paper will focus on the approximation method estimated via Gaussian density.

This paper is organized as follows. Section 2 reviews the three approximation methods. Section 3 describes the simulation design and reports the simulation results of the three approximation methods. Section 4 examines some empirical results using the three approximations. Section 5 offers some conclusions.

2. The Three Approximation Methods

The general diffusion process examined in this paper was investigated by CKLS and is given by

$$dr\left(t\right)=\left(\alpha+\beta r\left(t\right)\right)dt+\sigma r^{\gamma}\left(t\right)dB\left(t\right), \tag{1}$$

where $r(t)$ is the short-term interest rate, α, β, σ, and γ are unknown parameters, and $B(t)$ is the standard Brownian motion. Equation (1) encompasses many model specifications, as shown in CKLS (1992). They are reproduced in Table 1.

The solution of (1) is given by

$$r(t) = \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) + e^{\beta} r(t-1) + \int_{t-1}^{t} \sigma e^{\beta(t-\tau)} r^{\gamma}(\tau)\, dB(\tau). \tag{2}$$

MLE does not have a closed form, since we cannot find the transition density $f(r_t \mid r_{t-1})$, except in a few cases (with $\gamma = 0$ and $\gamma = \frac{1}{2}$). The idea behind two of the approximation methods examined is to transform equation (1) into a form where $f(r_t \mid r_{t-1})$ has a closed form. In particular, by transforming to an Ornstein-Uhlenbeck process, which has a linear drift and constant diffusion, the transition density is Gaussian and MLE can be easily applied. The Ornstein-Uhlenbeck process is given by

$$dx_t = (\alpha + \beta x_t)dt + \sigma dB_t,$$

with the solution

$$x_t = \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) + e^{\beta} x_{t-1} + \int_{t-1}^{t} \sigma e^{\beta(t-\tau)} dB(\tau),$$

and the transition density[1]

$$x_{t+\triangle} \mid x_t \sim N(-\frac{\alpha}{\beta} + (x_t + \frac{\alpha}{\beta})e^{\beta\triangle}, \frac{\sigma^2}{2\beta}(e^{\beta\triangle} - 1)).$$

Nowman (1997) converted equation (1) to an Ornstein-Uhlenbeck process by replacing the time-varying diffusion coefficient (i.e., σr_t^{γ}) with a constant σr_{t-1}^{γ}, which is a first-order approximation around r_{t-1}. Thus, he assumes that this diffusion coefficient remains constant from $t-1$ to t. Consequently, solution (2) collapses to

$$r(t) = \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) + e^{\beta} r(t-1) + r^{\gamma}(t-1) \int_{t-1}^{t} \sigma e^{\beta(t-\tau)} dB(\tau), \tag{3}$$

and the transition density is given by

$$r_{t+\triangle} \mid r_t \sim N(-\frac{\alpha}{\beta} + (r_t + \frac{\alpha}{\beta})e^{\beta\triangle}, \frac{\sigma^2 r_t^{2\gamma}}{2\beta}(e^{\beta\triangle} - 1)).$$

The approximated process becomes a heterogeneous Gaussian process, with variance at time t scaled by the short-term interest rate in the previous period. The performance of this approximation rests

[1] $N(\mu, \sigma^2)$ refers to a normal distribution with mean μ and variance σ^2.

on the assumption that the short-term interest rate remains fairly stable from the previous period to the current period. It works best if (i) the short-term interest rate is not too volatile over time, or (ii) the short-term interest rate is sampled at a higher frequency so that, with a shorter time interval between observations, the values of the observations are closer to each other.

Shoji and Ozaki's (1998) approximation method was originally designed to deal only with a model with a nonlinear drift and a constant diffusion. Their method, however, can be applied to solve the time-varying diffusion by first transforming the process into one with a unit diffusion coefficient. The interest rate, $r(t)$, is transformed so that

$$Y_t = G(r(t)) = \frac{r(t)^{1-\gamma}}{(1-\gamma)\sigma}. \tag{4}$$

Applying Ito's formula, the transformed process becomes

$$dY_t = \mu_Y(Y_t)\,dt + dB(t), \tag{5}$$

in which the drift coefficient of (5) is given by

$$\mu_Y(Y_t) = \frac{\alpha + \beta G^{-1}(Y_t)}{\sigma\left(G^{-1}(Y_t)\right)^{\gamma}} - \frac{1}{2}\frac{\partial\sigma\left(G^{-1}(Y_t)\right)^{\gamma}}{\partial(G^{-1}(Y_t))}.$$

The time-varying volatility in the disturbance term is shifted into the drift term. The transformed drift term is the original drift term scaled by the inverse of the volatility and adjusted by the sensitivity of the volatility to the change in the short-term interest rate. The drift term is then linearized. Assuming that $\frac{\partial\mu_Y(Y_{t-1})}{\partial Y_t}$ is constant, the process specified in equation (5) is approximated by

$$dY_t = (N_{t-1} + L_{t-1}Y_t)\,dt + dB(t), \tag{6}$$

where

$$\begin{aligned}
L_{t-1} &= \frac{\partial\mu_Y(Y_{t-1})}{\partial Y_t}, \\
&= \frac{-\gamma\alpha}{(\sigma(1-\gamma))^{\frac{1}{1-\gamma}}} Y_{t-1}^{\frac{-1}{1-\gamma}} + \frac{\beta}{1-\gamma}Y_{t-1} + \frac{\gamma}{2(1-\gamma)}Y_{t-1}^{-2}; \\
N_{t-1} &= \mu_Y(Y_{t-1}) - L_{t-1}Y_{t-1}, \\
&= Y_{t-1}^{\frac{-\gamma}{1-\gamma}}\left[\frac{\alpha}{(\sigma(1-\gamma))^{\frac{1}{1-\gamma}}}\right] - \frac{\gamma}{(1-\gamma)}Y_{t-1}^{-1}.
\end{aligned}$$

Thus the transformed process, (6), is an Ornstein-Uhlenbeck process with the solution:

$$Y_t = \frac{N_{t-1}}{L_{t-1}}\left(e^{L_{t-1}} - 1\right) + e^{L_{t-1}}Y_{t-1} + \int_{t-1}^{t} e^{L_{t-1}(t-\tau)}dB(\tau).$$

The disturbance term of the approximated process, $\int_{t-1}^{t} e^{L_{t-1}(t-\tau)} dB(\tau)$, follows a Gaussian distribution with mean zero and variance

$$\left(\frac{e^{2L_{t-1}}-1}{2L_{t-1}}\right).$$

The approximated process is heterogeneous in the mean reversion parameter, L_{t-1}, which is the marginal change in the transformed drift in response to a change in the transformed short-term interest rate. Thus, the transition density is given by

$$Y_{t+\triangle} \mid Y_t \sim N(-\frac{N_t}{L_t} + (Y_t + \frac{N_t}{L_t})e^{L_t\triangle}, \frac{(e^{L_t\triangle}-1))}{2L_t}).$$

Yu and Phillips (2001) utilize the continuous martingale property stated in the Dambis, Dubins-Schwarz theorem (see Revuz and Yor 1999). According to the theorem, with a suitable time change, all semi-martingales become Brownian motion. The sample for estimation is chosen such that, for any fixed constant $a > 0$, the time space between two consecutive observations is determined by h:

$$h = \inf\left\{s \mid \sigma^2 \int_0^s e^{2\beta(s-\tau)} r^{2\gamma}(t+\tau)\, d\tau \geq a\right\}. \tag{7}$$

Thus, in a discrete-time model, the observation $r(t)$ is not necessarily followed by $r(t+1)$: the immediate observation after $r(t)$ is $r(t+h)$, in which h can be any integer greater than or equal to 1. Also, the magnitude of h changes over time, depending on the level of the short-term interest rate and the volatility parameters, σ and γ. As discussed in Yu and Phillips (2001), a period of high volatility in the short-term interest rate with larger $\sigma^2 r^2(t)$ would lead to a smaller h and thus more frequent sampling. In implementing the estimation, the parameters σ^2, β, and γ are replaced by the estimates obtained from estimating Nowman's model.

The process r at time $t+h$ is given by

$$r(t+h) = \frac{\alpha}{\beta}\left(e^{\beta h}-1\right) + e^{\beta h} r(t) + \int_0^h \sigma e^{\beta(h-\tau)} r^{\gamma}(t+\tau)\, dB(\tau). \tag{8}$$

Utilizing the Dambis, Dubins-Schwarz theorem, the martingale, $\int_0^h \sigma e^{\beta(h-\tau)} r^{\gamma}(t+\tau)\, dB(\tau)$, can be written as a Brownian motion with a time change and

$$\int_0^h \sigma e^{(h-\tau)} r^{\gamma}(t+\tau)\, dB(\tau) \sim N(0,a),$$

in which a is the window width used in equation (7). Thus, the interest rate dynamics in (8) follows with

$$r_{t+h} \mid r_t \sim N(-\frac{\alpha}{\beta} + (r_t + \frac{\alpha}{\beta})e^{\beta h}, a).$$

In summary, the Yu and Phillips method differs from the previous two approximation methods in three respects. First, the process is homogeneous, with variance equal to the constant, a. Second, Yu and Phillips (2001) estimate only the drift coefficients, α and β. Their estimation method does not estimate σ and γ. Yu and Phillips (2001) use estimates of σ, γ, and β from Nowman's method in the time-change formula (7). Third, the observations are not evenly time-spaced. The sampling of observations depends on the volatility of the short-term interest rate and the window width, a. The time space, h, between two observations would be shorter with a higher volatility or a smaller window width, and vice versa.

In implementing the Yu and Phillips method, the time interval between two observations, h, is approximated by the discrete-time counterpart of the time-change formula (7),

$$h = \min\left\{ s \mid \sigma^2 \sum_{i=1}^{s} e^{2\beta(s-i)\triangle} r^{2\gamma}(t + i\triangle)\triangle \geq a \right\}. \tag{9}$$

The parameters in (9), β, γ, and σ, are replaced by the MLEs from Nowman's method. The window width, a, is obtained from the ML estimate of the constant volatility in Vasicek's (1977) model, which is an Ornstein-Uhlenbeck process with constant diffusion coefficient, σ,

$$dr_t = (\alpha + \beta r_t)\, dt + \sigma dB(t). \tag{10}$$

Thus, the process has constant volatility, $\tilde{a}$, and it can be estimated from the following equation via MLE,

$$r(t + \triangle) = \frac{\alpha}{\beta}(e^{\triangle\beta} - 1) + e^{\triangle\beta} r(t) + \varepsilon,$$

in which $\varepsilon \sim N(0, \tilde{a})$, where $\tilde{a} = \sigma^2$. In addition to the constant volatility, $\tilde{a}$, obtained from the Vasicek model in the simulation, other values of the window width are also examined, as will be explained in section 3.

