Yield and Inflation Differentials between Canada and the United States

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Introduction

It is well established that the term structure of interest rates reflects market expectations about future inflation and real interest rates. Extracting such information from the term structure is important for the implementation of monetary policy. In this paper, we propose an approach for extracting information about inflation expectations and inflation-risk premiums by exploiting both the co-movements among interest rates across the yield curve and the co-movements among those interest rates in two countries, Canada and the United States.

Research on the yield curve has attracted considerable attention from central banks because recent research has demonstrated empirical links between the yield curve and observed economic fundamentals. For example, the yield curve has been found to be able to predict real output growth (e.g., Cozier and Tkacz 1994; Estrella and Hardouvelis 1991) and inflation (e.g., Mishkin 1990; Day and Lange 1997; Engsted 1995) as well as to measure monetary policy stance (e.g., Rudebusch 1995; Macklem 1995).

One criticism of this research is that the estimated relationships are based on reduced forms that lack the benefit of structural restrictions imposed by equilibrium models. In this paper, a term-structure model is

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used as a tool to extract information about inflation from the yield curve by relating the underlying factors to macroeconomic fundamentals such as inflation and real returns. We believe that this line of research will help explain the term structure's key features, and determine the nature of the economic fundamentals that drive movements in the yield curve.

In this paper, we extend the two-factor term-structure model in Gong and Remolona (1997b) to a two-country setting by estimating the model jointly for Canada and the United States. The open-economy aspect is especially important to Canada because its bond yields are influenced by world financial markets, particularly the U.S. bond market. As a result, we attempt to consider explicitly the close link between the Canadian and U.S. financial markets.

In the model, yields in each country are determined by two unobserved, or latent, factors. One is specified as an inflation factor and the other as a portmanteau factor to represent real fundamentals. Since the factors are unobserved, one important question is how to identify them. In Gong and Remolona (1997b) and Jegadeesh and Pennacchi (1996), the inflation process is used to identify the inflation factor by empirically implementing a link between the term structure and observed inflation rates. In our model, the factors are identified by the assumption that the inflation factor is specific to each country, representing independent inflation expectations for the two countries, while the real factor is common to both countries, representing common real-rate expectations.[1] The underlying intuition is that a real shock originating in the United States will also affect Canada, or that some real shocks originating outside the two countries will affect both Canada and the United States, because of their similar economies and close economic links. However, inflation shocks in Canada may differ from those in the United States because, with floating exchange rates, Canada can pursue an independent monetary policy.

However, the assumption of a common real factor in a two-factor model may not be adequate to capture all the shocks experienced by the two countries. There could be real idiosyncratic shocks due to, for example, government debt and major political events that affect only Canadian yields, and that this model will not be able to capture. Later, we will discuss this issue and how future work can address it. We consider this paper as the first

1. In a two-sector model with one good more capital-intensive than the other, if there are both instantaneous factor mobility across sectors and sufficiently similar initial proportions of labour and capital among trading partners, trade in the two goods will lead to factor-price equalization and, therefore, real interest rate equality. As Obstfeld (1995) points out, this result holds even when some of these conditions are relaxed.

step in a research program to study the relationship between the yield curve and economic fundamentals in an open-economy setting.

It is well known that the most difficult challenge in the extraction of expectations from the yield curve has been to take account of time-varying risk premiums. Not only do we wish to allow risk premiums to vary over time, we would also like such premiums to meet the following considerations: that they arise from the pricing of an explicitly specified risk; that they satisfy the equilibrium condition of no arbitrage; and that they are related to expectations about fundamentals. In this paper, we try to take account of such risk premiums by means of the simplest possible term-structure model. Risks arise in the model because of revisions in expectations, and the model assumes that these risks are priced by the bond market. Square-root heteroscedastic shocks to the factors are possible sources of risks priced by the market, allowing both inflation-risk premiums and real-term premiums to vary over time. (With homoscedastic shocks, the risk premium will be constant over time.)

To estimate the model, we applied a Kalman filter to monthly data on zero-coupon bond yields for 2-year, 5-year, and 10-year maturities. The model's arbitrage conditions allow us to focus on interest rate movements that can be accounted for by consistent expectations processes. Because the model assumes no correlation between inflation and real rate expectations, we estimate the model only for those yields where such an assumption can be justified. The estimation procedure allows us to exploit the conditional density of bond yields without imposing special assumptions on measurement errors. The model's arbitrage conditions also serve as over-identifying restrictions. We estimate the model for two sample periods, January 1984 to December 1997 and February 1991 to December 1997. The first period starts after 1983 because of a change in monetary regime in the United States. The second period begins at the time of a change in inflation regime in Canada with the announcement of inflation targets in Canada on February 1991.[2] Once we obtain the parameter estimates of the model, we can back out from the model conditional forecasts of the unobserved factors, thus allowing us to conditionally decompose nominal bond yields into four components: expectations of real rates, real-term premiums, expectations of inflation, and inflation-risk premiums. We then examine the change of inflation expectations and inflation risks over time, as well as the

2. The data set is available from 1972 to 1997. However, it is commonly believed that there are two different monetary regimes with different operating procedures in the United States: 1972 to 1979 and 1984 to 1997; see, for example, Brown and Dybvig (1986); Evans and Watchel (1993); and Jeffrey (1997). The Bank of Canada and the Department of Finance jointly announced targets for inflation reduction on 26 February 1991; see Bank of Canada Review (1991, 3).

relationship between Canada–U.S. yield differentials and the associated inflation differentials.

The results of our estimation show that the model is capable of extracting useful information from the yield curves, especially for Canada. This suggests that it is important to exploit additional information contained in the U.S. yield curve to study the Canadian term structure, and that the assumption of a common real factor and independent inflation factors is plausible. We also find a close relationship between the yield differentials and inflation differentials of Canada and the United States, suggesting that a significant portion of the yield differential could be explained by differentials in inflation expectations and inflation-risk premiums.

The rest of the paper is organized as follows. Section 1 presents the two-country, two-factor model. Section 2 discusses the data and estimation. Section 3 reports and discusses the empirical results. The conclusion summarizes and suggests future research.

1 Affine Yield Two-Country Two-Risk Two-Factor Model

1.1 The affine class of term-structure models

In this paper, we construct a two-country, two-factor affine model based on the class of term-structure models proposed by Duffie and Kan (1996). In this class of models, the interest rates and prices of bonds are linear, or affine, functions of a small number of factors. The dynamics of these factors are described by a generalized square-root diffusion process. The major advantage of working with this class of models is that it is tractable but capable of capturing many shapes of the yield curve. The affine term-structure model nests many well-known models, such as the one-factor models of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), and the two-factor model of Longstaff and Schwartz (1992).

Following the recent work by Campbell, Lo, and MacKinlay (1997) and Gong and Remolona (1997a), we use a discrete-time approach to specify this class of affine models. This allows us to avoid the pitfalls of estimating a continuous-time process with discrete-time data (see Aït-Sahalia, 1996). These models consist of a dynamic model for the stochastic processes of the factors, and a model for bond prices (or yields) as functions of the factors and the time to maturity. Thus, by combining both the time-series and cross-section dimensions of these models, they can be estimated with both time-series and cross-section data.[3] This permits us to fully exploit the cross-sectional restrictions imposed by the term-structure model, and permits us to identify the market price of risk. The basic two-factor model is similar to the one in Gong and Remolona (1997b).

1.1.1 The pricing kernel

The pricing-kernel approach relies on a no-arbitrage condition. In the case of zero-coupon bonds, the real price of an *n*-period bond is given by[4]

$$P_{nt} = E_t[P_{n-1,\,t+1}M_{t+1}], \tag{1}$$

where M_{t+1} is the stochastic discount factor. This pricing equation says that the price of the *n*-period bond is equal to the expected discount value of the bond's next period price. It rules out arbitrage opportunities by applying the same discount factor to all bonds. In what follows, we will model P_{nt} by modelling the stochastic process of M_{t+1}.

In an affine-yield model, the distribution of the stochastic discount factor M_{t+1} is conditionally lognormal, and bond prices are jointly lognormal with M_{t+1}. This helps maintain model tractability. Taking logs of (1), we get

$$p_{nt} = E_t[m_{t+1} + p_{n-1,\,t+1}] + \frac{1}{2}\mathrm{Var}_t[m_{t+1} + p_{n-1,\,t+1}], \tag{2}$$

where lower-case letters denote the logs of the corresponding upper-case letters. For example,

$$p_{t+1} = \log(P_{t+1}).$$

Since there are two factors, $x_{1,\,t}$ and $x_{2,\,t}$, that forecast m_{t+1}, an affine-yield model that satisfies the Duffie–Kan (1996) conditions can be written as[5]

$$-p_{nt} = A_n + B_{1n}x_{1t} + B_{2n}x_{2t}, \tag{3}$$

which is a linear function of the factors. As the *n*-period bond yield is

$$y_{nt} = -\frac{p_{nt}}{n},$$

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3. There is a growing body of literature that estimates the term-structure model using panel data.

4. See Campbell, Lo, and MacKinlay (1997) and Fung and Tkacz (1997) for a brief description of the derivation of the pricing equation.

5. See Campbell, Lo, and MacKinlay (1997, 441).

yields will also be linear in the factors. Note that both the intercept A_n and factor loadings B_{1n} and B_{2n} are time-invariant functions of the time to maturity n. The basic problem here is to specify the coefficients A_n, B_{1n}, and B_{2n} by solving (3) based on the stochastic processes of $x_{1,\,t}$ and $x_{2,\,t}$, and verify that (2) holds.

We will consider two affine yield two-factor models, one for Canada and one for the United States, satisfying the Duffie–Kan conditions.

1.2 The U.S. model

The pricing kernel in this model is assumed to be driven by two factors. One factor reflects the expectations of inflation that are specific to the United States, and the other is a real factor common to the United States and Canada. The negative of the log-stochastic discount factor is forecast by the two factors that enter into the forecasting relationship additively:

$$-m_{t+1} = x_{1t} + x_{2t} + w_{t+1} \tag{4}$$

where w_{t+1} represents the unexpected change in the log-stochastic discount factor and will be related to risk. The shock has a mean of 0 and a variance that will be specified to depend on the stochastic processes of the two factors $x_{1,\,t}$ and $x_{2,\,t}$. Each of these factors follows a univariate AR(1) process with heteroscedasticity shocks (depending on its own level) described by this square-root process:

$$x_{1t+1} = (1-\phi_1)\mu_1 + \phi_1 x_{1t} + x_{1t}^{1/2} u_{1t+1} \tag{5}$$

$$x_{2t+1} = (1-\phi_2)\mu_2 + \phi_2 x_{2t} + x_{2t}^{1/2} u_{2t+1}, \tag{6}$$

where $(1-\phi_1)$ and $(1-\phi_2)$ are the rates of mean reversion $(0 < \phi_1, \phi_2 < 1)$, μ_1 and μ_2 are the long-run means to which the factors revert, and $u_{1,\,t+1}$ and $u_{2,\,t+1}$ are shocks with means of 0 and with volatilities σ_1^2 and σ_2^2. The shocks are assumed to be uncorrelated.[6]

To model both inflation-risk and real-term premiums, the shock to m_{t+1} is specified to be proportional to the shocks to $x_{1,\,t+1}$ and $x_{2,\,t+1}$:

$$w_{t+1} = \lambda_1 x_{1t}^{1/2} u_{1,\,t+1} + \lambda_2 x_{2t}^{1/2} u_{2,\,t+1}, \tag{7}$$

6. The assumption that shocks to the expectation of real return are orthogonal to those of inflation expectations is not unreasonable because the expectation of real return is likely to be driven by real activity. Fama and Gibbons (1992) employ a similar orthogonality assumption to extract estimates of expected real returns from ex post inflation and short-term rates.

where λ_1 represents the market price of inflation risk and λ_2 the market price of real risk. Here risks arise from revisions in expectations, and the model assumes that these risks are priced by the bond market. Following Cox, Ingersoll, and Ross (1985) and Campbell, Lo, and MacKinlay (1997), we specify the volatilities of the shocks to be proportional to the square root of the respective factors. Such square-root diffusions have several advantages; in particular, they induce time-varying risk premiums while keeping yields linear in the factors so that the model remains tractable.

