The Zero Lower Bound on Interest Rates and Monetary Policy in Canada

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Introduction

This paper constructs a limited-dependent rational-expectations (LD-RE) model to examine the time-series implications of the non-negativity constraint on interest rates for the term structure and monetary policy in Canada. Nominal interest rates are bounded below by zero because agents would rather hoard the currency than lend it at a loss (Fisher 1896). Wolman (1999) and McCallum (2000) formalize this idea in terms of familiar optimization models, where money enters the utility function or reduces the time and/or cost involved in making transactions. In this case, the interest rate is strictly positive if the marginal benefit of holding real money balances is strictly positive, and can be zero only if there is a quantity beyond which additional real money balances provide no extra services. Hence, the values that the nominal interest rate can take are limited to the interval $[0, \infty)$.

This paper models econometrically the lower bound on interest rates by treating the short-term interest rate as a limited-dependent variable and then derives the time-series implications of this bound for long-term interest rates under the pure expectations hypothesis of the term structure. Limited dependency is a device that forces agents to consider explicitly the zero lower bound when constructing their forecasts, even if all observations of the nominal interest rate to date are strictly positive.[1] Closed-form analytical results are obtained for the simpler case of a two-period bond and normally distributed disturbances. Numerical results are obtained for longer maturities under more general distributional assumptions by using a frequency simulator to compute the forecasts of the non-linear model.

The main implications of the non-negativity constraint are the following. First, the zero lower bound induces a non-linear and convex relation between the long-term interest rate and the level and standard deviation of the short-term interest rate. Second, the response of long-term interest rates to changes in the short-term rate is asymmetric. A decrease in the short-term rate produces a smaller response (in absolute value) in the long-term rate than does an increase of the same magnitude. Third, the response of long-term rates to changes in the short-term rate (whether an increase or a decrease) is smaller in the neighbourhood of the zero lower bound, especially for longer maturities. All of these results—coupled with the observation that when the short-term interest rate is low, the scope for further interest rate cuts is limited by the zero lower bound—imply that the power of monetary policy to affect long-term interest rates through the term structure is considerably reduced at low interest rates. The magnitude of the effects just described diminishes as the short-term interest rate rises above zero, and it is negligible when the short-term rate is at a safe distance from the non-negativity constraint.

Previous research by Ruge-Murcia (2002) examines Japanese data and finds non-linear and asymmetric effects in line with the LD-RE model. In addition, the non-linear LD-RE model delivers smaller forecast errors than a benchmark linear model, both in sample and out of sample. Since nominal interest rates are at historically low levels in Canada, it is of practical interest to examine whether the implications of the zero lower bound outlined above are also empirically relevant for the recent Canadian experience. However, as we will see, at current short-term interest rates, the predictions of the LD-RE model coincide with those of the linear forecasting model that ignores the effect of the zero lower bound on expectations. This observation supports the conclusion that although Canadian interest rates are low by historical standards, they are sufficiently high above the zero lower bound that the predictions of the LD-RE model are not verified.

1. LD-RE models have been used in previous work by Shonkwiler and Maddala (1985) and Holt and Johnson (1989) to study the determination of commodity prices in price-support schemes, by Baxter (1990) to study adjustable-peg exchange rate regimes, and by Pesaran and Samiei (1992, 1995) and Pesaran and Ruge-Murcia (1999) to study exchange rates subject to two-sided limits.

The paper is organized as follows. Section 1 introduces a simple time-series process for the one-period bond that describes how interest rates are bounded below by zero, derives the implications of the non-linear model for a two-period bond when shocks are normally distributed, and outlines a frequency simulator to compute the conditional expectations of the non-linear process in more general cases. Section 2 examines the empirical predictions of the model using data from Canada, and the final section concludes.

1 The LD-RE Model of the Term Structure

This section presents a time-series model for the one-period nominal interest rate that captures the idea that nominal interest rates are bounded below by zero, and then derives the implications of the zero lower bound for longer-term maturities using the pure expectations hypothesis of the term structure.[2]

The model for the one-period nominal interest rate is based on Fischer Black's interpretation of currency and interest rates as options (Black 1995). Black argues that currency is an option in the sense that were the bond return negative, agents could hold currency instead. This means that the observed nominal interest rate, r_t, may be interpreted as an option on r_t^* with a strike price of zero, where r_t^* is what the interest rate would be in the absence of the currency option. The latter is the "shadow" interest rate and may be positive or negative. The observed and shadow interest rates are related by

$$r_t = max(r_t^*, 0), \tag{1}$$

where r_t and r_t^* are the one-period observed and shadow nominal interest rates, respectively. Equation (1) can be written as

$$r_t = \begin{cases} r_t^*, & if \quad r_t^* > 0 \\ 0, & otherwise \end{cases} \tag{2}$$