3. Simulation

The simulation of sample paths follows that of Chapman and Pearson (2000). Data used in the simulation study are generated by assuming that the interest rate follows the diffusion process specified in the CIR model,

$$dr(t) = (\alpha + \beta r(t))\, dt + \sigma r^{\frac{1}{2}}(t)\, dB(t). \tag{11}$$

The process has the advantage that the transition density is known and follows a non-central χ^2 distribution. More specifically, the distribution of $2cr\,(t+\triangle)$, conditional on $2cr\,(t)$, is given by

$$\chi^2\,(2cr\,(t)\,, 2q+2, 2\lambda)\,,$$

where

$$\begin{aligned} c &= \frac{-2\beta}{(\sigma^2\,(1-e^{\beta\triangle}))}, \\ \lambda &= cr(t)e^{\beta\triangle}, \\ q &= \frac{2\alpha}{\sigma^2}-1. \end{aligned}$$

The degrees of freedom are $2q+2$ and the non-centrality parameter is 2λ. Note that $2q$ must be an integer to serve as the degrees of freedom. Three time intervals, $\triangle$, are considered: 1/12, 1/52, and 1/250. The values of the parameters α, β, and σ follow the set-up in Yu and Phillips (2001) and are given in Table 2. The degrees of freedom of the non-central χ^2 distribution are 384, 98, and 8, respectively, in the daily, weekly, and monthly interval. Each experiment is generated by 1,000 replications.

Two models are examined in this paper. In the first model, the short-term interest rate process is assumed to follow the process specified in CIR (1985); i.e, γ is assumed to be known and is set to 1/2. The transformed drift in Shoji and Ozaki's (1998) method is given by

$$\mu_Y\,(Y_t) = \left(2\alpha/\sigma^2 - 1/2\right)/Y_t + \beta Y_t/2.$$

In this first model, because the transition density is known to be non-central χ^2, the exact MLE is also implemented. In this way, the accuracy of the estimated parameters can be examined using the three approximation methods.

In the second model, the level-effect parameter, γ, is assumed to be unknown. The transformed drift in Shoji and Ozaki's (1998) method is given by

$$\mu_Y\,(Y_t) = \frac{\alpha + \beta(Y_t\,(1-\gamma)\,\sigma)^{\frac{1}{1-\gamma}}}{\sigma(Y_t\,(1-\gamma)\,\sigma)^{\frac{\gamma}{1-\gamma}}} - \frac{\gamma}{2}\,(Y_t\,(1-\gamma))^{-1}\,.$$

All the nonlinear optimizations associated with all three methods, as well as the exact MLE in the CIR model, are implemented in SAS using the trust region method of their nonlinear optimization routines.

For the Yu and Phillips method, window widths other than $\tilde{a}$ are used, because the window width is an exogenous parameter in the model and the choice of that width may affect finite-sample estimation. By examining other values of the window width, it is possible to determine, via simulation, how changes in the window width affect the empirical distribution of the parameters α and β in the nonlinear differential equation. Yu and Phillips (2001) use only the constant volatility $\tilde{a}$ as a window width, because it "reflects the average volatility in the data." The window width used in this study is of the form $c\tilde{a}$. The values of c are chosen to be equally spaced in the interval [0.1,2].[2]

3.1 The first model: the CIR model

3.1.1 Empirical distribution of parameters

Tables 4, 5, and 6 show the statistics of the empirical distributions of parameters in the CIR model with daily, weekly, and monthly intervals using the exact MLE, Nowman, and Shoji and Ozaki methods. The three sets of results share some common features. First, the statistics for the empirical distribution of parameters using Nowman's method and Shoji and Ozaki's method are quite close to those of the exact MLE. The exact MLE gives only slightly less biased estimates of the parameters. The improvement in bias using exact MLE over the two approximation methods for α and β is less than 1 per cent for the monthly interval. For the weekly interval and the daily interval, the improvements in bias for α are around 1.2 per cent and 2 per cent, respectively, whereas those for β are around 1.2 per cent and 2 per cent, respectively. The second common feature is that the statistical properties of the parameters estimated by Nowman's method and Shoji and Ozaki's method are quite close to each other in all three time intervals. The differences in bias of α, β, and σ between these two methods are less than 1 per cent, respectively. The third feature is that both approximation methods provide very good estimates of σ. The bias of σ in all three time intervals is less than 1 per cent and the MSEs in all three time intervals are also very small. This feature justifies the use of σ estimated from Nowman's method in the Yu and Phillips method. The fourth common feature is that estimates of α and β using Nowman's method and Shoji and Ozaki's method are not accurate. The sample means of the two parameters deviate quite substantially from

[2]For c larger than 2, the effective sample size becomes very small. In most cases, less than half of the observations in a sample are used. Therefore, values larger than 2 are not used, so as not to lose too much information.

their true values. For α, the deviations of the mean of the two approximation methods are around 54 per cent, 46 per cent, and 85 per cent in daily, weekly, and monthly intervals, respectively. The deviations of β for the two approximation methods are around 54 per cent, 46 per cent, and 91 per cent in the daily, weekly, and monthly intervals, respectively. The two approximation methods work quite badly in estimating these two drift parameters. The bias of estimated α and β is large because of the autocorrelated nature of the short-term interest rate process. In the case of first-order autocorrelation, the slope parameter estimator of β is downwards biased and the bias becomes larger as β approaches one.

For the Yu and Phillips method, the values of window widths examined are of the form $c\tilde{a}$; i.e., the constant volatility estimated from Vasicek's model was multiplied by a constant c. Table 3 shows the summary statistics of the constant volatility, $\tilde{a}$, from Vasicek's model. The range of constants, c, studied are from 0.1 to 2. The smallest window width examined is 0.1 and each successive window width is a 0.1 increment of the previous window width. This setting allows analysis of how the empirical distribution of estimates and model fit vary as we vary the window width. Because 20 window widths are studied for each parameter, only the mean and the percentage of bias of each parameter are shown, to facilitate presentation. Figures 1 to 6 show the bias (percentage) of the empirical distributions of the parameters using different window widths with the Yu and Phillips method. One general result with successively larger window widths is that the bias of the sample mean first drops and then gets larger. As the window width is enlarged further, say to $c = 1.5$, the bias can go up or down. For example, the bias of α and β drops 10 per cent as the window width increases from $c = 1.3$ to $c = 1.5$ in the weekly frequency.

As can be seen from Figures 1 to 6, the Yu and Phillips method can reduce the bias of α and β quite significantly if an appropriate window width is chosen. For example, if $c = 0.7$ is chosen in the monthly interval, the bias of α is less than half of that with Nowman's method and the bias of β is about two-thirds that of Nowman's method. Choosing the window width at $c = 0.9$ for the weekly and daily interval produces a similar effect. However, the issue as to which window width is appropriate needs to be resolved. As shown in the simulation study, the bias of parameters is not monotonically increasing in the window width. Thus, it is possible that if a large window width is chosen, the bias of parameter estimates would be smaller. However, a large window width utilizes very little information from the sample. For example, as the window width is enlarged to $c = 1.3$

and beyond, the Yu and Phillips procedure takes up less than half of the observations. Another problem with using a large window width is that the fit of the model is poor, as will be shown in the next section. Therefore, it seems that the choice of window width should be guided not only by the magnitude of bias reduction but also by the fit of the model.

3.1.2 Fit of model

Figures 7 to 9 show the fit of the three approximation methods. Two measures of fit are examined: the average likelihood and the MSE of the model. The three figures show the sample mean of the average likelihood and the MSE of the model. The average likelihood is the likelihood of the model divided by the number of observations used in the sample. The average likelihood is used because a different number of observations are picked up by different window widths and the aim is to have a standardized measure of model fit. The other measure of model fit, the MSE of the model, is defined as

$$\frac{\sum_{t=1}^{T} (y_t - \hat{y}_t)^2}{T - k},$$

in which k denotes the number of parameters.

Several observations can be made regarding these figures. The first observation is that, for Nowman's method and Shoji and Ozaki's method, the fit of the model measured in terms of the average likelihood is similar, though Shoji and Ozaki's method consistently has the lowest MSE in all frequencies. The second observation is that, if the fit is compared across sampling frequencies, it worsens as the time interval becomes larger. The sample mean of the average likelihood of the two approximation methods with daily data is about 5.5, but it declines to 3.28 with weekly data and 0.9 with monthly data. Similarly, the MSE of the model using the two approximation methods increases as the time interval between observations becomes larger.

The third observation is that the model fit, measured in terms of average likelihood, is not as good for the Yu and Phillips method as it is for the other two approximation methods for all the window widths examined. This is especially true for the monthly interval: the average likelihood of using $c = 1$ as the window width is about half that of Nowman's method. Another feature of the average likelihood is that it is very small at the smallest window width and then increases for larger windows, but then falls again for large windows. For example, at the monthly interval, the sample mean of the average likelihood is -0.39 with $c = 0.3$. It increases to 0.48 with $c = 0.7$, and then drops to -0.17 with $c = 2$. Judging from the average likelihood alone, it seems that a window width with the highest average likelihood should be chosen. However, if a window width based on the MSE of the model is chosen, it will lead to a totally different conclusion. The use of the MSE as a measure of model fit can be motivated by the argument that the window width is an exogenous value but it acts as the variance of the Gaussian process in the Yu and Phillips method. Therefore, the exogenously imposed window width affects the average likelihood and, as a result, the model's performance should be judged by the distance between the actual and fitted value. One common feature of the three figures is that the MSE of the model is increasing in window width, which implies that the fit of the model, measured in terms of the MSE, worsens as the window width enlarges. For monthly data, the MSE increases from 0.18 with $c = 0.3$ to 0.44 with $c = 2$. Thus, in choosing a window width, if the MSE is used as a measure of model fit, it is necessary to justify the trade-off between the bias of parameter estimates and the fit of the model.

Despite the contradictory conclusion in choosing the optimal window width, the two measures of model fit have one common feature: the model fit is poor with large window widths. For monthly intervals, the average likelihood with $c = 2$ is among the lowest and the MSE with $c = 2$ is the highest in the range of window widths examined. Similar observations can be made for both the daily and weekly intervals. This common feature of the average likelihood and the MSE is useful in that it provides a means to screen out window widths that are too large.

3.2. The second model: the general nonlinear diffusion process

3.2.1 Empirical distribution of parameters

The second model estimates a general nonlinear diffusion process, since it also estimates the level parameter, γ. Tables 7, 8, and 9 show the statistics of the parameters' empirical distribution for Nowman's method and Shoji and Ozaki's method with the three sampling frequencies. These tables are of interest for several reasons. First, the bias of σ is, in general, larger than that of the previous model: the biases of σ with Nowman's method and Shoji and Ozaki's method are near to zero in the CIR model in all three frequencies, but in the nonlinear diffusion process models they are around 12 per cent and 7 per cent with daily and weekly data. Second, the bias of γ is quite small: the biases of the sample mean of γ are less than 2 per cent in all three time intervals, which indicates that the estimation of γ is quite accurate using the two methods. Third, the statistics of α and β in the second model are almost the same as in the first model for Nowman's method and Shoji and Ozaki's method. In estimating α and β, the performance of the two methods is similar in the nonlinear diffusion process.