Since a bond trades at par at maturity, normalization gives $\log(P_{0,t}) \equiv p_{0,t} = 0$. Thus the one-period yield is

$$y_{1,t} = -p_{1,t} = \left(1 - \frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1t} + \left(1 - \frac{1}{2}\lambda_2^2\sigma_2^2\right)x_{2t}, \tag{8}$$

which is also linear in the factors, with the coefficients

$$A_1 = 0,\ B_{1,1} = 1 - \frac{1}{2}\lambda_1^2\sigma_1^2 \text{ and } B_{2,1} = 1 - \frac{1}{2}\lambda_2^2\sigma_2^2.$$

We can also verify that the price of an *n*-period bond is linear in the factors with the coefficients[7] given by:

$$A_n = A_{n-1} + (1-\phi_1)\mu_1 B_{1,n-1} + (1-\phi_2)\mu_2 B_{2,n-1} \tag{9}$$

$$B_{1n} = 1 + \phi_1 B_{1,n-1} - \frac{1}{2}(\lambda_1 + B_{1,n-1})^2 {\sigma_1}^2 \tag{10}$$

$$B_{2n} = 1 + \phi_2 B_{2,n-1} - \frac{1}{2}(\lambda_2 + B_{2,n-1})^2 \sigma_2^2. \tag{11}$$

The coefficients B_{1n} and B_{2n} are factor loadings; the coefficient A_n represents the pull of the factors to their long-run means. Equations (9) through (11) impose cross-sectional restrictions to be satisfied by eight parameters: the rates of mean reversion $1-\phi_1$ and $1-\phi_2$, the long-run means μ_1 and μ_2, the prices of risks λ_1 and λ_2, and the volatilities σ_1 and σ_2.

1.3 The Canadian model

The Canadian model follows the same set-up as the U.S. model, except that those variables and coefficients that are specific to the Canadian model are denoted with an asterisk $(*)$. Thus, the negative of the log-stochastic discount factor is:

$$-m_{t+1}{}^* = x_{1t}{}^* + x_{2t} + w_{t+1}{}^*, \tag{12}$$

7. See Appendix 1 for the derivations of these coefficients.

where $w_{t+1}{}^*$ represents the unexpected change in the log stochastic discount factor and will be related to risk. The shock has a mean of 0 and a variance that will be specified to depend on the stochastic processes of the factors $x_{1t}{}^*$ and x_{2t}. Since the second factor is common to both countries, we need only specify the process for the first factor:

$$x_{(1,t)+1}{}^* = (1-\phi_1{}^*)\mu_1{}^* + \phi_1{}^* x_{1t}{}^* + (x_{1t}{}^*)^{1/2} u_{1t}{}^*, \tag{13}$$

where all the variables are defined similarly to those in the U.S. model.

The shock to $m_{t+1}{}^*$ is specified to be proportional to the shock to $x_{1,t+1}{}^*$ and $x_{2,t+1}$:

$$w_{t+1}{}^* = \lambda_1{}^* x_{1t}{}^{*1/2} u_{1,t+1}{}^* + \lambda_2{}^* x_{2t}^{1/2} u_{2,t+1}. \tag{14}$$

Here the price of risk of the real factor is specified to be different than that in the U.S. model.

Since the Canadian model shares the real factor of the U.S. model, the price of an n-period bond is given by:

$$-p_{nt}{}^* = A_n{}^* + B_{1n}{}^* x_{1t}{}^* + B_{2n}{}^* x_{2t}. \tag{15}$$

Note that we allow the loading of the real factor $B_{2n}{}^*$ to be different between the two countries because the prices of risk of the real factors are allowed to be different. We call this more general model the four-risk model. We will also examine the case in which both countries have the same price of real risk by setting $\lambda_2 = \lambda_2{}^*$ and, hence, $B_{2n} = B_{2n}{}^*$. This model, which has the same real risk, we call the three-risk model. We will report results for both models to examine whether financial markets in the two countries price real risk in the same way given the assumption of a common real shock.

The one-period yield is

$$y_{1t}{}^* = -p_{1t}{}^* = \left(1 - \frac{1}{2}\lambda_1{}^{*2}\sigma_1{}^{*2}\right)x_{1t}{}^* + \left(1 - \frac{1}{2}\lambda_2{}^{*2}\sigma_2^2\right)x_{2t}. \tag{16}$$

This yield is also linear in the factors, with the coefficients

$$A_1{}^* = 0,\ B_{1,1}{}^* = 1 - \frac{1}{2}\lambda_1{}^{*2}\sigma_1{}^{*2} \text{ and } B_{2,1}{}^* = 1 - \frac{1}{2}\lambda_2{}^{*2}\sigma_2^2.$$

We can also verify that the price of an n-period bond is linear in the factors with the coefficients[8] given by:

$$A_n{}^* = A_{n-1}{}^* + (1-\phi_1{}^*)\mu_1{}^* B_{1,n-1}{}^* + (1-\phi_2)\mu_2 B_{2,n-1}{}^*, \tag{17}$$

8. See Appendix 1 for the derivations of these coefficients.

$$B_{1n}^* = 1 + \phi_1^* B_{1,\, n-1}^* - \frac{1}{2}(\lambda_1^* + B_{1,\, n-1}^*)^2 {\sigma_1^*}^2, \tag{18}$$

$$B_{2n}^* = 1 + \phi_2 B_{2,\, n-1}^* - \frac{1}{2}(\lambda_2^* + B_{2,\, n-1}^*)^2 \sigma_2^2. \tag{19}$$

Again, the coefficients B_{1n}^* and B_{2n}^* are factor loadings, while the coefficient A_n^* represents the pull of the factors to their long-run means. Equations (17) through (19) impose cross-sectional restrictions to be satisfied by eight parameters: the rates of mean reversion $1-\phi_1^*$ and $1-\phi_2$, the long-run means μ_1^* and μ_2, the prices of risks λ_1^* and λ_2^*, and the volatilities σ_1^* and σ_2.

1.4 Expected inflation and the inflation factor

In order to identify the inflation factor, Gong and Remolona model the market's perception of the inflation process. Their identification relies on the assumption of rational expectations and a simple inflation process perceived by market participants so that actual inflation can be expressed as a function of the inflation factor. In our two-country model, we identify inflation by assuming a common real factor. We can therefore decompose nominal bond yields into four components: expectations of real rates, real-term premiums, expectations of inflation, and inflation-risk premiums.

Note that the short rate in (8) is a risk-free rate because there is no need for revisions in expectations in one period. Hence, we can decompose the short rate into the inflation expectations and the expectations of real return according to the Fisher equation. Since $x_{1,\,t}$ is the inflation factor, the first term on the right-hand side of (8) is the inflation expectation. Thus:

$$E_t(\pi_{t+1}) \equiv \left(1 - \frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1,\,t}. \tag{20}$$

Similarly, the Canadian expected inflation is given by

$$E_t(\pi^*_{t+1}) \equiv \left(1 - \frac{1}{2}(\lambda_1^*)^2(\sigma_1^*)^2\right)x_{1,\,t}^*. \tag{21}$$

1.5 Inflation-risk and real-term premiums

The U.S. inflation-risk premium and real-term premium can be derived from the expected excess return on an *n*-period bond:

$$E_t(p_{n-1,\,t+1}) - p_{nt} - y_{1t} = -\lambda_1 B_{1,\,n-1}\sigma_1^2 x_{1t} - \frac{1}{2}B_{1,\,n-1}^2\sigma_1^2 x_{1t}$$

$$-\lambda_2 B_{2,\,n-1}\sigma_2^2 x_{2t} - \frac{1}{2}B_{2,\,n-1}^2\sigma_2^2 x_{2t}, \tag{22}$$

where the terms with x_{1t} represent the inflation-risk premium and the terms with x_{2t} represent the real-term premium. The two terms not containing λ_1 or λ_2 represent Jensen's inequality. Note that both the inflation-risk and real-term premiums will depend on maturity and vary over time with the respective factors.

Similarly, the Canadian inflation-risk premium and real-term premium can be derived from the expected excess return on an *n*-period bond:

$$E_t(p_{n-1,\,t+1}{}^*) - p_{nt}{}^* - y_{1t}{}^* = -\lambda_1{}^* B_{1,\,n-1}{}^* \sigma_1{}^{*2} x_{1t}{}^* - \frac{1}{2} B_{1,\,n-1}{}^{*2} \sigma_1{}^{*2} x_{1t}{}^* - \lambda_2{}^* B_{2,\,n-1}{}^* \sigma_2^2 x_{2t} - \frac{1}{2} B_{2,\,n-1}{}^{*2} \sigma_2^2 x_{2t} . \qquad (23)$$

2 Data and Estimation

2.1 Data

Some recent work on term-structure models, for example, Duffie and Singleton (1997) and Gong and Remolona (1997c), have found that a third factor may be needed to fit the entire yield curve and to explain the hump in the volatility curve. Therefore, we focus on fitting the 2-year to 10-year range of the yield curve because inflation expectations and inflation risks tend to have larger and more persistent influences on these yields than on the shorter-term yields.

2.1.1 Canadian data

The Canadian monthly data set spans the period January 1972 to December 1997, and consists of zero-coupon rates derived from the constant maturity par-value yields on federal bonds used in Day and Lange (1997).[9]

9. The par-value yields are constructed using the Bell method. There are two standardized ways to express the term structure in the literature: to report a par yield curve consisting of yield to maturity on par bonds; or to report a spot-rate curve consisting of yields to maturity on zero-coupon bonds. Either way of expressing the term structure requires estimating the term structure from yields to maturity on non-par coupon bonds. However, the par yield and the spot rate can be derived from each other once constructed. For the range of bond yields studied in this paper, only Canadian par yield data is available at the moment. The 10-year par-value yield is from Boothe (1991) up to 1989 and then spliced with the data base at the Bank of Canada. Both use the Bell model.

2.1.2 U.S. data

Monthly data on zero-coupon yields of 2-year to 10-year bonds are from McCulloch and Kwon (1993), and supplemented by data from the Federal Reserve Bank of New York. In the case of the Federal Reserve data, each zero curve is generated by fitting a cubic spline to prices and maturities of about 160 outstanding coupon-bearing U.S. Treasury securities. The securities are limited to off-the-run Treasuries to eliminate the most-liquid securities and reduce the possible effect of liquidity premiums.

Summary statistics for year-over-year CPI inflation and the zero-coupon yields for maturities of 2, 5, and 10 years for the two countries are reported in Table 1. The CPI inflation is constructed from the year-over-year percentage change in seasonally adjusted CPI (excluding indirect and tobacco taxes). The estimation runs from January 1984 to December 1997. We also consider a subsample of February 1991 to December 1997 to investigate the effect of the announcement of inflation-reduction targets in Canada in February 1991. As Table 1 shows, both average bond yields and inflation, as well as their volatilities, decrease from the first sample period to the more recent sample period. The average inflation and yield differentials over the two sample periods are reported in Table 2. The average inflation differentials between Canada and the United States are negative in both periods, but larger in magnitude in the second period. Average bond yield differentials are lower in the second period in the 2-year maturity, the same in the 5-year maturity, and higher in the second period in the 10-year maturity.