Figures 10 to 15 show the performance of the Yu and Phillips method with values of window width other than $\tilde{a}$. As in the CIR model, only the empirical distribution of the mean and the percentage of bias are shown. The values of constant c examined and the constant volatility, $\tilde{a}$, are the same as in the CIR model. The general pattern of results in the nonlinear diffusion model is similar to those in the CIR model. Choosing a suitable window width can reduce bias quite significantly. Also, the biases of both α and β first decrease and then increase as the window width is enlarged. At large window width values, the bias can go up or down. Thus, as in the case of the CIR model, a window width cannot be chosen on the basis of a reduction in the bias of parameters alone. It is important to avoid choosing a large window, which loses too many observations and has a poor model fit.

3.2.2 Fit of model

Figures 16 to 18 show the fit of the nonlinear model. The addition of the parameter γ in the estimation does not seem to affect the fit of the nonlinear diffusion model. The fit of the nonlinear model, measured in terms of the average likelihood and the MSE, is similar to those of the CIR model. Several features of the model fit are similar to that of the CIR model. First, Nowman's method and Shoji and Ozaki's method give a similar average likelihood. Again, Shoji and Ozaki's method always has the smallest MSE. Second, the two approximation methods work less well as the approximation error, the time interval between observations, increases. Third, measured in terms of the average likelihood, the two approximation methods outperform the Yu and Phillips method for all values of the window width examined. Fourth, the Yu and Phillips method, in general, gives a lower average likelihood with both a small and a large window width. Finally, the MSE of the Yu and Phillips method increases with the window width. Thus, the nonlinear diffusion process is the same as the CIR model, in that a large window width gives a poor model fit.

4. Empirical Results

The one-month yield of the Canadian treasury bill and the one-month sterling interbank middle rate are used to conduct empirical analysis. The Canadian dataset is obtained from the CANSIM database and it contains 270 observations from January 1980 to June 2002. The U.K. dataset was used in Nowman's (1997) paper and it contains 242 observations of the one-month sterling interbank middle rate from March 1975 to March 1995.

Table 10 shows the results of the CIR model using Canadian data. It reports the estimates, their standard errors, and the average likelihood of the three approximation methods, along with the exact MLE. For the Yu and Phillips method, only a selection of representative window widths and their results are provided. Regarding the fit of the three approximation methods, Shoji and Ozaki's method gives the highest average likelihood, 0.09. For the Yu and Phillips method, the average likelihood, in general, increases and then drops as the window width increases. Confirming the results in a simulation study, all window widths give a lower average likelihood than the other two approximation methods. Regarding estimates, the parameter estimates of Nowman's method and the exact MLE are similar. However, estimates of α and β from both methods are not statistically different from zero at the 5 per cent significance level. For the Yu and Phillips method, the estimates of β with window widths $c = 0.3$, $c = 0.5$, and $c = 0.7$ are similar to those of Nowman's method, but the estimates of α with these window widths are lower than those of Nowman's method. Again, the estimates of α are not statistically different from zero for all window widths examined.

Table 11 reports results with the nonlinear diffusion process using Canadian data. For the fit of the model, Shoji and Ozaki's method again gives the highest average likelihood, 0.14, and it is much higher than that for the CIR model. For the Yu and Phillips method, the average likelihood first increases and then decreases as the window width is enlarged. It reaches the maximum, -0.74, at window width $\tilde{a}$ estimated from Vasicek's model. This again confirms the pattern of average likelihood in the simulation study. While none of the parameter estimates of α are statistically different from zero at the 5 per cent significance level, virtually all the estimates of β are significant. With the addition of γ, the estimates of α and β vary widely with the window width for the Yu and Phillips method when compared with those in the CIR model. With a window width of $2\tilde{a}$, the estimates of α, 1.26, and β, -0.26, are the largest (in absolute value). This window width coincides with the worst fit of all the cases examined. On the other hand, window width $\tilde{a}$ gives the lowest estimate of α, 0.14, and β, -0.12.

Table 12 reports results of the CIR model using the U.K. short-term interest rate. The exact MLE gives the highest average likelihood. The model fit of the Yu and Phillips method conforms with the simulation study: the model fit measured in terms of the average likelihood first increases and then drops as the window width becomes larger. The estimates of the three parameters with Nowman's method, Shoji and Ozaki's method, and the exact MLE are quite close to each other. With the exception of the $0.7\tilde{a}$ window width and the exact MLE, all estimates of α are statistically different from zero for all approximation methods. In addition, the estimates of β, σ, and γ are statistically significant for all estimation methods. For the Yu and Phillips method, the pattern of results with successively larger window width is similar to the simulation study. The estimates of α and β first drop (in absolute value) as the window width increases. Estimates of α drop from 3.83 to 2.49 and estimates of β drop (in absolute value) from -0.37 to -0.27 as the window width increases from $0.3\tilde{a}$ to $0.7\tilde{a}$. As the window width enlarges further, the estimates of the two parameters, in general, increase, but they can go up or down. Table 13 shows the results of the nonlinear diffusion process using the U.K. short-term interest rate. The results are similar to the CIR model and will not be discussed separately.

5. Conclusion

This paper has compared the performance of three Gaussian approximation methods. Two models were examined: the CIR model and the nonlinear diffusion process. The performance of the approximation methods has been measured in terms of the empirical distributions of parameters, the sample mean of the average likelihood, and the MSE of the models. Several major conclusions can be drawn from the simulation study. First, the performance of Nowman's method and Shoji and Ozaki's method in terms of the three measures is similar. Second, for the daily frequency, the bias and the MSE of α and β estimates using the Yu and Phillips method with the window width $\tilde{a}$ are lower than for the other two methods. Third, the window width used by Yu and Phillips has a critical influence on the estimation outcome, with a large window width being associated with a poor model fit. An empirical study using Canadian and U.K. short-term interest rates was also implemented. It was found that among the three approximation methods (i) Shoji and Ozaki's method gives the best model fit for both countries and (ii) the estimates of drift parameters for both countries using the Yu and Phillips method are determined by the window width, with a large window width giving a poor model fit. The last finding conforms with the findings in the simulation study.

The findings in this study suggest that, although the Yu and Phillips method can reduce the bias of α and β estimates, an inappropriate window width leads to severely biased estimates. Because there is no way of knowing the optimal window width, caution should be exercised in applying the Yu and Phillips method. One possible way to narrow the choice of window width is to estimate the model via the other two approximation methods first and then use the estimates as a benchmark for choosing window width.

Future research could examine iterating with the Yu and Phillips estimator—that is, once the first-round estimates of α and β are found, again find new h_j's using the new β and proceed.

The closeness of the approximation methods to the exact MLE seems to indicate that the CIR model's transition density can be well approximated by a Gaussian density. This confirms results in Ait-Sahalia (1999) and Jiang and Knight (2001), where Edgeworth/Gram-Charlier expansions around a normal density are used to approximate the likelihood.

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Table 1: Alternative One-Factor Short-Term Interest Rate Model

	Model
Merton (1973)	$dr(t) = \alpha dt + \sigma dB$
Vasicek (1977)	$dr(t) = (\alpha + \beta r(t))dt + \sigma dB$
Cox, Ingersoll, and Ross (1985)	$dr(t) = \alpha dt + \sigma r(t)^{1/2} dB$
Dothan (1978)	$dr(t) = \sigma r dB$
Geometric Brownian motion	$dr(t) = \beta r(t)dt + \sigma r dB$
Brennan and Schwartz (1979)	$dr(t) = (\alpha + \beta r(t))dt + \sigma r dB$
Cox, Ingersoll, and Ross (1980)	$dr(t) = \sigma r^{3/2} dB$
Constant elasticity of variance	$dr(t) = \beta r(t)dt + \sigma r^{\gamma} dB$
CKLS (1992)	$dr(t) = (\alpha + \beta r(t))dt + \sigma r^{\gamma} dB$

Table 2: Parameters and Sample Size

	Daily	Weekly	Monthly
α	6	3	0.72
β	-1	-0.5	-0.12
σ	0.25	0.35	0.6
Sample size	2000	1000	500

Table 3: Summary Statistics of Constant Volatility from Vasicek Model

	Mean	Variance	MSE
Daily	0.0386	0.0000*	0.0413
Weekly	0.0826	0.0000*	0.0905
Monthly	0.4172	0.0038	0.6957

Table 4: Empirical Distribution of Parameters in the CIR Model for Daily Data

	Parameters	Mean	Variance	MSE	Bias%	Bias
Exact MLE	α	9.23	18.28	28.73	53.84	3.23
	β	-1.54	0.51	0.8	53.95	-0.54
	σ	0.25	0.00*	0.00*	-0.27	0.00*
Nowman	α	9.27	18.57	29.24	54.42	3.27
	β	-1.55	0.52	0.82	54.53	-0.55
	σ	0.25	0.00*	0.00*	0.04	0.00*
Shoji and Ozaki	α	9.26	18.57	29.24	54.42	3.26
	β	-1.55	0.52	0.82	54.52	-0.55
	σ	0.25	0.00*	0.00*	0.04	0.00*

*The Variances and MSE are less than 10^{-5}.

Table 5: Empirical Distribution of Parameters in the CIR Model for Weekly Data

	Parameters	Mean	Variance	MSE	Bias%	Bias
Exact MLE	α	4.34	3.59	5.39	44.62	1.34
	β	-0.73	0.10	0.15	45.07	-0.23
	σ	0.35	0.00	0.00*	-0.60	0.00*
Nowman	α	4.38	3.72	5.61	45.84	1.38
	β	-0.73	0.11	0.16	46.30	-0.23
	σ	0.35	0.00*	0.00*	0.10	0.00*
Shoji and Ozaki	α	4.37	3.72	5.61	45.82	1.37
	β	-0.73	0.11	0.16	46.27	-0.23
	σ	0.35	0.00*	0.00*	0.10	0.00*

*The Variances and MSE are less than 10^{-5}.

Table 6: Empirical Distribution of Parameters in the CIR Model for Monthly Data

	Parameters	Mean	Variance	MSE	Bias%	Bias
Exact MLE	α	1.31	0.49	0.84	82.26	0.59
	β	-0.23	0.01	0.03	88.5	-0.11
	σ	0.6	0.00*	0.00*	-0.81	0.00*
Nowman	α	1.33	0.51	0.88	84.4	0.61
	β	-0.23	0.02	0.03	90.81	-0.11
	σ	0.6	0.00*	0.00*	0.14	0.00*
Shoji and Ozaki	α	1.33	0.51	0.89	84.71	0.61
	β	-0.23	0.01	0.03	90.99	-0.11
	σ	0.6	0.00*	0.00*	0.06	0.00*

*The Variances and MSE are less than 10^{-5}.

Table 7: Empirical Distribution of Parameters in Nonlinear Diffusion Process for Daily Data

	Parameters	Mean	Variance	MSE	Bias%	Bias
Nowman	α	9.25	18.49	29.09	54.23	3.25
	β	-1.54	0.52	0.81	54.34	-0.54
	σ	0.28	0.02	0.02	12.1	0.03
	γ	0.5	0.07	0.07	-0.82	0.00*
Shoji and Ozaki	α	9.26	18.58	29.19	54.25	3.26
	β	-1.54	0.52	0.82	54.36	-0.54
	σ	0.28	0.02	0.02	12.62	0.03
	γ	0.49	0.07	0.07	-1.79	-0.01

*The Variances and MSE are less than 10^{-5}.

Table 8: Empirical Distribution of Parameters in Nonlinear Diffusion Process for Weekly Data

	Parameters	Mean	Variance	MSE	Bias%	Bias
Nowman	α	4.38	3.72	5.62	45.89	1.38
	β	-0.73	0.11	0.16	46.34	-0.23
	σ	0.38	0.02	0.02	7.97	0.03
	γ	0.49	0.04	0.04	-1.95	-0.01
Shoji and Ozaki	α	4.38	3.73	5.63	45.92	1.38
	β	-0.73	0.11	0.16	46.37	-0.23
	σ	0.38	0.02	0.02	7.21	0.03
	γ	0.49	0.04	0.04	-1.28	-0.01

*The Variances and MSE are less than 10^{-5}.

Table 9: Empirical Distribution of Parameters in Nonlinear Diffusion Process for Monthly Data

	Parameters	Mean	Variance	MSE	Bias%	Bias
Nowman	α	1.33	0.51	0.88	84.24	0.61
	β	-0.23	0.01	0.03	90.61	-0.11
	σ	0.61	0.01	0.01	2.17	0.01
	γ	0.49	0.01	0.01	-1.27	-0.01
Shoji and Ozaki	α	1.33	0.51	0.89	84.8	0.61
	β	-0.23	0.02	0.03	91.07	-0.11
	σ	0.61	0.01	0.01	1.14	0.01
	γ	0.5	0.01	0.01	-0.18	0.00*

*The Variances and MSE are less than 10^{-5}.

Table 10: Empirical Study using the Canadian Short-Term Interest Rate with the CIR Model

	α	β	σ	Average likelihood
Exact MLE	0.67	-0.14	0.77	0.02
	(0.83)	(0.12)	(0.03)	
Nowman	0.68	-0.14	0.78	0.04
	(0.59)	(0.08)	(0.02)	
Shoji and Ozaki	0.79	-0.16	0.76	0.09
	(0.58)	(0.08)	(0.02)	
Yu and Phillips				
$\tilde{a}$	0.46	-0.16		-0.75
	(0.54)	(0.07)		
$0.3\tilde{a}$	0.65	-0.15		-1.41
	(0.40)	(0.05)		
$0.5\tilde{a}$	0.53	-0.14		-0.73
	(0.46)	(0.06)		
$0.7\tilde{a}$	0.47	-0.15		-0.73
	(0.49)	(0.06)		
$1.3\tilde{a}$	0.72	-0.18		-1.06
	(0.57)	(0.08)		
$1.5\tilde{a}$	0.83	-0.2		-1.43
	(0.58)	(0.08)		
$2\tilde{a}$	0.72	-0.18		-1.52
	(0.60)	(0.08)		

Note: Standard errors in parentheses

Table 11: Empirical Study using the Canadian Short-Term Interest Rate with the Nonlinear Diffusion Process

	α	β	σ	γ	Average likelihood
Nowman	0.68	-0.14	0.58	0.65	0.05
	(0.55)	(0.08)	(0.07)	(0.06)	
Shoji and Ozaki	0.60	-0.13	0.43	0.77	0.14
	(0.5)	(0.08)	(0.05)	(0.06)	
Yu and Phillips					
$\tilde{a}$	0.14	-0.12			-0.74
	(0.53)	(0.07)			
$0.3\tilde{a}$	0.46	-0.13			-1.52
	(0.39)	(0.04)			
$0.5\tilde{a}$	0.38	-0.12			-0.87
	(0.43)	(0.05)			
$0.7\tilde{a}$	0.21	-0.12			-0.76
	(0.48)	(0.06)			
$1.3\tilde{a}$	0.21	-0.14			-0.97
	(0.56)	(0.08)			
$1.5\tilde{a}$	1.08	-0.23			-1.03
	(0.63)	(0.08)			
$2\tilde{a}$	1.26	-0.26			-1.40
	(0.66)	(0.09)			

Note: Standard errors in parentheses

Table 12: Empirical Study using the U.K. Short-Term Interest Rate with the CIR Model

	α	β	σ	Average likelihood
Exact MLE	3.37	-0.33	0.88	-0.30
	(2.02)	(0.19)	(0.04)	
Nowman	3.42	-0.34	0.89	-0.60
	(1.47)	(0.14)	(0.03)	
Shoji and Ozaki	3.33	-0.33	0.87	-0.55
	(1.44)	(0.14)	(0.03)	
Yu and Phillips				
$\tilde{a}$	3.00	-0.35		-0.91
	(1.32)	(0.12)		
$0.3\tilde{a}$	3.83	-0.37		-1.73
	(0.92)	(0.08)		
$0.5\tilde{a}$	3.50	-0.35		-0.90
	(1.12)	(0.10)		
$0.7\tilde{a}$	2.49	-0.27		-0.77
	(1.18)	(0.11)		
$1.3\tilde{a}$	4.43	-0.47		-1.30
	(1.40)	(0.13)		
$1.5\tilde{a}$	3.88	-0.41		-1.59
	(1.42)	(0.13)		
$2\tilde{a}$	4.48	-0.48		-1.81
	(1.50)	(0.14)		

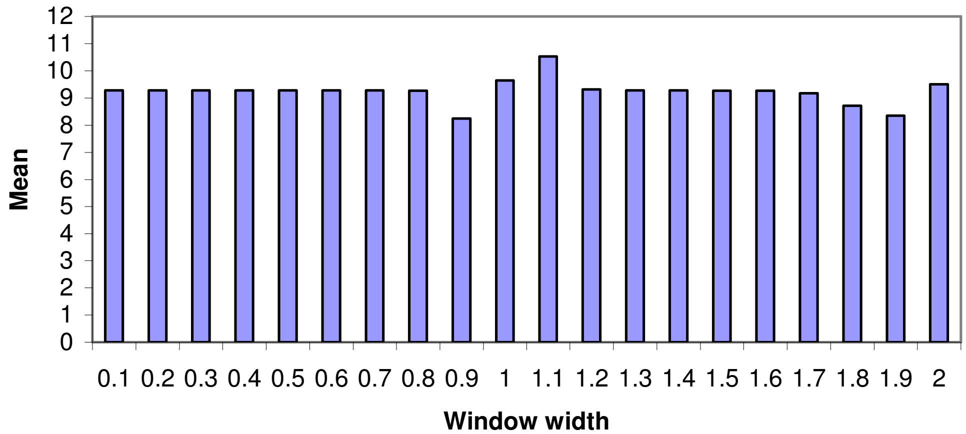
Note: Standard errors in parentheses

Table 13: Empirical Study using the U.K. Short-Term Interest Rate with the Nonlinear Diffusion Process

	α	β	σ	γ	Average likelihood
Nowman	3.56	-0.35	1.45	0.29	-0.59
	(1.56)	(0.14)	(0.39)	(0.11)	
Shoji and Ozaki	3.17	-0.31	0.68	0.60	-0.55
	(1.40)	(0.14)	(0.18)	(0.11)	
Yu and Phillips					
$\tilde{a}$	3.48	-0.39			-0.96
	(1.38)	(0.13)			
$0.3\tilde{a}$	3.87	-0.38			-1.75
	(0.93)	(0.08)			
$0.5\tilde{a}$	3.87	-0.38			-0.92
	(1.20)	(0.11)			
$0.7\tilde{a}$	3.30	-0.33			-0.69
	(1.29)	(0.12)			
$1.3\tilde{a}$	4.93	-0.49			-1.46
	(1.44)	(0.13)			
$1.5\tilde{a}$	3.72	-0.38			-1.56
	(1.42)	(0.13)			
$2\tilde{a}$	4.08	-0.43			-1.67
	(1.57)	(0.14)			

Note: Standard errors in parentheses

Figure 1. Mean and Bias (%) of α in CIR Model with Daily Frequency



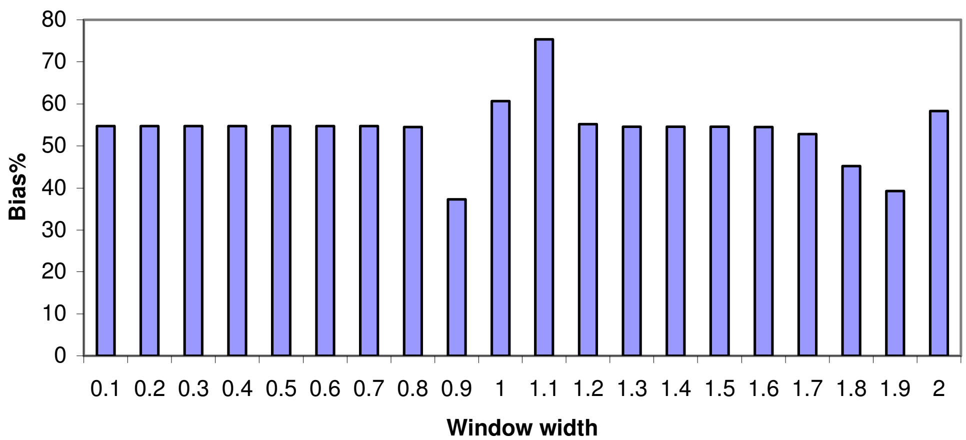
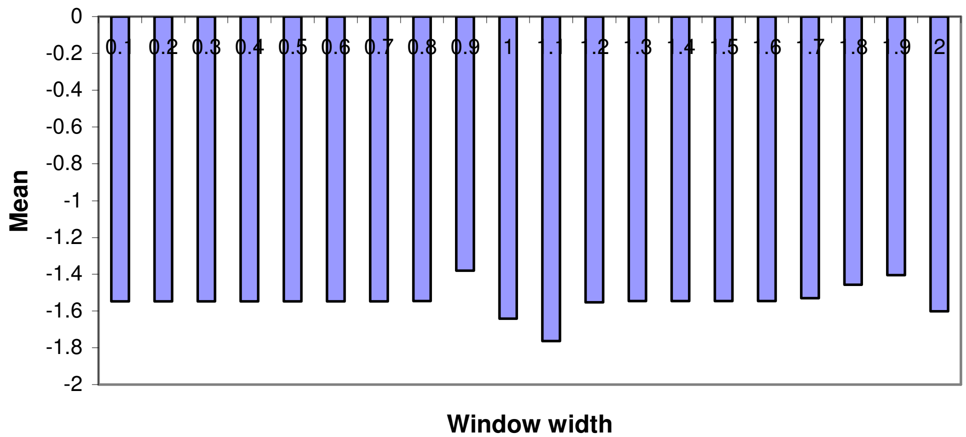