Figure 1a plots the U.S. and Canadian 5-year yields, and Figure 1b plots the CPI inflation over the whole sample period. Canadian yields were above U.S. yields between 1986 and 1996. Before 1991, inflation in Canada and the United States was very similar. However, the sharp drop in commodity prices in 1986 led to a much lower inflation rate in the United States than in Canada because Canada had higher energy taxes. A decline in oil prices, for example, did not lead to a big drop in energy prices and, hence, CPI inflation. Interestingly, 5-year bond yields in Canada and the United States had been at the same level since 1984. The sharp fall in U.S. inflation coincided with a sharp decline in U.S. bond yields below those in Canada. After 1988, inflation in the United States moved back to the same level as in Canada. The introduction of inflation-reduction targets in 1991 resulted in a sharp drop in inflation in Canada, and since 1992 it has been lower than U.S. inflation. However, bond yields remained higher in Canada from 1986 to late 1996.

Table 1

Summary Statistics

First sample: January 1984 to December 1997

	United States			Canada		
Variable	**Mean**	**Standard deviations**	**First order auto-correlation**	**Mean**	**Standard deviations**	**First order auto-correlation**
CPI inflation	3.43	1.03	0.97	3.13	1.35	0.99
2-year bond yield	6.99	1.94	0.98	8.06	2.11	0.97
5-year bond yield	7.61	1.84	0.98	8.45	1.77	0.97
10-year bond yield	8.05	1.72	0.98	8.96	1.59	0.97

Second sample: February 1991 to December 1997

	United States			Canada		
Variable	**Mean**	**Standard deviations**	**First order auto-correlation**	**Mean**	**Standard deviations**	**First order auto-correlation**
CPI inflation	2.96	0.66	0.96	1.95	0.83	0.96
2-year bond yield	5.54	0.96	0.95	6.38	1.45	0.94
5-year bond yield	6.28	0.78	0.93	7.12	1.22	0.95
10-year bond yield	6.85	0.74	0.94	7.89	1.02	0.95

Table 2

Average Canada–U.S. Inflation and Yield Differentials

	January 1984 to December 1997	February 1991 to December 1997
CPI inflation	−0.30	−1.01
2-year bond yield	1.08	0.84
5-year bond yield	0.84	0.84
10-year bond yield	0.91	1.05

Figure 1a

U.S. and Canadian 5-Year Yields, January 1984 to December 1997

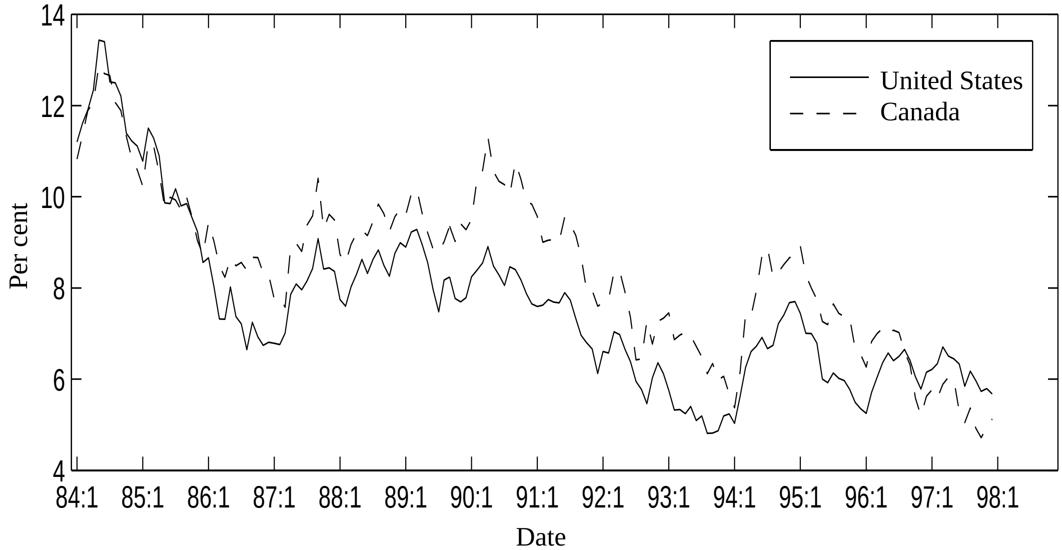
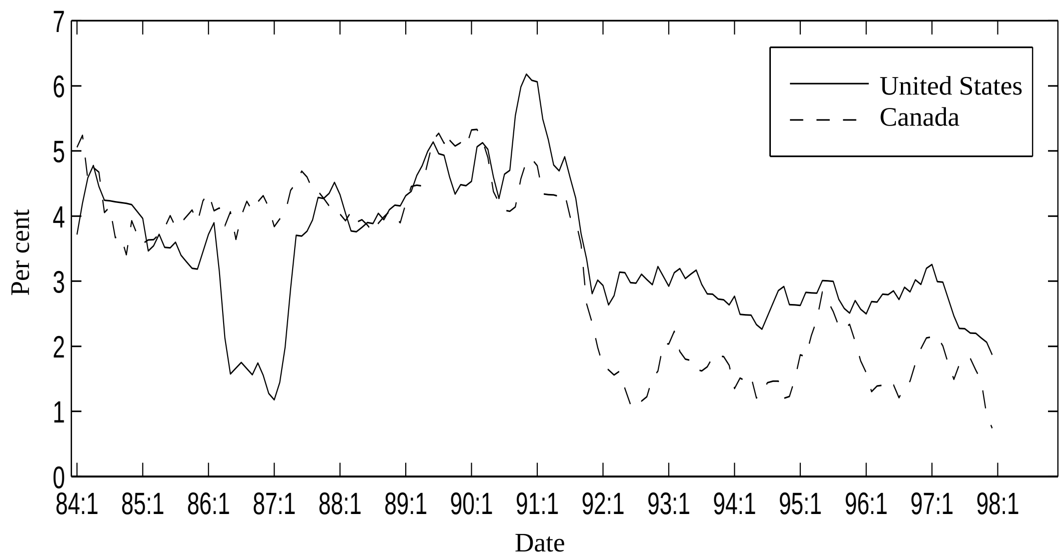


Figure 1b

U.S. and Canadian Inflation, January 1984 to December 1997



2.2 Kalman filtering and maximum-likelihood estimation

Estimation of the model is based on a subset of the available yields that covers the medium maturity spectrum. Since the factors are treated as latent variables, they can be backed out using the Kalman filter. Estimation is then by maximum likelihood, based on the conditional means and variances of the processes of the factors.[10] In applying the Kalman filter in our estimation, we have to write our models in linear state-space form. The measurement and transition equations are given by:

$$y_t = A + HX_t + v_t \tag{24}$$

$$X_{t+1} = C + FX_t + u_{t+1}. \tag{25}$$

In our model, the yields, which are affine functions of the factors, serve as the measurement equations. The factors' stochastic processes, which are AR(1) processes, are the transition equations. Thus:

$$\begin{bmatrix} y_{lt} \\ y_{mt} \\ y_{nt} \\ y_{lt}^* \\ y_{mt}^* \\ y_{nt}^* \end{bmatrix} = \begin{bmatrix} a_l \\ a_m \\ a_n \\ a_l^* \\ a_m^* \\ a_n^* \end{bmatrix} + \begin{bmatrix} b_{1l} & b_{2l} & 0 \\ b_{1m} & b_{2m} & 0 \\ b_{1n} & b_{2n} & 0 \\ 0 & b_{2l}^* & b_{1l}^* \\ 0 & b_{2m}^* & b_{1m}^* \\ 0 & b_{2n}^* & b_{1n}^* \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{1t}^* \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \\ v_{4t} \\ v_{5t} \\ v_{6t} \end{bmatrix} \tag{26}$$

where

$$y_{lt}, y_{mt}, y_{nt} \text{ and } y_{lt}^*, y_{mt}^*, y_{nt}^*$$

are zero-coupon yields at time t with maturities l, m, and n in the United States and Canada, respectively. The coefficients in the equation are:

$$a_k = \frac{A_k}{k},\ b_{1\phi k} = \frac{B_{lk}}{k},\ \text{and } b_{2k} = \frac{B_{mk}}{m},\ k=l, m, n,$$

which are given by equations (9) through (11), whereas those coefficients with an asterisk are the Canadian counterparts given by equations (17)

10. De Jong (1997) discusses some empirical problems related to the estimation of the parameters by maximum-likelihood and quasi-maximum-likelihood estimation. However, he argues that for parameters typically found in estimates of the term-structure model, the simulation results in Lund and Anderson (1997) suggest that the bias in the QML estimator is not particularly large.

through (19). The v_{it} variables are measurement errors distributed with zero-mean and standard deviations e_i, where $i = 1, 2, \ldots, 6$.

The transition equations correspond to equations (5), (6), and (13):

$$\begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{1t}^* \end{bmatrix} = \begin{bmatrix} (1-\phi_1)\mu_1 \\ (1-\phi_2)\mu_2 \\ (1-\phi_1^*)\mu_1^* \end{bmatrix} + \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_1^* \end{bmatrix} \begin{bmatrix} x_{1,\,t-1} \\ x_{2,\,t-1} \\ x_{1,\,t-1}^* \end{bmatrix} + \begin{bmatrix} x_{1,\,t-1}^{1/2} u_{1t} \\ x_{2,\,t-1}^{1/2} u_{2t} \\ (x_{1,\,t-1}^*)^{1/2} u_{1t}^* \end{bmatrix}, \tag{27}$$

where the shocks u_{lt}, u_{2t}, and u_{lt}^* are distributed normally with mean-zero and standard errors σ_1, σ_2, and σ_1^*. Note that in standard linear state-space models, no restrictions link the measurement equations and the transition equations. In our model, however, the arbitrage conditions serve as over-identifying restrictions that link the coefficients of these two equations. The arbitrage conditions are given by equations (9) through (11) and equations (17) through (19), with initial values set by equations (8) and (16).

3 Results

3.1 Parameter estimates

Table 3 reports the parameter estimates for the three-risk and four-risk models over the two sample periods. The $(1-\phi)$ variables measure the rates of mean reversion; our parameter estimates suggest rather fast mean reversion. The σ variables measure the factors' volatilities and the μ variables are the long-run means of the factors. λ_1 and λ_1^* are the prices of inflation risks, and λ_2 measures the price of real risk. The prices of inflation risks are higher in the United States in the three-risk model, but lower or the same in the four-risk model. Although the four-risk model allows the prices of real risks to be different between the two countries, it turns out that they are very similar. These results suggest that both bond markets may have priced the real risk in a similar way, since there is a common real shock.