Figure 2. Mean and Bias (%) of β in CIR Model with Daily Frequency



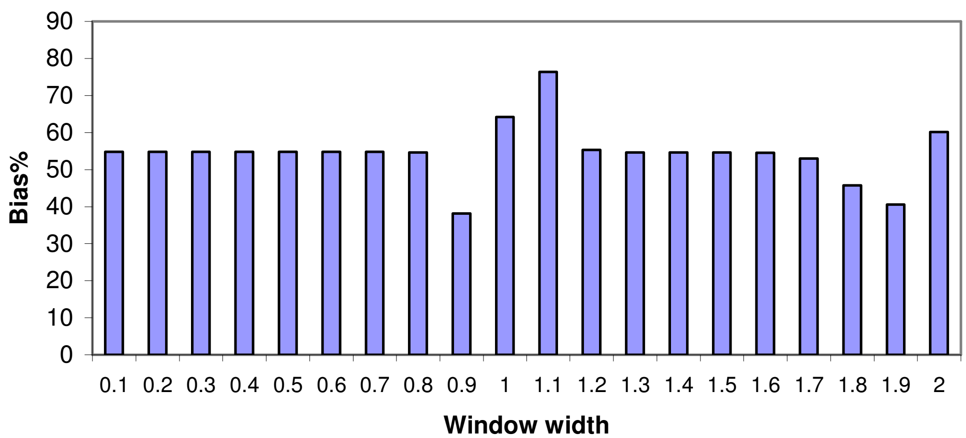
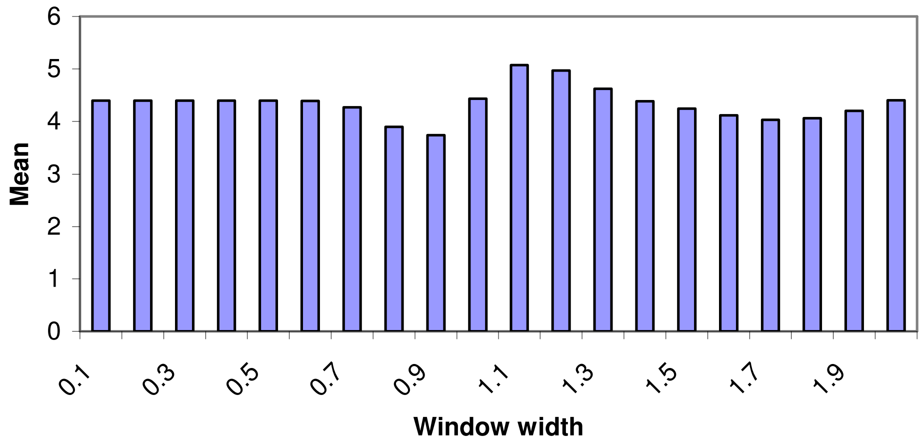


Figure 3. Mean and Bias (%) of α in CIR Model with Weekly Frequency



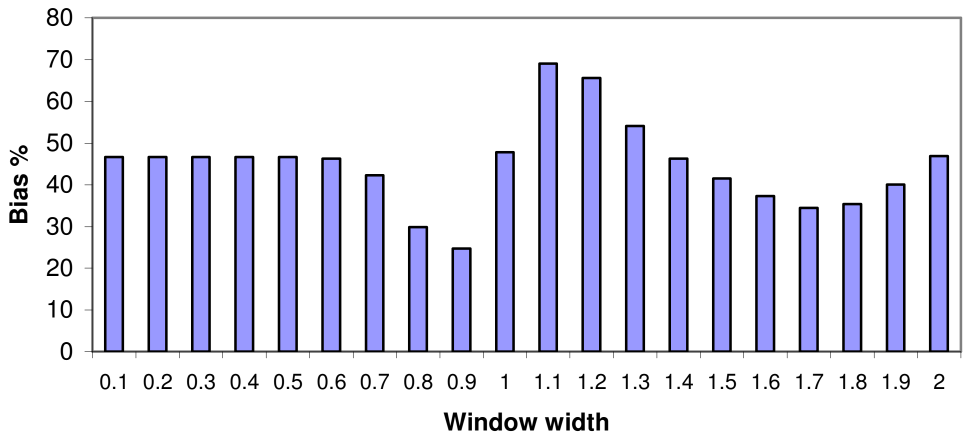
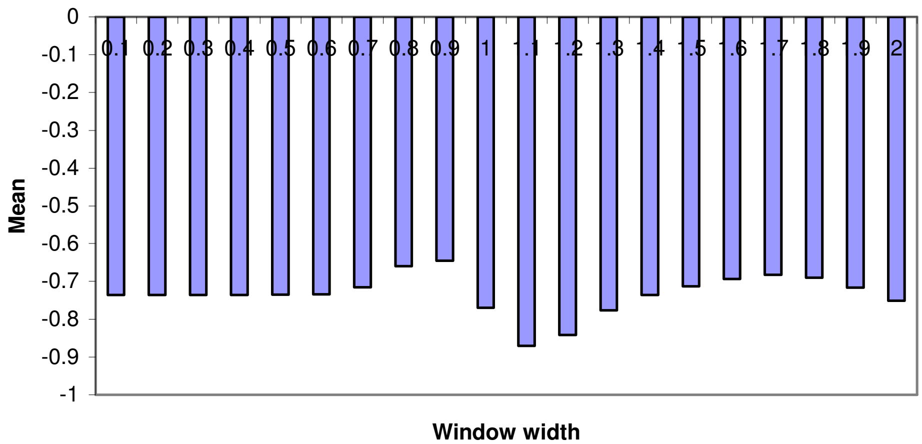


Figure 4. Mean and Bias (%) of β in CIR Model with Weekly Frequency



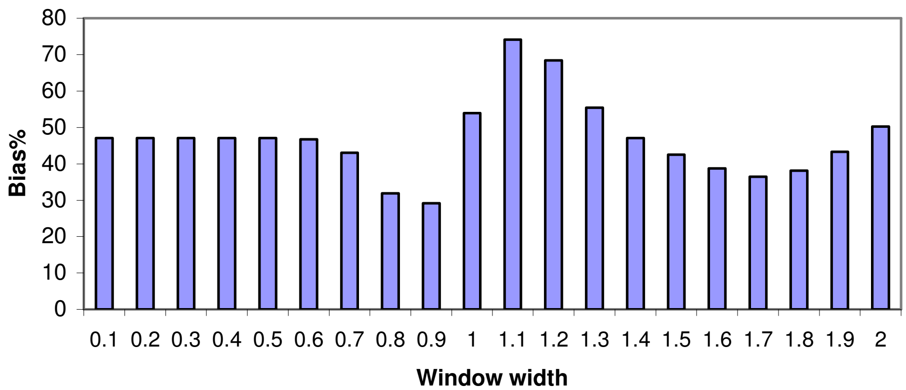
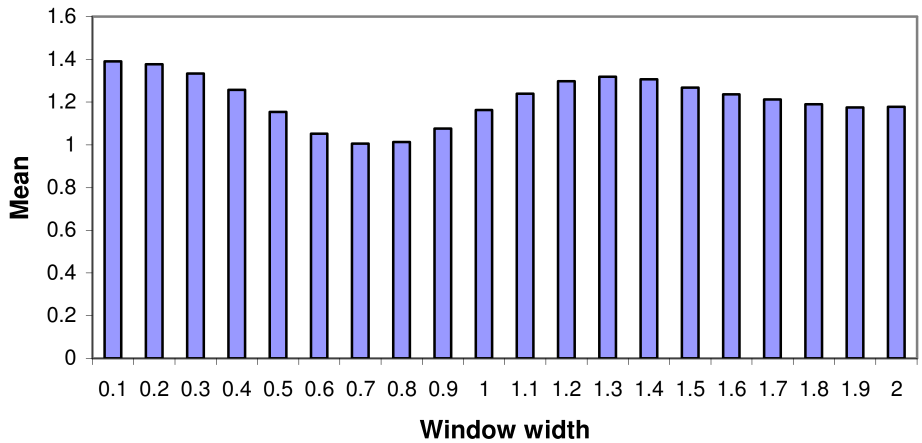


Figure 5. Mean and Bias (%) of α in CIR Model with Monthly Frequency



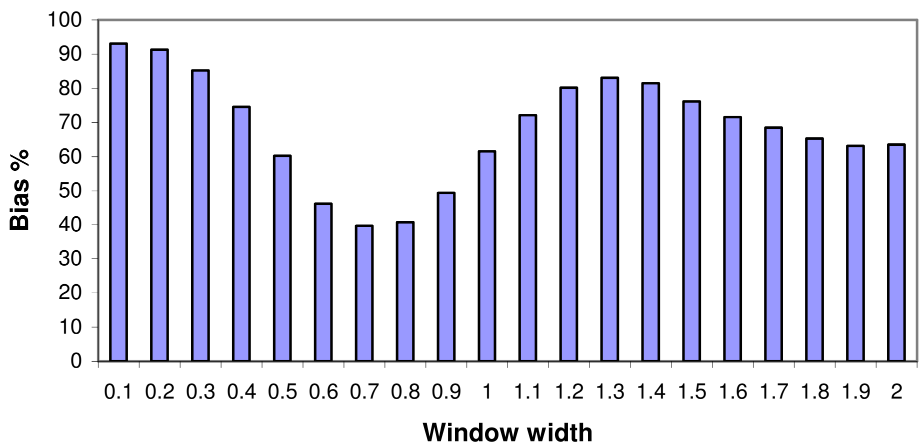
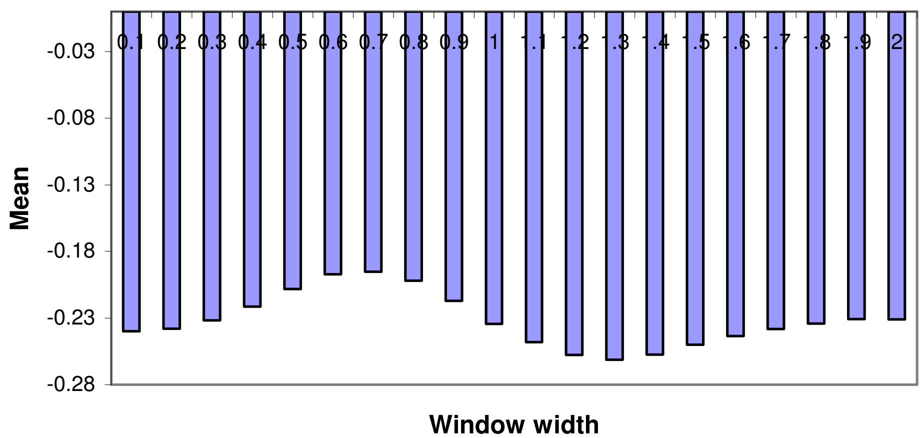


Figure 6. Mean and Bias (%) of β in CIR Model with Monthly Frequency



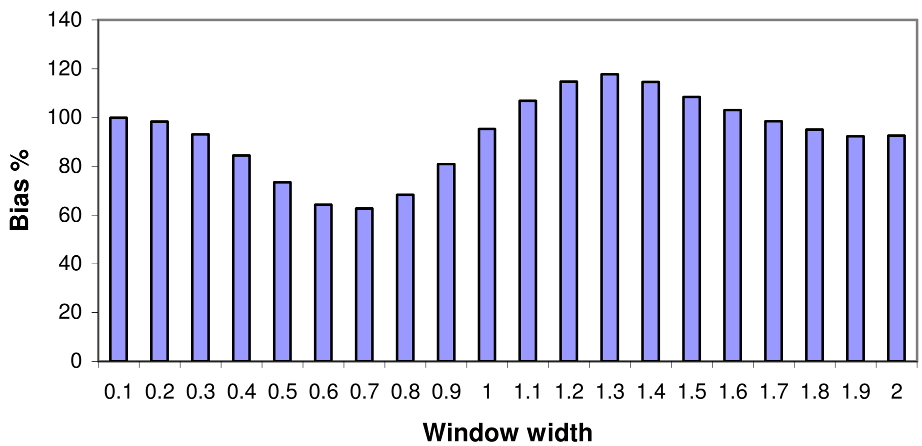
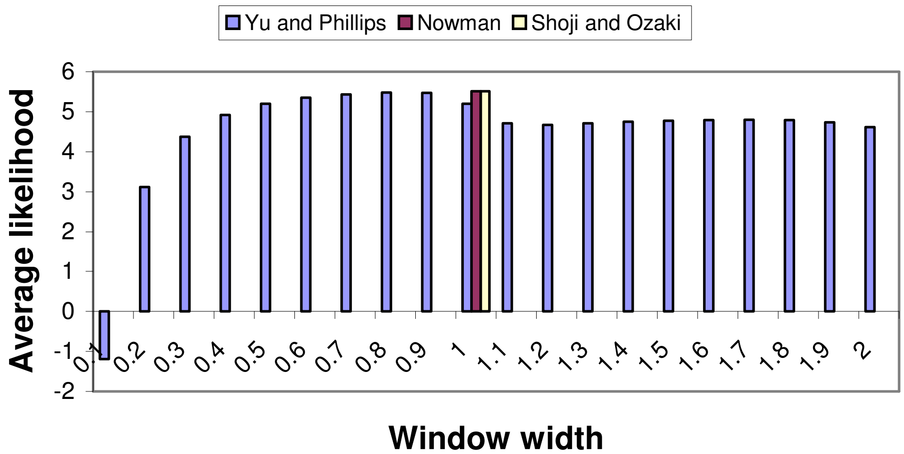


Figure 7. Fit of the Three Approximation Methods: CIR Model with Daily Frequency



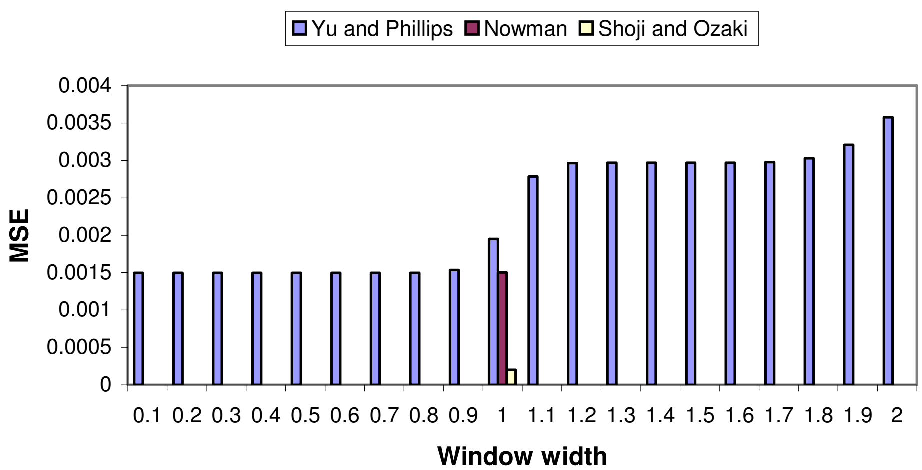
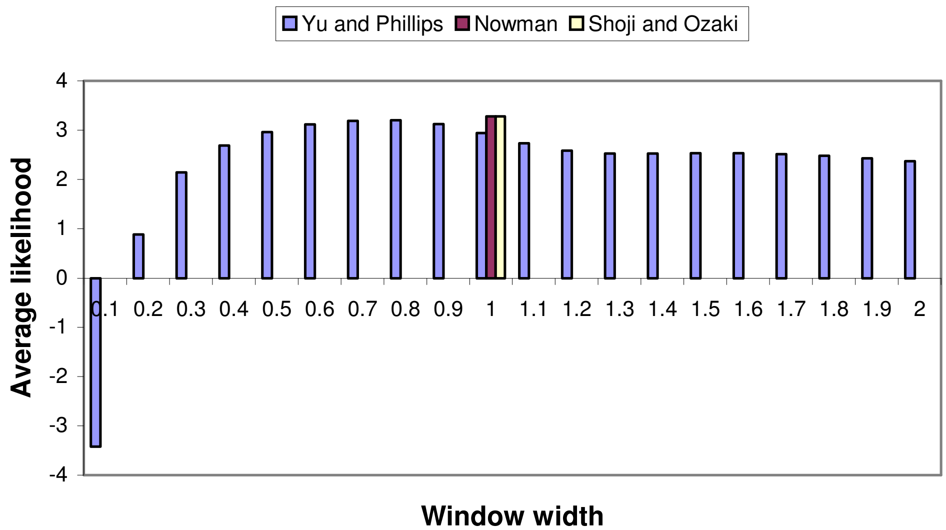


Figure 8. Fit of the Three Approximation Methods: CIR Model with Weekly Frequency



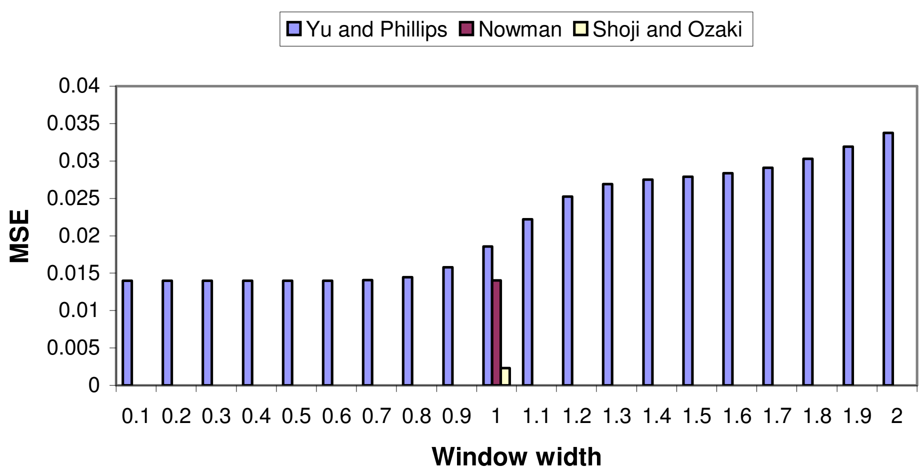


Figure 9. Fit of the Three Approximation Methods: CIR Model with Monthly Frequency