In evaluating the model, we rely on the implications of the parameters for inflation expectations and risk premiums rather than individual estimates. To do this, we back out from the model conditional forecasts of the inflation factor, derive the implied expectations and risk

Table 3

Parameter Estimates

	Three-risk model		Four-risk model	
	Sample January 1984 to December 1997	Sample February 1991 to December 1997	Sample January 1984 to December 1997	Sample February 1991 to December 1997
Inflation parameters				
ϕ_1	0.79	0.89	0.77	0.70
ϕ_1*	0.86	0.86	0.72	0.71
μ_1	3.68	0.99	1.61	1.93
μ_1*	5.78	6.78	1.32	3.14
λ_1	–1.33	–1.42	–1.03	–1.05
λ_1*	–0.92	–0.97	–1.09	–1.05
σ_1	0.73	0.84	1.01	0.85
σ_1*	1.32	1.26	0.84	0.87
Real return parameters				
ϕ_2	0.65	0.68	0.71	0.73
μ_2	7.71	7.71	3.27	2.78
λ_2	–0.97	–1.04	–1.03	–0.98
λ_2*			–1.04	–0.99
ϕ_2	0.86	0.77	0.84	1.00
Standard deviation of measurement errors				
e_1	1.08	0.52	0.29	1.48
e_2	1.29	2.88	1.31	1.20
e_3	1.65	1.34	1.50	0.64
e_4	0.68	1.95	0.41	0.21
e_5	1.45	1.38	1.39	2.88
e_6	2.19	1.42	1.30	1.09
Mean log likelihood				
	6.83	5.63	27.73	30.37

premiums, and then examine how these implied variables behave over time. In particular, we can examine how they vary over time in light of the behaviour of the yield differentials over the same period. We will focus our discussion on the full sample period of January 1984 to December 1997.

3.2 Inflation expectations

Figures 2a and 2b plot the actual inflation and inflation expectations in the United States, and Figures 3a and 3b plot those in Canada. One-period-ahead inflation expectations are backed out from the model's

Figure 2a

U.S. Actual and Expected Inflation, January 1984 to December 1997, Three-Risk Model

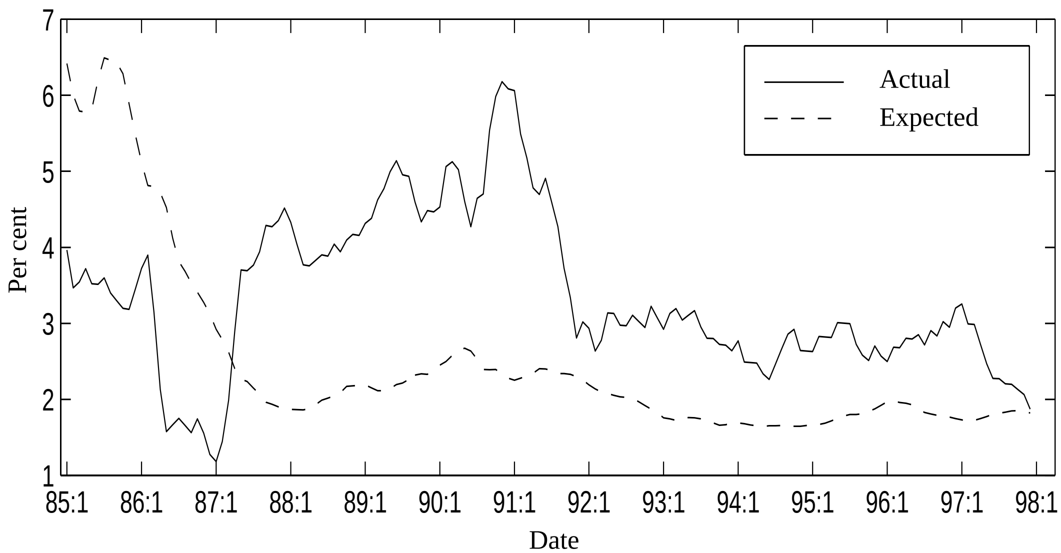


Figure 2b

U.S. Actual and Expected Inflation, January 1984 to December 1997, Four-Risk Model

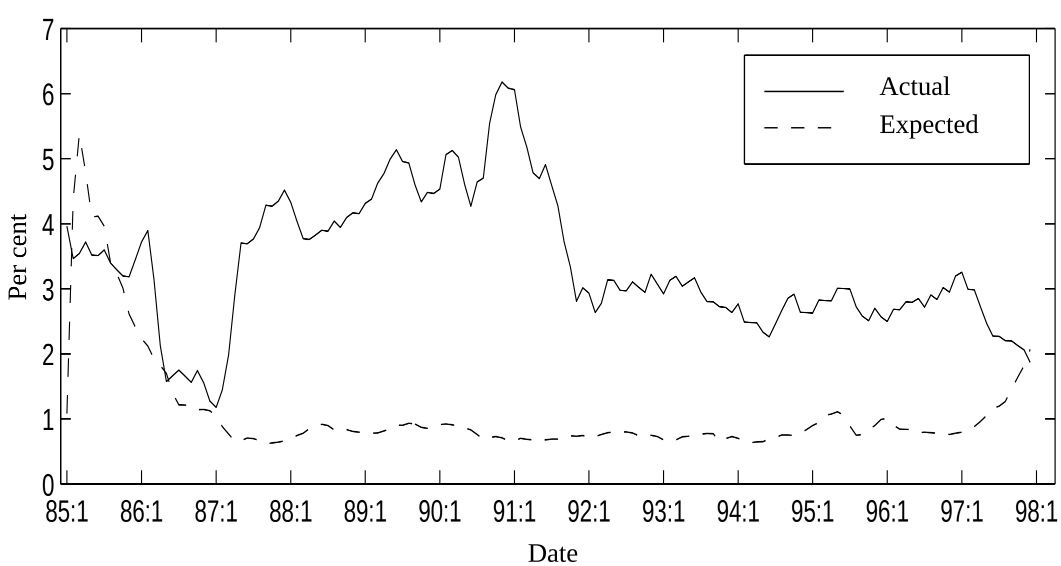


Figure 3a

Canadian Actual and Expected Inflation, January 1984 to December 1997, Three-Risk Model

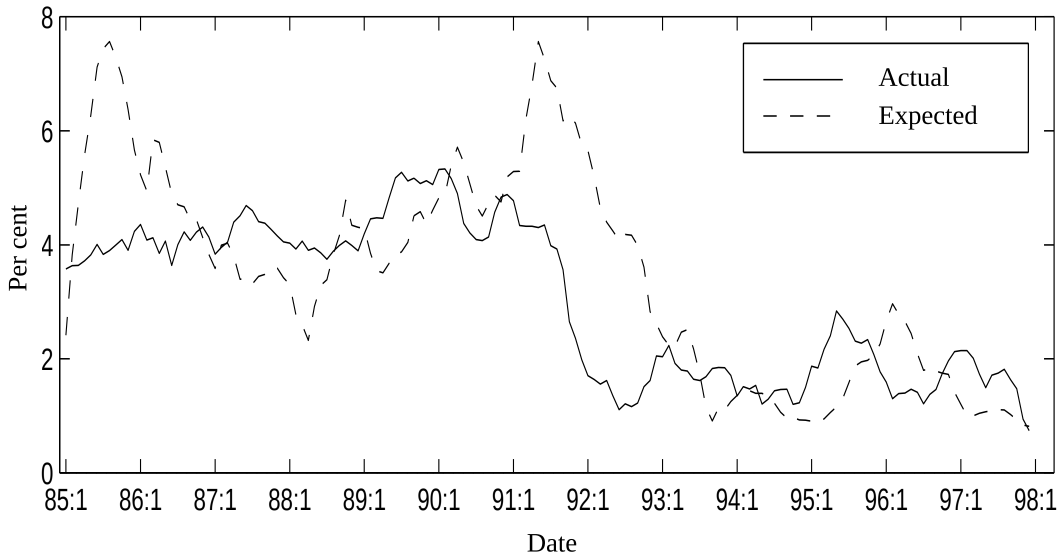
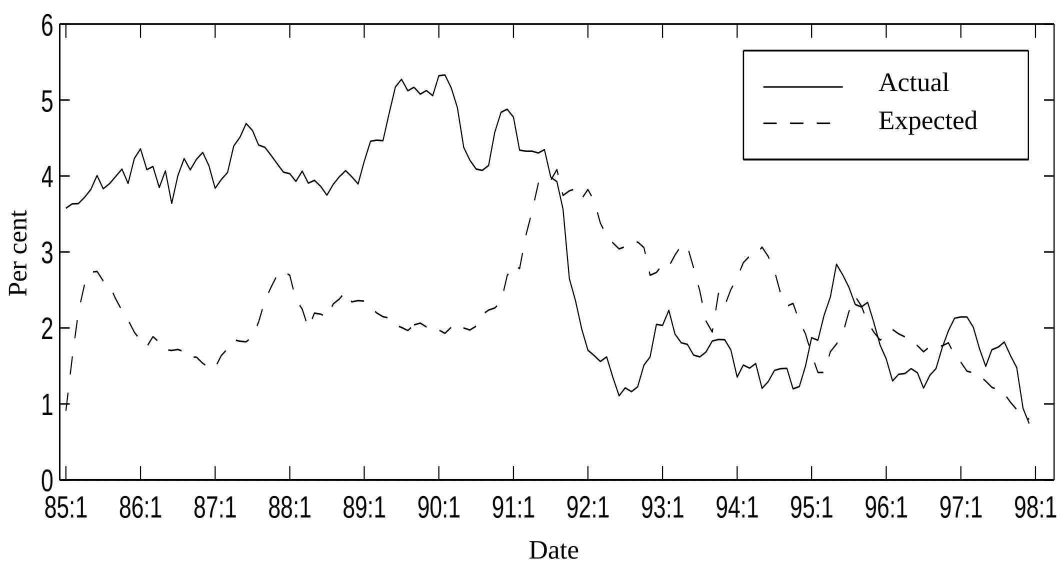


Figure 3b

Canadian Actual and Expected Inflation, January 1984 to December 1997, Four-Risk Model



conditional forecasts of x_{1t} and x_{1t}^*, and equations (20) and (21). We can then calculate the 12-month inflation expectations by accumulating them over the appropriate horizon. In Figure 2, the inflation expectations in the United States followed a similar downward trend as actual inflation before 1987. However, after 1987, inflation expectations flattened out, and were substantially below actual inflation due to the fast mean reversion of the inflation factor as implied by a low ϕ_1 in both models.

In Canada, inflation expectations moved very closely with actual inflation in the three-risk model, except between 1985 and 1987; the Canadian dollar came under extreme downward pressure in late 1985 and early 1986. As the Bank of Canada reacted strongly to support the dollar, the assumption of an independent inflation factor may not work as well as in other periods. In the four-risk model, ϕ_1^* is smaller than that in the three-risk model, which implies faster mean reversion. Thus, the inflation expectations measure tends to move around its mean with relatively little variation compared with actual inflation. However, since 1995, it tracked actual inflation well, as actual inflation stabilized at between 1 and 2 per cent. In fact, in both models, inflation expectations have tracked actual inflation very well since 1995, as actual inflation became stable and tended to converge with its long-run mean.

The results for inflation expectations suggest that the model does a better job in estimating expected inflation in Canada than the United States, and that the three-risk model gives a better fit than the four-risk model because of a larger persistence parameter, ϕ_1^*. In the model, inflation is determined by the inflation factor alone. It is possible that the inflation process is more complicated in the United States and cannot be adequately explained by one factor only. In future work we would also like to compare the estimated inflation expectations in our model with survey data of inflation forecasts or inflation forecasts from other models.

3.3 Inflation risks

Revisions in inflation expectations are a source of risk that appears to have been priced by the bond market in the 1980s and 1990s. Since the magnitudes of the revisions are related to the level of the expectations, risk premiums vary over time. The estimates of the prices of risks, λ_1, allow us to calculate inflation-risk premiums by applying the model's conditional forecasts of x_{1t} and x_{1t}^* to the relevant terms in equations (22) and (23). In Figures 4a and 4b, we graph the estimated inflation-risk premiums for the 5-year yield in Canada and the United States for the two models. In both models, the Canadian inflation-risk premium varies substantially over time, while the U.S. inflation-risk premium varies substantially only at the two ends of the sample period.