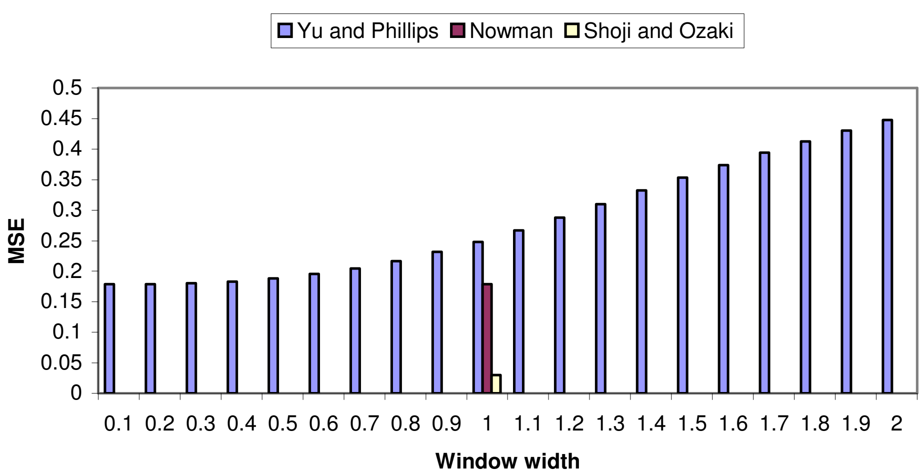


Figure 10. Mean and Bias (%) of α in Nonlinear Model with Daily Frequency

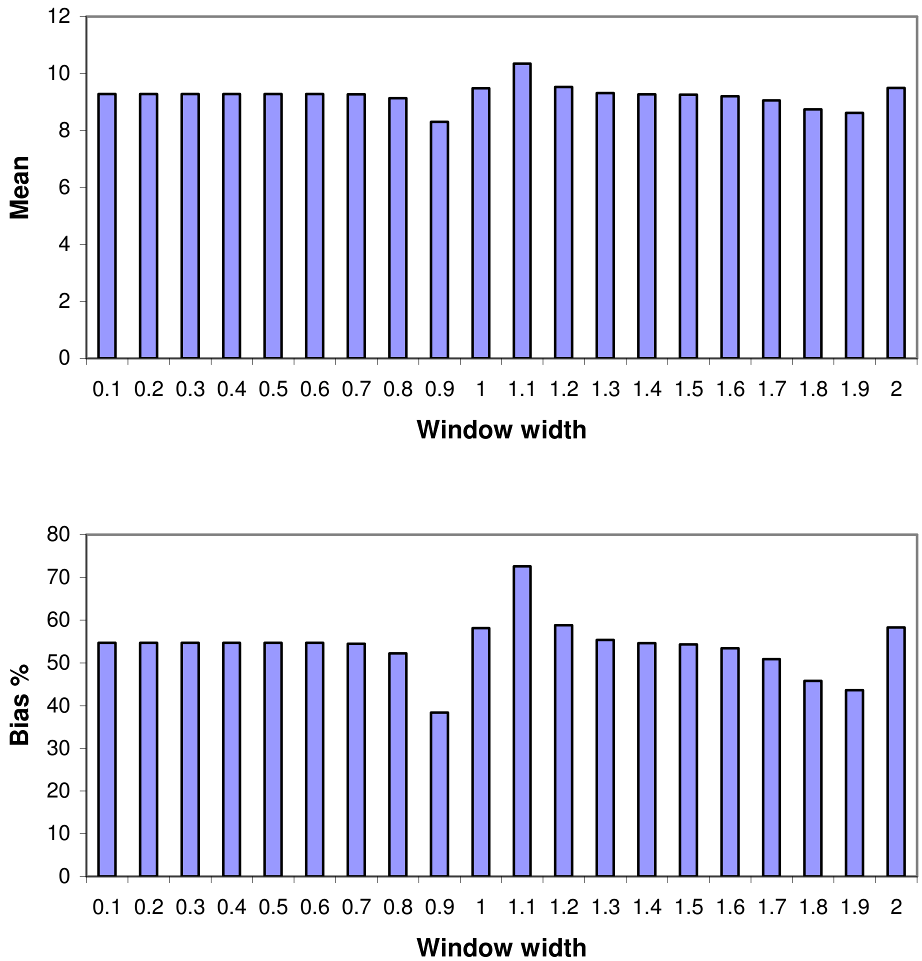


Figure 11. Mean and Bias (%) of β in Nonlinear Model with Daily Frequency

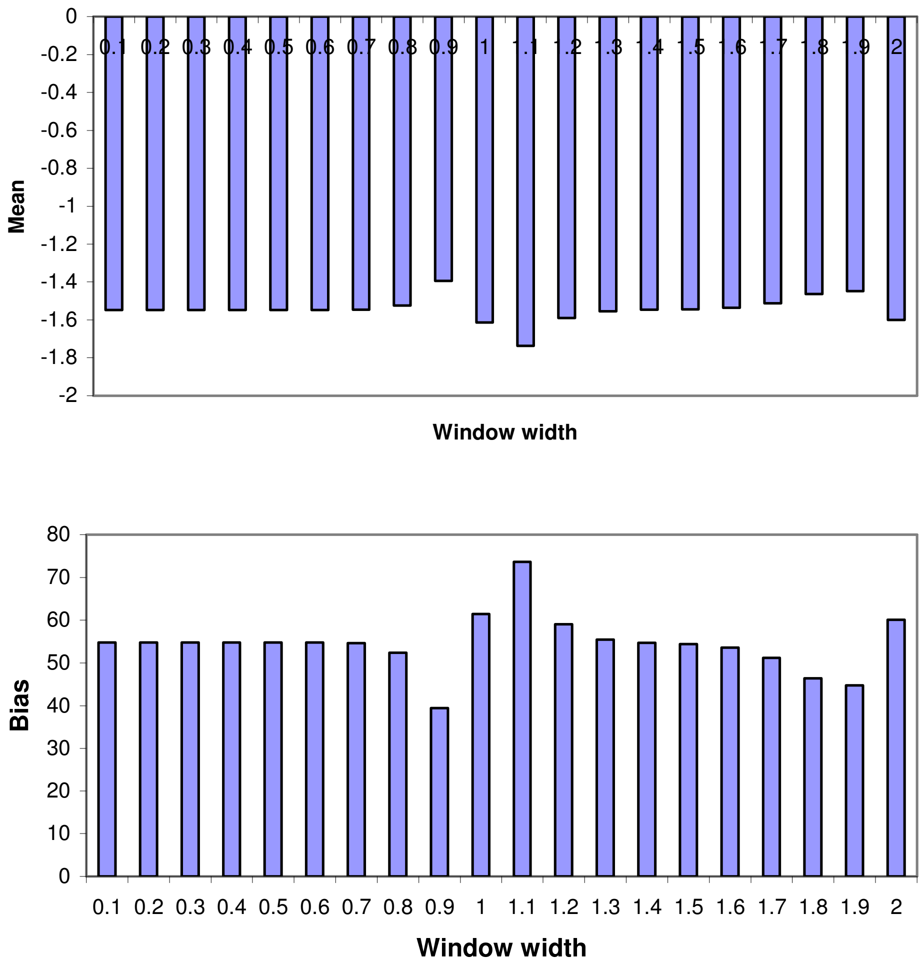
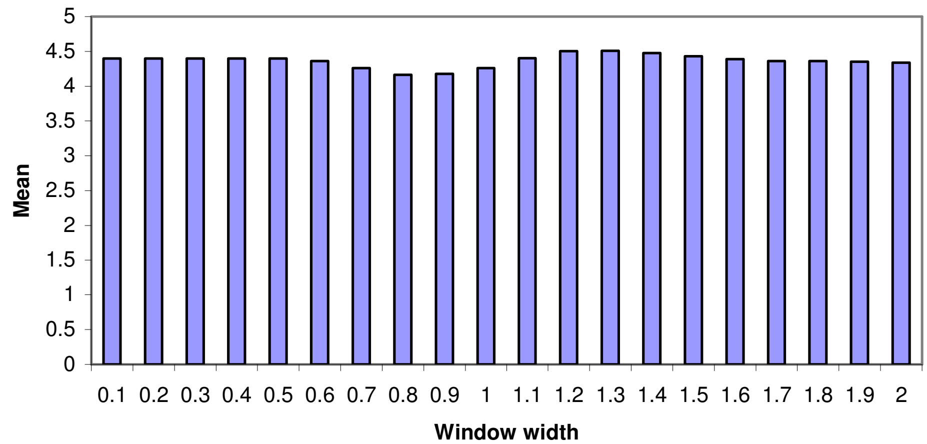


Figure 12. Mean and Bias (%) of α in Nonlinear Model with Weekly Frequency



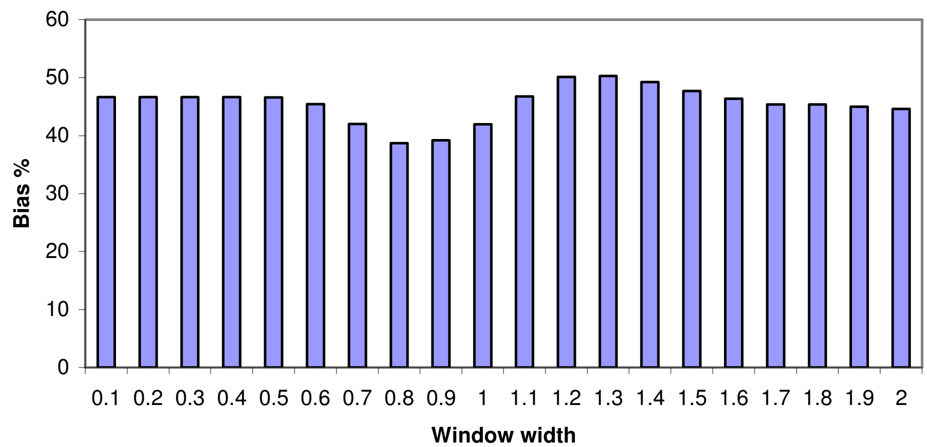
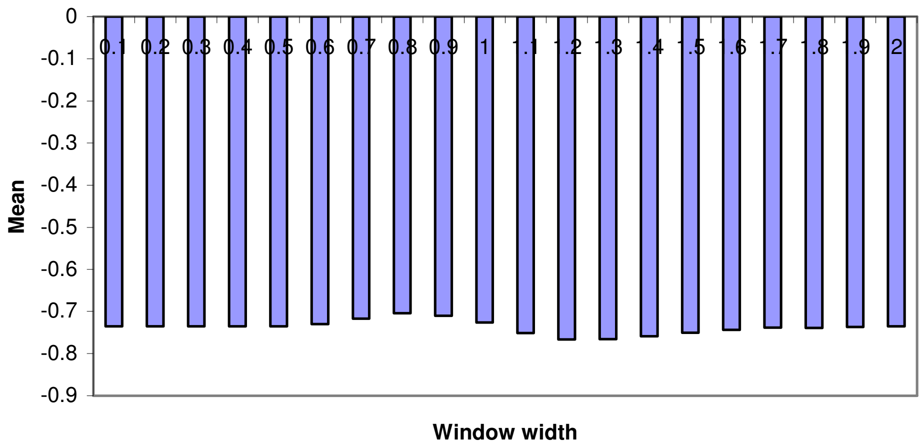


Figure 13. Mean and Bias (%) of β in Nonlinear Model with Weekly Frequency



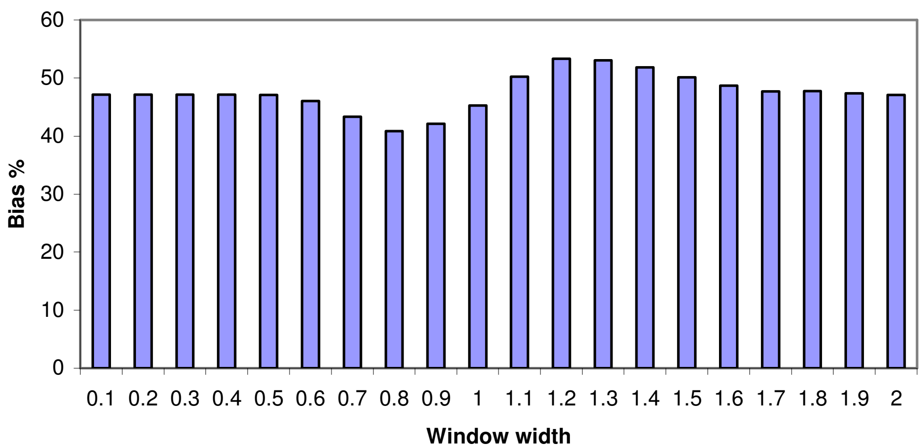
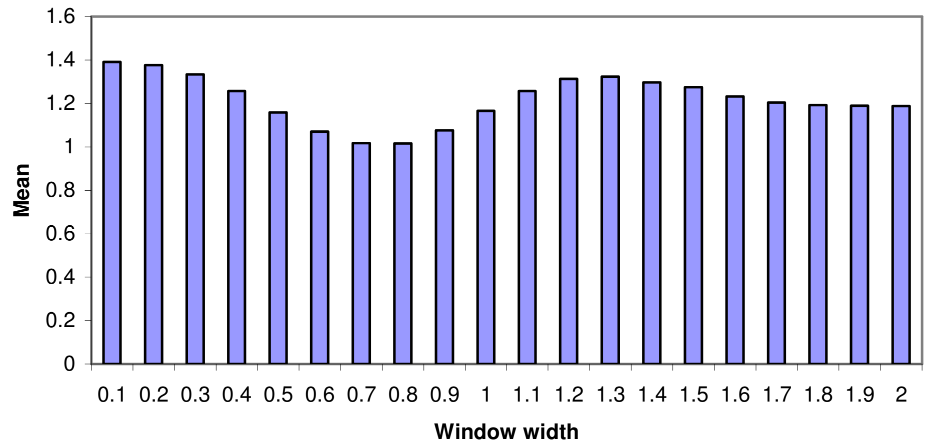


Figure 14. Mean and Bias (%) of α in Nonlinear Model with Monthly Frequency



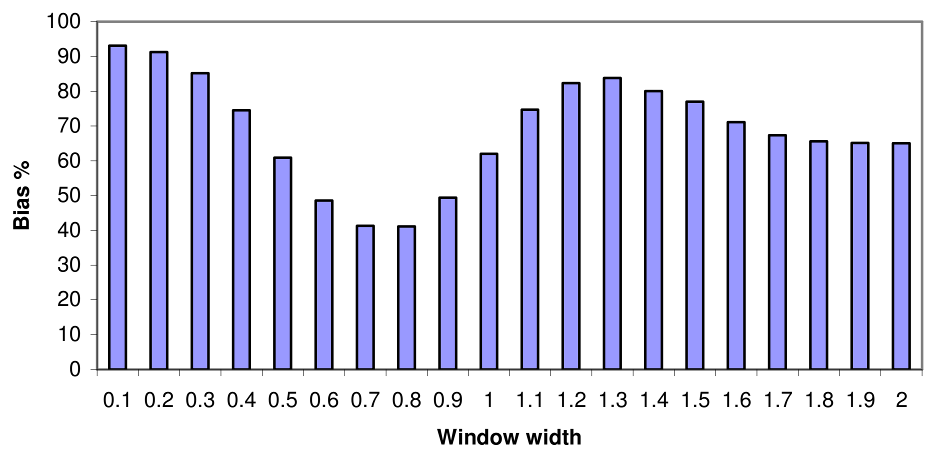


Figure 15. Mean and Bias (%) of β in Nonlinear Model with Monthly Frequency

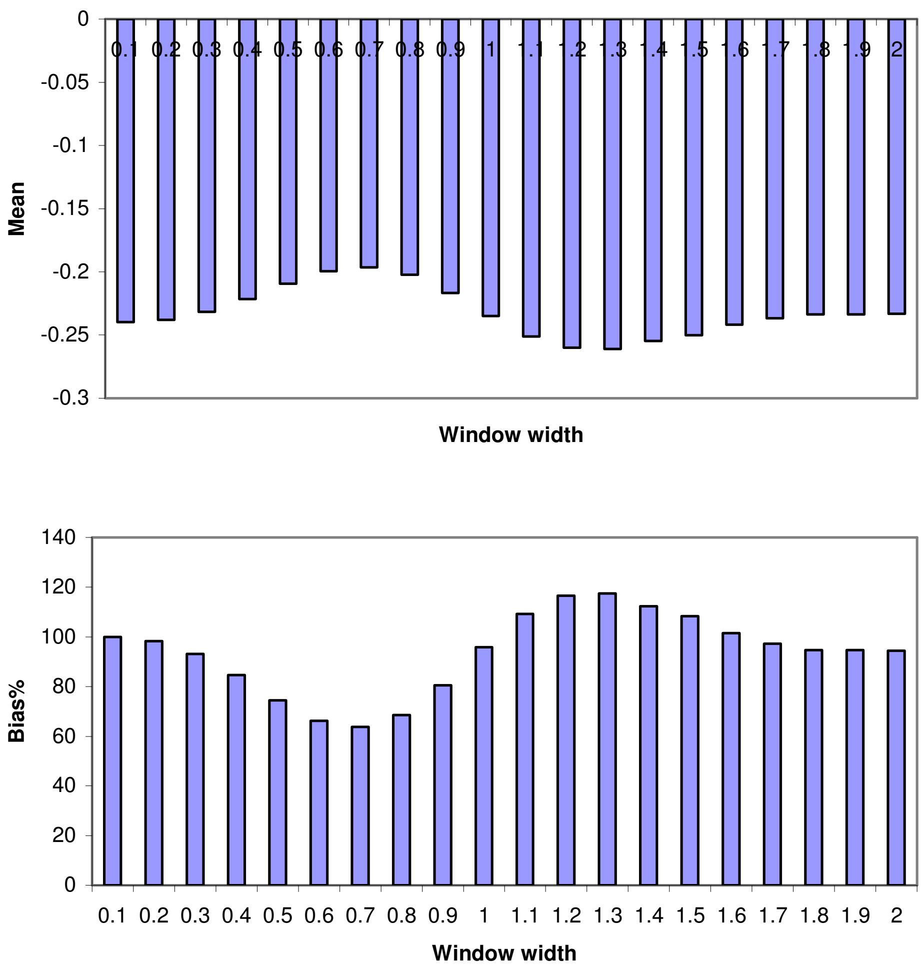
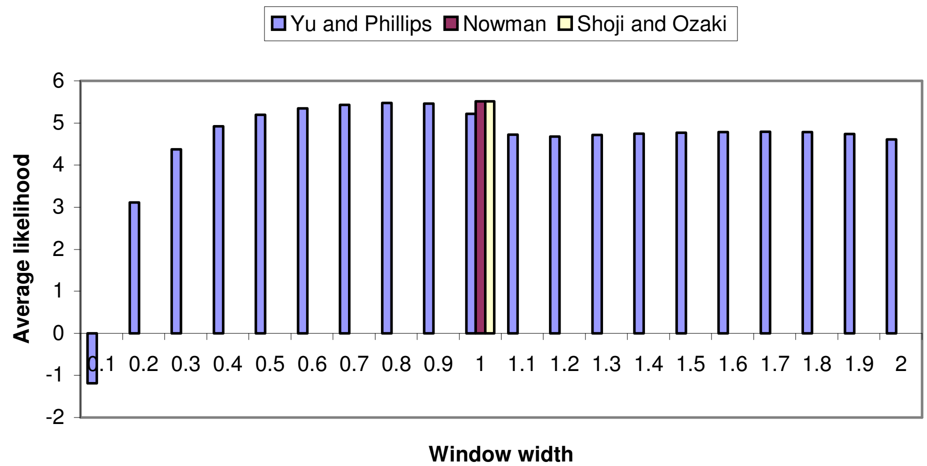


Figure 16. Fit of the Three Approximation Methods: Nonlinear Model with Daily Frequency



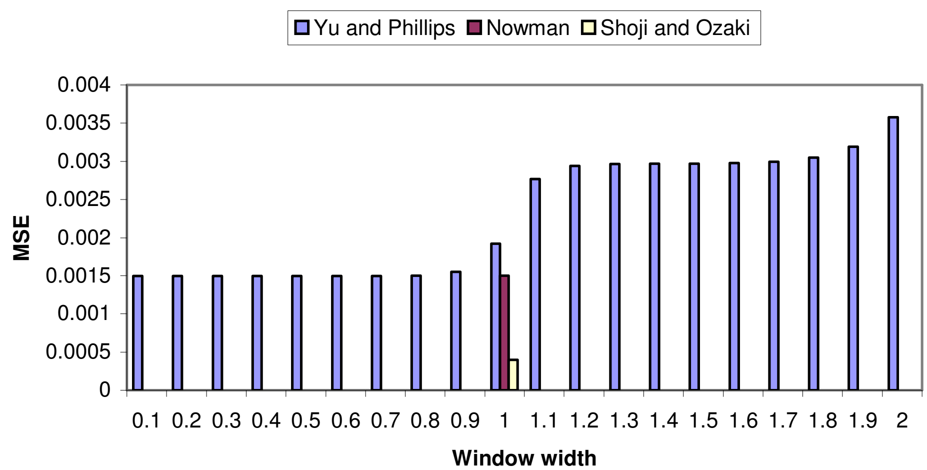
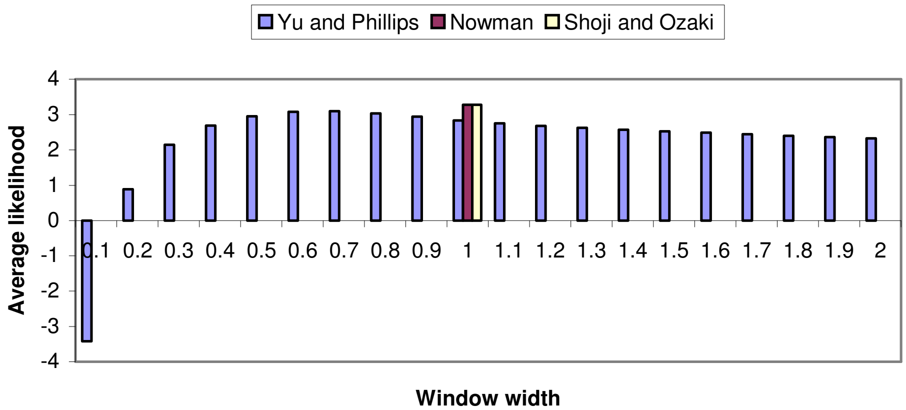


Figure 17. Fit of the Three Approximation Methods: Nonlinear Model with Weekly Frequency



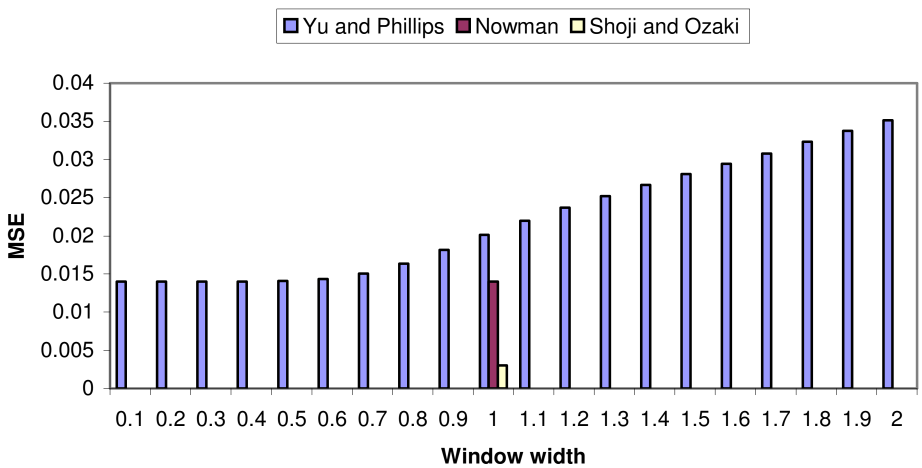
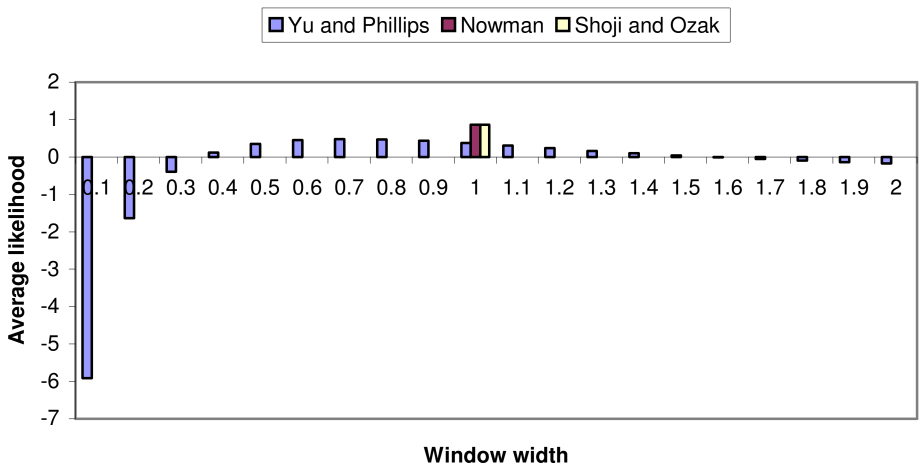
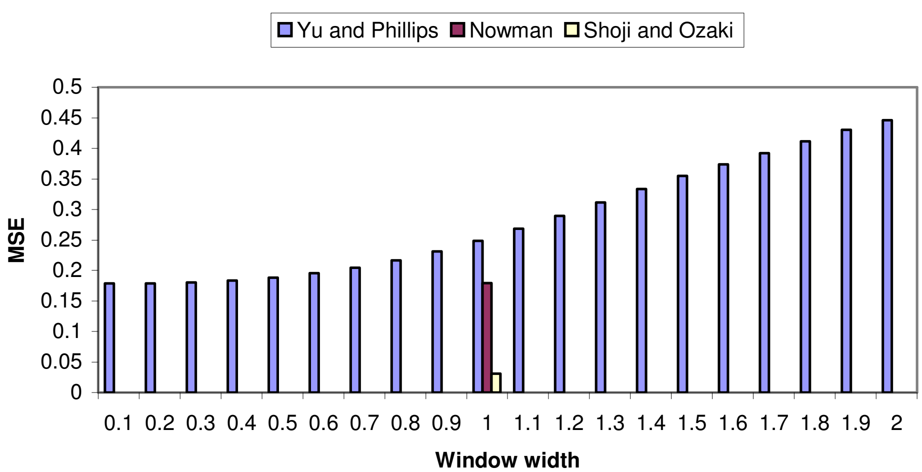


Figure 18. Fit of the Three Approximation Methods: Nonlinear Model with Monthly Frequency





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