Figure 4a

U.S. and Canadian Inflation-Risk Premiums, January 1984 to December 1997, Three-Risk Model

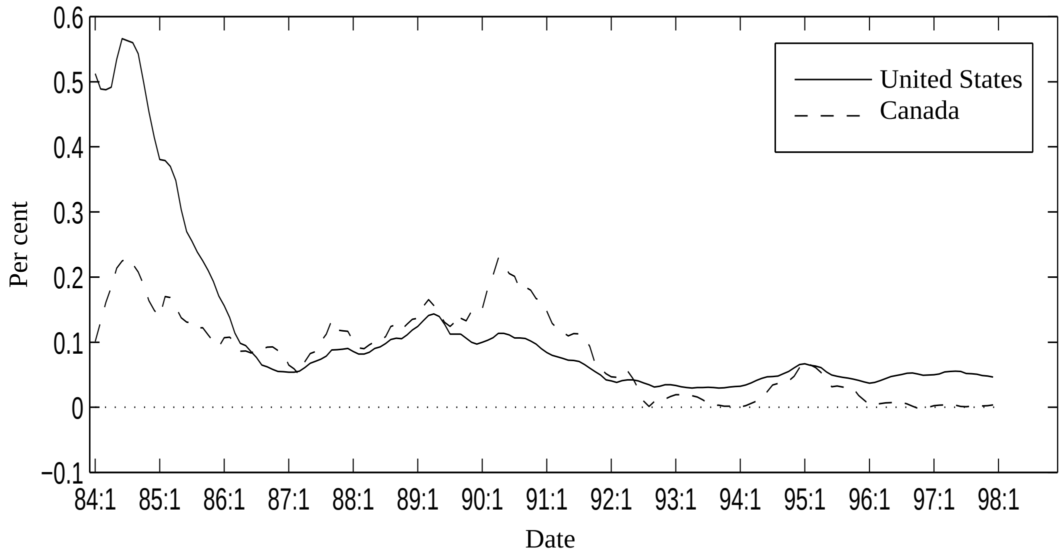
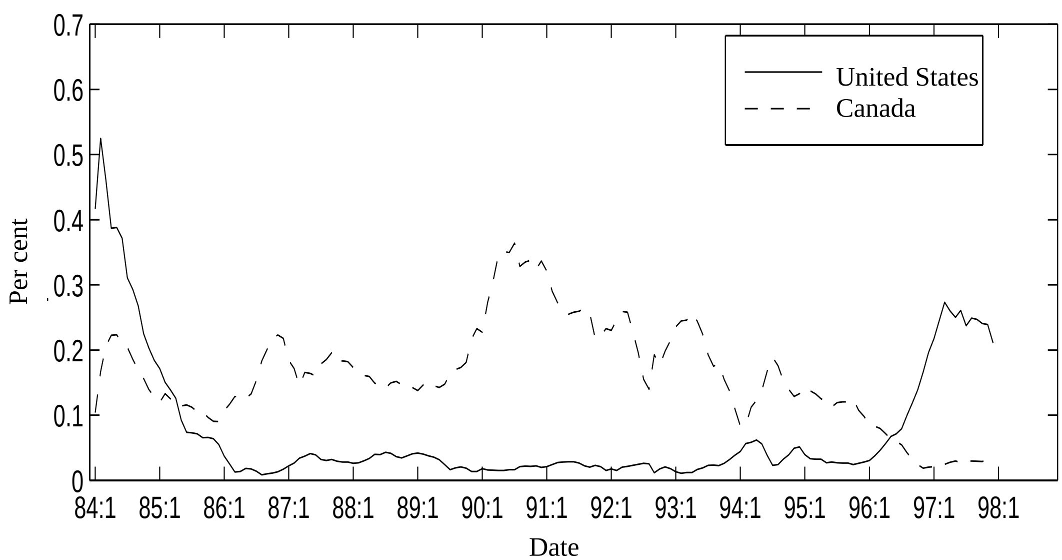


Figure 4b

U.S. and Canadian Inflation-Risk Premiums, January 1984 to December 1997, Four-Risk Model



Campbell and Shiller (1996) estimate the size of the inflation-risk premium in the United States, defined as the average excess return on an inflation-sensitive asset that is attributable to its inflation sensitivity, using two different methods. In the first method, they assume that the average excess return on a nominal 5-year bond over a comparatively riskless asset such as a nominal 3-month Treasury bill is entirely accounted for by its inflation-risk premium. Over the sample period 1953 to 1994, they estimate a risk premium of 70 to 100 basis points on a 5-year nominal bond.[11] In the second method, they use asset-pricing theory to try to judge what risk premium is implied by the covariance of bond returns with relevant state variables. They use the return on a proxy for the market portfolio, such as a value-weighted stock index, and the growth rate of aggregate consumption. They obtain an implied risk premium of about 90 to 150 basis points. Thus, Campbell and Shiller suggest that a best guess might be 50 to 100 basis points for a 5-year zero-coupon bond. Gong and Remolona (1997b) estimate the inflation-risk premium in the United States to be time-varying, ranging from about 50 to about 150 basis points.

In our three-risk model, the inflation-risk premiums in Canada and the United States are small. Over the sample period 1984 to 1997, the average inflation-risk premium is about 8 basis points in Canada and 12 basis points in the United States, with a differential of about 4 basis points. The lower average inflation-risk premium in Canada is mainly due to the substantially lower inflation-risk premium from 1984 to 1986. As discussed in the previous section, the model does not work well in this period, as the Bank of Canada reacted to the weakness of the Canadian dollar. In any case, our estimates of the inflation-risk premiums are much lower than the estimates in Campbell and Shiller and Gong and Remolona. In the four-risk model, the average inflation-risk premium is 16 basis points in Canada and 7 basis points in the United States. Although the inflation-risk differentials are small in the two models, they seem to track well with the yield differentials over the sample period; we will discuss this in the next subsection. Thus, we think that even though the inflation-risk differentials are biased downward for both countries compared with estimates in other studies, their differentials may be in the right order of magnitude.

Next, we examine whether these inflation-risk premiums are consistent with Canada's inflation experiences by focusing on several key events that are believed to have major impacts on inflation and inflation expectations. These include the Eric J. Hanson Memorial Lecture delivered by Governor Crow in January 1988, the joint announcement of inflation

11. This estimate could be interpreted as the upper bound for the inflation-risk premium because of the possible presence of a real-risk premium.

reduction targets by the Bank of Canada and the Department of Finance in February 1991, and the second inflation target announcement when Governor Thiessen was appointed in December 1993.

First, we discuss the inflation risk in the three-risk model depicted in Figure 4a. Although price stability was mentioned as the goal of monetary policy in the Hanson lecture, there was not a significant drop in inflation-risk premiums between 1988 and 1991. Indeed, they went up slightly during the period. However, the movement in the inflation risk is consistent with movements in bond yields and actual inflation in the same period. The 5-year bond yield went up along with the rising U.S. bond yield as actual inflation continued to rise. In other words, the bond market did not perceive a significant decline in inflation risk in spite of the Bank's stated commitment to price stability, because actual inflation did not begin to drop. However, the Bank gained credibility over time as actual inflation started to fall in the 1990s, particularly after the announcement of inflation-reduction targets in February 1991. In mid-1991, the inflation-risk premium began to decline. The announcement of the second inflation target in December 1993 had little impact on the inflation risk, as it was almost 0 by mid-1992. However, the inflation-risk premium began to rise in early 1994, when the U.S. Federal Reserve started tightening. The risk began to come down again in late 1994, and stabilized at around 0 from 1996 onward.

The four-risk model in Figure 4b generally tells a similar story. The increase in inflation-risk premiums from 1988 to late 1990 is more substantial than in the three-risk model, suggesting that there might be a significant buildup of inflation risks in the period. However, the introduction of inflation-reduction targets seems to have had an effect on inflation-risk premiums, as they started moving in a downward trend shortly after. Since 1997, inflation-risk premiums have been flat at a very small value, substantially below those of the United States.

3.4 Yield differentials and inflation differentials

Figures 5a and 5b plot the Canada–U.S. differentials in 5-year bond yields and one-year inflation expectations in the three-risk and four-risk models, respectively. In both models, inflation-expectations differentials were higher than yield differentials between 1991 and 1993, although they followed similar trends. In the three-risk model depicted in Figure 5a, inflation differentials were significantly lower than yield differentials from late 1992 to 1996 and were negative between mid-1993 and mid-1995. However, inflation differentials have been almost the same as yield differentials since 1996. In the four-risk model depicted in Figure 5b, the inflation differentials also tracked yield differentials closely. Since 1996, however, inflation differentials have been higher than yield differentials but

Figure 5a

Canada–U.S. Actual Yield and Inflation-Expectations Differentials in 5-year Yields, Three-Risk Model

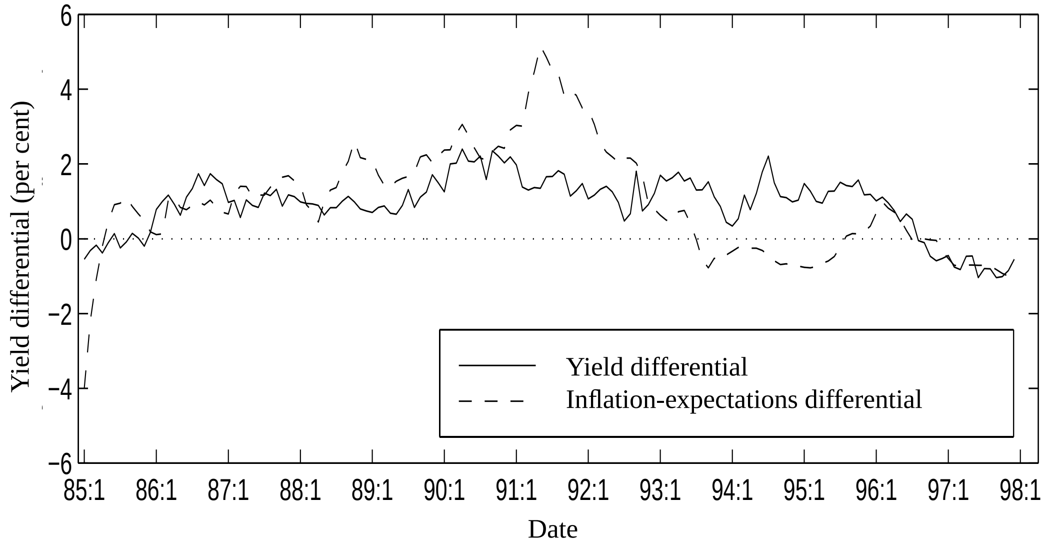
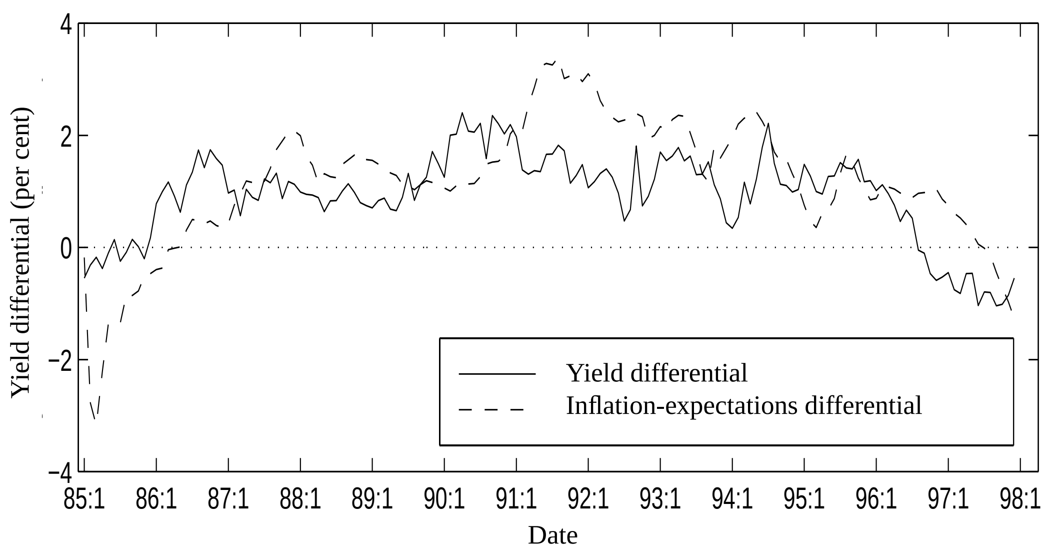


Figure 5b

Canada–U.S. Actual Yield and Inflation-Expectations Differentials in 5-year Yields, Four-Risk Model



have followed similar trends. In the period 1985 to 1997, average inflation-expectations differentials are 0.98 per cent in the three-risk model and 1.17 per cent in the four-risk model. They are very close to the average five-year bond yield differentials of 0.84 per cent in the same period. Overall, estimates for inflation expectations differentials in the three-risk model seem to be slightly better than those in the four-risk model.

Figures 6a and 6b plot the Canada–U.S. yield differentials and inflation-risk differentials in 5-year bonds in the three-risk and four-risk models respectively. Note that yield differentials follow the left-hand scale, while inflation-risk differentials follow the right-hand scale. In the three-risk model, until mid-1995, inflation-risk differentials moved closely together with yield differentials, though the inflation differentials were much smaller in magnitude. They were negative before early 1986 and after early 1992. Yield differentials were also negative before early 1986, and turned negative again after mid-1996. After late 1995, risk differentials were flat at just below 0 per cent, but yield differentials continued to decline. In the four-risk model, inflation-risk differentials track yield differentials very closely over the entire sample period. The 4-risk model fits especially well after mid-1996. Inflation-risk differentials became negative in early 1996 and yield differentials followed suit shortly after. Thus, we find that inflation-risk differentials in the four-risk model track yield differentials better than the three-risk model, which is consistent with the finding that inflation risks in the four-risk model seem to better describe Canada's inflation experience.

The results suggest that in the 1980s, yield differentials generally had the same sign as differentials in inflation expectations, inflation risks and actual inflation. However, in the 1990s, even though actual inflation has been lower in Canada since 1992, yield differentials only became negative after mid-1996. As reported in Table 2, the actual CPI inflation differential is –1.01 per cent from 1991 through 1997. This suggests that yield differentials do not follow the same sign as actual inflation differentials, but rather differentials in inflation expectations and inflation risks. Thus, lower observed inflation does not necessarily mean a lower inflation risk. The four-risk model depicted in Figure 5b shows that both inflation expectations and inflation risk were higher in Canada for most of the early 1990s. Canada has only had lower inflation expectations since mid-1997 and lower inflation risks since mid-1996. As a result, yields remained higher in Canada for most of the early 1990s and became lower only after mid-1996.

3.5 Subsample period February 1991 to December 1997

Because the announcement of inflation-reduction targets in Canada in February 1991 may imply a new monetary regime of inflation targeting, we examine whether this will affect our full-sample analysis by considering the

Figure 6a

Canada–U.S. Actual Yield and Inflation-Risk Differentials in 5-Year Yields, Three-Risk Model

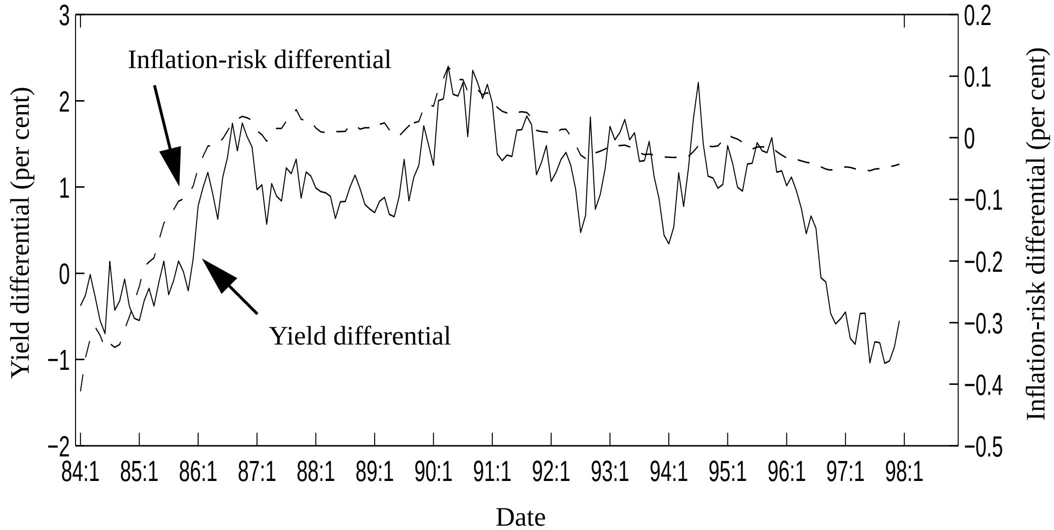
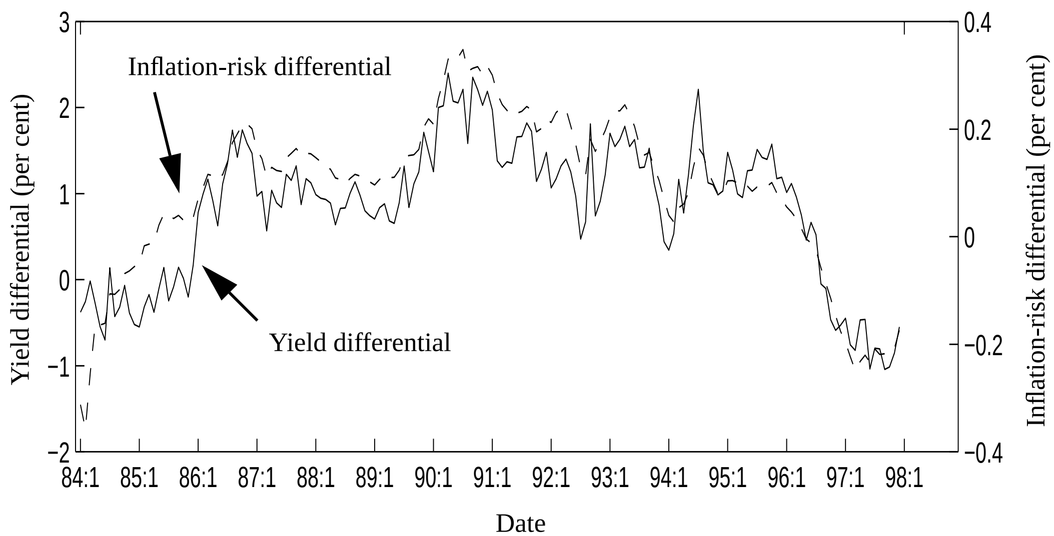


Figure 6b

Canada–U.S. Actual Yield and Inflation-Risk Differentials in 5-Year Yields, Four-Risk Model



subsample from February 1991 to December 1997. In Table 2, yield differentials between Canada and the United States do not differ very much in this subsample period relative to the full sample. However, actual inflation is substantially lower in Canada in the subperiod. Parameter estimates for the subsample are similar to the full sample. The price of inflation risk is lower in the subsample in the four-risk model, which is consistent with a target of low inflation in Canada. However, the three-risk model gives the opposite results, with a higher price of inflation risk in the subsample period.

Figures 7 and 8 show the results in the subsample period from February 1991 to December 1997. In the three-risk model depicted in Figure 7a, inflation-risk premiums were higher in Canada until late 1996. The average inflation risks for the period were 4 basis points in the United States and 11 basis points in Canada, with a differential of 7 basis points. Conversely, Canada has had a lower risk premium most of the time since late 1991 in the four-risk model (see Figure 7b). The average inflation risks are 8 and 7 basis points in the United States and Canada, respectively, with a differential of only 1 basis point. This result is in line with the full-sample results, which show that the inflation-risk differentials are very small.

Figure 8a plots actual inflation and inflation expectations in Canada. Actual inflation was stable in this period, staying within the Bank's inflation target range of 1 to 3 per cent. Inflation expectations estimated from both the three-risk and four-risk models were capable of tracking actual inflation well since 1993. Figure 8b plots the Canadian–U.S. yield differentials and inflation-expectations differentials. Inflation-expectations differentials were negative before mid-1992 in the three-risk model, but followed an upward trend. In the four-risk model, inflation differentials were negative before mid-1993, and also followed an upward trend from early 1992. Yield differentials, on the contrary, were positive until mid-1996, but followed a downward trend. The results are not as good as those in the full sample, which suggests that the subsample may be too short or perhaps that the change in regime in 1991 in Canada may not be very important in our analysis.

3.6 Actual and implied yield differentials

One question often asked in working with term-structure models is how well the implied yield curve from the model fits the actual average yield curve over the sample period. In a two-country model, it may be more appropriate to look at how well the three factors reproduce the shape of the average yield differential curve; if the model is misspecified, it will affect the implied yield curves in the two countries in more or less the same way. Figures 9a and 9b plot the actual and implied Canada–U.S. yield differentials across maturities up to 10 years. The actual yield differential

Figure 7a

Inflation-Risk Premiums, February 1991 to December 1997, Three-Risk Model

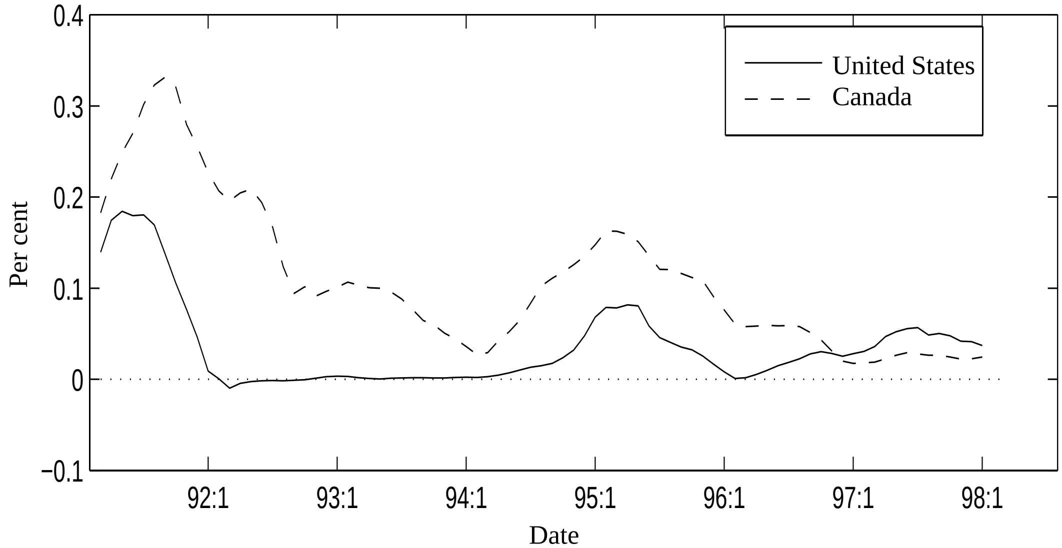


Figure 7b

Inflation-Risk Premiums, February 1991 to December 1997, Four-Risk Model

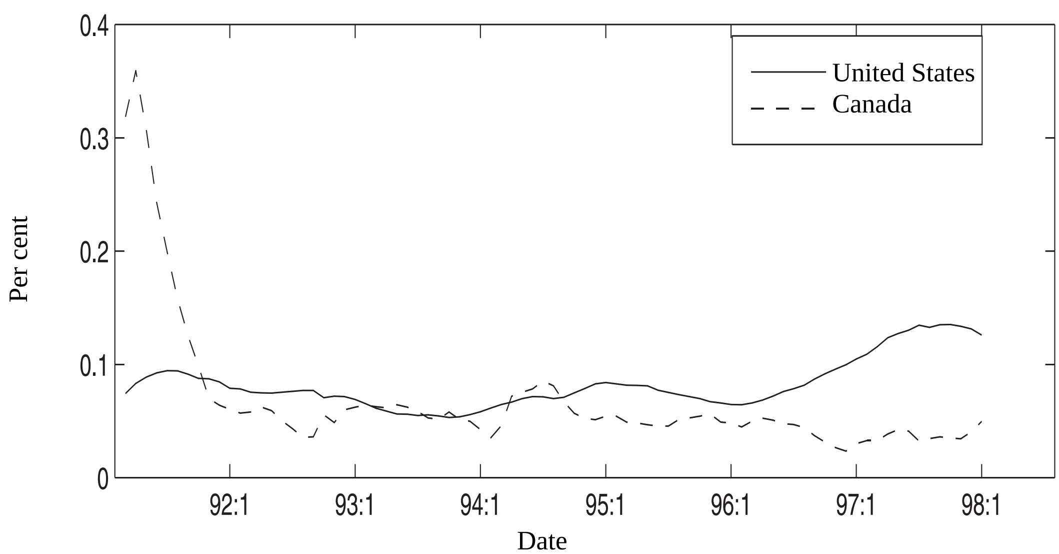


Figure 8a

Canadian Inflation Expectations, February 1991 to December 1997, Three-Risk and Four-Risk Models

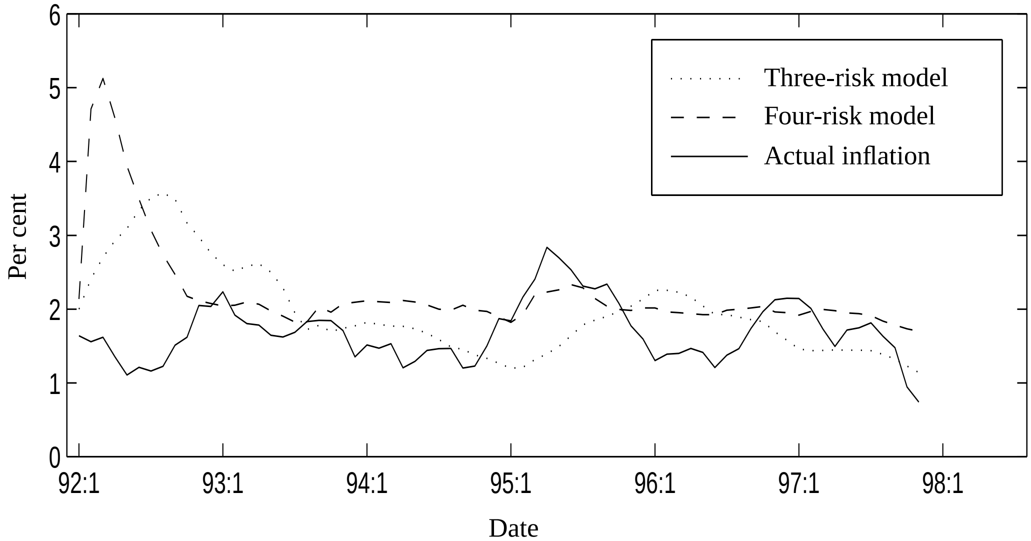


Figure 8b

Canada–U.S. Actual Yield and Inflation-Expectations Differentials in 5-Year Yields, Four-Risk Model

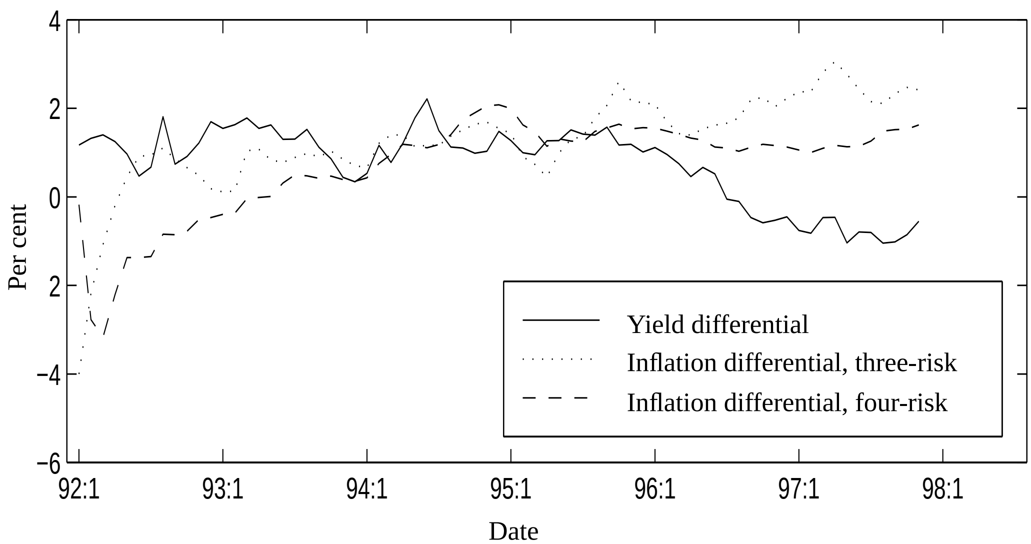


Figure 9a

Actual and Implied (Average) Canada–U.S. Yield Differentials, Three-Risk Model

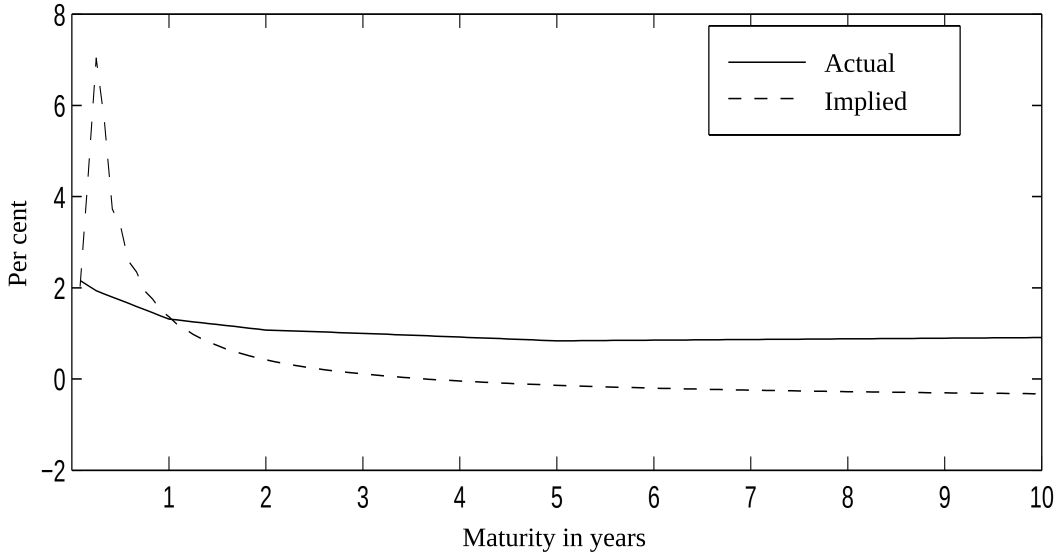
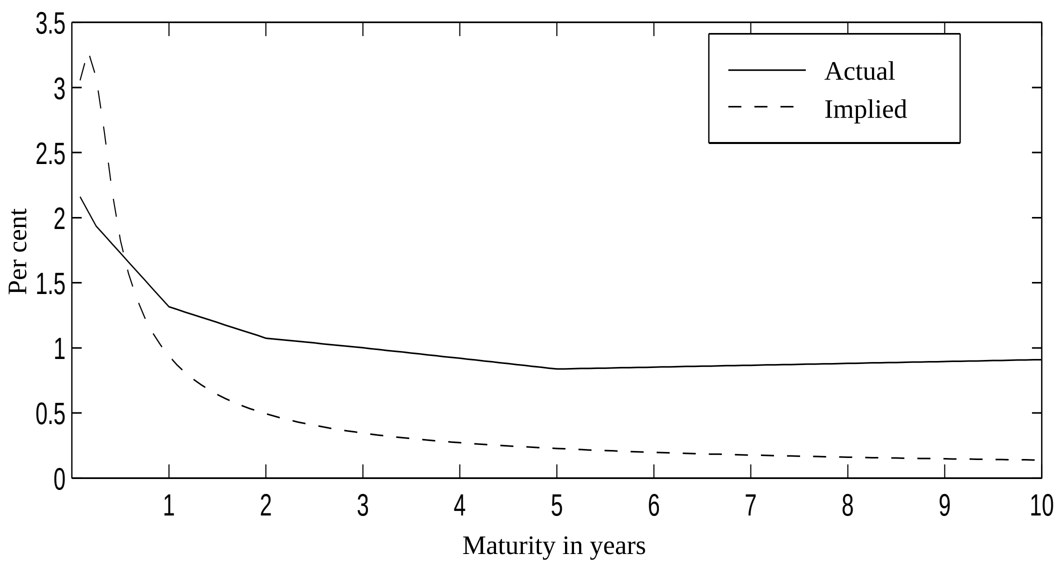


Figure 9b

Actual and Implied (Average) Canada–U.S. Yield Differentials, Four-Risk Model



curve is a *U* shape, a yield differential “smile”—higher at the short and long maturities, but smaller in the middle. The differential is smallest at the 5-year yield. Although the implied yield-differential curve also has an inverted *U* shape, it is below the average curve from the 1-year maturity onwards. Since we do not include yields at the short end in our estimation, the model also missed the shape at the short end.

Conclusions

In this paper, we construct a two-country, two-factor affine term-structure model to estimate inflation expectations and inflation-risk premiums in Canada and the United States using bond yields of 2-, 5-, and 10-year maturities. The results suggest that there is useful and substantial information that can be extracted from the yield curve. We also find that there is a close relationship between inflation differentials and yield differentials between Canada and the United States. There is not enough evidence to determine whether the three-risk model or the four-risk model is more suitable.[12] The market appears to price real risks roughly the same way in the two countries. We will re-examine this issue in the future by performing more vigorous tests.

A few other issues also deserve further investigation. First, the mean-reversion parameters estimated in the paper suggest rapid mean reversion. As we interpret the factors as inflation and real return factors, the fact that the inflation factor follows such a fast mean-reverting process may pose a problem. Second, in future work, we could allow for real idiosyncratic shocks that affect only Canadian yields by including an additional real factor that is specific to Canada. Finally, although we do not include actual inflation in the estimation, the inflation expectations and risk premiums estimated in the model are very reasonable, except between 1985 and 1987. In future work, we would like to examine whether including actual inflation will give a better fit in this period. Moreover, we would like to improve the fit of U.S. inflation.

This paper should only be seen as a first step in a research program to examine the relationship between the yield curve and economic fundamentals by exploiting the co-movements among interest rates across the yield curve and the co-movements among those interest rates across countries.

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12. A log-likelihood ratio test rejects the three-risk model in favour of the four-risk model.

Appendix 1: Recursive Restrictions

We start with the general pricing equation:

$$p_{nt} = E_t[m_{t+1} + p_{n-1,\, t+1}] + \frac{1}{2}\mathrm{Var}_t[m_{t+1} + p_{n-1,\, t+1}].$$

The short rate is derived by setting $p_{0,\, t} = 1$:

$$y_{1t} = -p_{1,\, t} = -E_t(m_{t+1}) - \frac{1}{2}\mathrm{Var}_t(m_{t+1}),$$

$$= \left(1 - \frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1t} + \left(1 - \frac{1}{2}\lambda_2^2\sigma_2^2\right)x_{2t},$$

showing the short rate to be linear in the factors.

Now we guess that the price of an *n*-period bond is affine:

$$-p_{nt} = A_n + B_{1n}x_{1t} + B_{2n}x_{2t}.$$

We verify that A_n, B_{1n}, and B_{2n} exist, and that they satisfy the general pricing equation:

$$-p_{nt} = -E_t[m_{t+1} + p_{n-1,\, t+1}] - \frac{1}{2}\mathrm{Var}_t[m_{t+1} + p_{n-1,\, t+1}].$$

$$= (A_{n-1} + (1-\phi_1)\mu_1 B_{1,\, n-1} + (1-\phi_2)\mu_2 B_{2,\, n-1})$$

$$+ \left(1 + \phi_1 B_{1,\, n-1} - \frac{1}{2}(\lambda_1 + B_{1,\, n-1})^2 {\sigma_1}^2\right)x_{1t}$$

$$+ \left(1 + \phi_2 B_{2,\, n-1} - \frac{1}{2}(\lambda_2 + B_{2,\, n-1})^2 {\sigma_2}^2\right)x_{2t}.$$

Now by matching coefficients we have

$$A_n = A_{n-1} + (1-\phi_1)\mu_1 B_{1,\, n-1} + (1-\phi_2)\mu_2 B_{2,\, n-1}$$

$$B_{1n} = 1 + \phi_1 B_{1,\, n-1} - \frac{1}{2}(\lambda_1 + B_{1,\, n-1})^2 {\sigma_1}^2$$

$$B_{2n} = 1 + \phi_2 B_{2,\, n-1} - \frac{1}{2}(\lambda_2 + B_{2,\, n-1})^2 {\sigma_2}^2.$$

Appendix 2: Kalman Filtering Procedure[1]

For the state-space models in Section 2, the measurement and transition equations can be written in the following matrix form.

Measurement equation:

$$y_t = A + HX_t + v_t,$$

where $v_t \sim N(0, R)$.

Transition equation:

$$X_{t+1} = C + FX_t + u_{t+1},$$

where $u_{t+1|t} \sim N(0, Q_t)$.

The Kalman filter procedure of this state–space model is the following.

1. First, initialize the state-vector S_t.

The recursion begins with a guess $S_{1|0}$, usually given by

$$\hat{S}_{1|0} = E(S_1).$$

The associated mean square error (MSE) is

$$P_{1|0} \equiv E[(S_1 - \hat{S}_{1|0})(S_1 - \hat{S}_{1|0})'] = Var(S_1).$$

The initial state S_1 is assumed to be $N(\hat{S}_{1|0}, P_{1|0})$.

2. Next, forecast y_t:

Let I_t denote the information set at time *t*. Then

$$\hat{y}_{t|t-1} = A + HE[S_t | I_{t-1}] = A + H\hat{S}_{t|t-1}.$$

The forecasting MSE is

$$E[(y_t - \hat{y}_{t|t-1})(y_t - \hat{y}_{t|t-1})'] = HP_{t|t-1}H' + R.$$

1. See also Hamilton (1994) for a more complete description of the procedure.

3. To update the inference about S_t given I_t:

Knowing y_t helps to update $S_{t|t-1}$ by the following: Write

$$S_t = \hat{S}_{t|t-1} + (S_t - \hat{S}_{t|t-1})$$

$$y_t = A + H\hat{S}_{t|t-1} + H(S_t - \hat{S}_{t|t-1}) + v_t.$$

We have the following joint distribution:

$$\begin{bmatrix} S_t | I_{t-1} \\ y_t | I_{t-1} \end{bmatrix} \sim N\left(\begin{bmatrix} \hat{S}_{t|t-1} \\ A + H\hat{S}_{t|t-1} \end{bmatrix}, \begin{bmatrix} P_{t|t-1} & P_{t|t-1}H' \\ HP_{t|t-1} & HP_{t|t-1}H' + R \end{bmatrix} \right).$$

Thus,

$$\hat{S}_{t|t} \equiv E[S_t | y_t, I_{t-1}] = \hat{S}_{t|t-1}$$
$$+ P_{t|t-1}H'(HP_{t|t-1}H' + R)^{-1}$$
$$(y_t - HS_{t|t-1} - A)$$

$$P_{t|t} \equiv E[(S_t - \hat{S}_{t|t})(S_t - \hat{S}_{t|t})']$$
$$\equiv P_{t|t-1} - P_{t|t-1}H'(HP_{t|t-1}H' + R)^{-1}HP_{t|t-1}.$$

4. To forecast S_{t+1} given I_t:

$$\hat{S}_{t+1|t} = E[S_{t+1} | I_t] = F\hat{S}_{t|t}$$

$$P_{t+1|t} = E[(S_{t+1} - \hat{S}_{t+1|t})(S_{t+1} - \hat{S}_{t+1|t})']$$
$$= FP_{t|t}F' + Q.$$

5. To calculate the maximum likelihood estimation of parameters, the likelihood function can be constructed recursively

$$\log L(Y_T) = \sum_{t=1}^{T} \log f(y_t | I_{t-1})$$

where $f(y_t | I_{t-1}) = (2\pi)^{-0.5} |H'P_{t|t-1}H + R|^{-0.5} \times$

$$\exp\left\{-\frac{1}{2}(y_t - A - H\hat{S}_{t|t-1})'(H'P_{t|t-1}H + R)^{-1}(y_t - A - H\hat{S}_{t|t-1})\right\}$$

for $t = 1, 2, \ldots, T$.

Parameter estimates can then be calculated based on the numerical maximization of the likelihood function.

References

Aït-Sahalia, Y. 1996. "Testing Continuous Time Models of the Spot Interest Rate." *Review of Financial Studies* 9 (2): 385–426.

Backus, D. and S. Zin. 1994. "Reverse Engineering the Yield Curve." Salomon Brothers Working Paper 5-94-5.

Bank of Canada. 1991. "Targets for Reducing Inflation: Announcements and Background Material." *Bank of Canada Review* (March): 3.

Boothe, P. 1991. "Interest Parity, Cointegration, and the Term Structure in Canada and the United States." *Canadian Journal of Economics* 24 (August): 595–603.

Brown, S. and P. H. Dybvig. 1986. "The Empirical Implications of the Cox, Ingersoll, Ross Theory of the Term Structure of Interest Rates." *The Journal of Finance* 41 (3): 617–630.

Campbell, J. and R. Shiller. 1996. "A Scorecard for Indexed Government Debt." National Bureau of Economic Research Working Paper 5587.

Campbell, J., A. Lo, and C. MacKinlay. 1997. *The Econometrics of Financial Markets.* Princeton, New Jersey: Princeton University Press.

Cox, J., J. Ingersoll, and S. Ross. 1985. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53 (2): 385–407.

Cozier, B. and G. Tkacz. 1994. "The Term Structure and Real Activity in Canada." Bank of Canada Working Paper 94-3.

Day, J. and R. Lange. 1997. "The Structure of Interest Rates in Canada: Information Content about Medium-term Inflation." Bank of Canada Working Paper 97-10.

de Jong, F. 1997. "Time-series and Cross-section Information in Affine Term Structure Models." Tilburg University Centre for Economic Research Discussion Paper 9786.

Duffie, D. and R. Kan. 1996. "A Yield-Factor Model of Interest Rates." *Mathematical Finance* 6 (4): 379–406.

Duffie, D. and K. Singleton. 1997. "An Econometric Model of the Term Structure of Interest-Rate Swap Yields." *Journal of Finance* 52 (September): 1287–1321.

Engsted, T. 1995. "Does the Long-term Interest Rate Predict Future Inflation? A Multi-country Analysis." *Review of Economics and Statistics* 77 (1): 42–54.

Estrella, A. and G. Hardouvelis. 1991. "The Term Structure as a Predictor of Real Economic Activity." *Journal of Finance* 46 (2): 555–76.

Evans, M. and P. Wachtel. 1993. "Inflation Regimes and the Sources of Inflation Uncertainty." *Journal of Money, Credit, and Banking* 25 (3), Part 2: 475–511.

Fama, E. and M. Gibbons. 1992. "Inflation, Real Returns and Capital Investment." *Journal of Monetary Economics* 9 (3): 297–323.

Fung, B. and G. Tkacz. 1997. "A One-Factor General Equilibrium Model of the Canadian Term Structure." Unpublished. Bank of Canada.

Gong, F. and E. Remolona. 1997a. "Two Factors along the Yield Curve." *The Manchester School Supplement*: 1–31.

———. 1997b. "Inflation Risk in the United States Yield Curve: The Usefulness of Indexed Bonds." Working Paper 9637. Federal Reserve Bank of New York.

Gong, F. and E. Remolona. 1997c. “A Three-Factor Econometric Model of the United States Term Structure.” Federal Reserve Bank of New York Staff Report 19.

Hamilton, J. 1994. *Time Series Analysis*. Princeton: Princeton University Press.

Jeffrey, A. 1997. “An Empirical Examination of the Path-Dependent Term Structure Model.” Mimeograph. Yale School of Business.

Jegadeesh, N. and G. Pennacchi. 1996. “The Behavior of Interest Rates Implied by the Term Structure of Eurodollar Futures.” *Journal of Money, Credit, and Banking* 28 (3): 426–46.

Longstaff, F. and E. Schwartz. 1992. “Interest Rate Volatility and the Term Structure: A Two-factor General Equilibrium Model.” *The Journal of Finance* 47 (September): 1259–82.

Lund, J. and Anderson, T. 1997. “Estimating Continuous Time Stochastic Volatility Models of the Short-Term Interest Rate. *Journal of Econometrics* 77: 343–77.

Macklem, T. 1995. “The Yield Spread, the Transmission Mechanism and Monetary Stance.” Bank of Canada. Mimeograph.

McCulloch, J. and H. Kwon. 1993. “U.S. Term Structure Data, 1947–1991.” Ohio State University Working Paper 93-6.

Mishkin, F. 1990. “What Does the Term Structure Tell Us About Future Inflation?” *Journal of Monetary Economics* 25 (January): 77–95.

Obstfeld, M. 1995. “International Capital Mobility in the 1990s.” In *Understanding Interdependence: The Macroeconomics of the Open Economy*, edited by P. Kenen. Princeton, New Jersey: Princeton University Press.

Rudebusch, G. 1995. “Federal Reserve Interest Rate Targeting, Rational Expectations, and the Term Structure.” *Journal of Monetary Economics* 35 (April): 245–74.

Vasicek, O. 1977. “An Equilibrium Characterization of the Term Structure.” *Journal of Financial Economics* 5 (2): 177–88.