2. This section draws on Ruge-Murcia (2002).

that corresponds to the familiar formulation of a limited-dependent variable censored at zero, with r_t^* the associated latent variable.[3]

By the pure expectations hypothesis of the term structure of interest rates, the nominal return on an n-period zero-coupon bond must equal the average expected return on the sequence of n one-period bonds held over its lifetime,

$$R_t^{(n)} = (1/n)[r_t + E(r_{t+1}|I_t) + \ldots + E(r_{t+n-1}|I_t)] + \theta_t, \qquad (3)$$

where $R_t^{(n)}$ is the nominal return on the n-period bond, I_t is the non-decreasing set of information available to market participants at time t and is assumed to include observations of the variables up to and including period t, $E(r_{t+s}|I_t)$ is the conditional expectation of the nominal return on the one-period bond acquired at time $t+s$ for $s = 1, \ldots, n-1$, and θ_t is a serially uncorrelated stochastic term that includes a liquidity premium and has variance σ_θ^2.

To give empirical content to the theory, let us specify the following process for the shadow nominal interest rate:

$$r_t^* = \alpha + \psi(L)r_t + \beta x_t + \varepsilon_t, \qquad (4)$$

where α is a constant intercept, L is the lag operator, $\psi(L)$ represents the polynomial

$$\sum_{j=1}^{p} \psi_j L^j,$$

β is a $1 \times m$ vector of parameters, x_t is an $m \times 1$ vector of explanatory variables, and ε_t is a disturbance term with zero mean and variance σ_ε^2, serially uncorrelated, and uncorrelated with θ_t. Wolman (1999) considers a deterministic version of equations (2)–(4), where r_t^* is the central bank's desired short-term nominal interest rate and arises from a Taylor-type policy rule.

The explanatory variables in x_t may be generated by the linear stochastic process

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3. At least two models in the literature also address explicitly the non-negativity constraint on nominal interest rates. Cox, Ingersoll, and Ross (1985) construct a continuous-time model where the volatility of the short-term interest rate is proportional to the square root of its level. Other authors specify the process of the short-term interest rate in logarithms. In this case, the log function imposes directly the non-negativity constraint.

$$x_t = AH_{t-1} + w_{1,t}, \tag{5}$$

where A is an $m \times b$ matrix of coefficients, H_t is a $b \times 1$ vector of predetermined variables possibly including past values of x_t, and $w_{1,t}$ is an $m \times 1$ vector of random disturbances assumed independently and identically distributed (i.i.d.) $(0, \Omega^{1/2})$ and uncorrelated with θ_t and ε_t.

The following proposition derives the conditional expectations of the short-term interest rate when r_t is subject to non-negativity.

Proposition 1. Assume that the short-term interest rate follows the limited-dependent process (equation 2) where r_t^* is determined according to equation (4). Assume that the explanatory variables, x_t, follow the process (equation 5). Define the composite error term

$$u_{s,t+s} = \varepsilon_{t+s} + \sum_{k=1}^{s-1} \psi_{s-k}\mu_{k,t+k} + \beta w_{s,t+s}, \tag{6}$$

where $\mu_{k,t+k} = r_{t+k} - E(r_{t+k}|I_t)$ and $w_{s,t+s} = x_{t+s} - E(x_{t+s}|I_t)$ with cumulative distribution and density functions denoted by $F_s(\cdot)$ and $f_s(\cdot)$, respectively. Define the variable

$$c_{t+s} = -E(r^*_{t+s}|I_t), \tag{7}$$

where

$$E(r^*_{t+s}|I_t) = \alpha + \sum_{k=1}^{min\{p, s-1\}} \psi_k E(r_{t+s-k}|I_t)$$

$$+ \sum_{j=s}^{p} \psi_j L^j r_{t+s} + \beta E(x_{t+s}|I_t). \tag{8}$$

Then, the conditional expectation of the short-term nominal interest rate at time $t+s$ constructed at time t given by

$$E(r_{t+s}|I_t) = [E(r^*_{t+s}|I_t) + E(u_{s,t+s}|I_t, u_{s,t+s} > c_{t+s})]$$

$$[1 - F_s(c_{t+s})]. \tag{9}$$

Proof. Use the definitions of $u_{s,t+s}$ and c_{t+s} to write the process of the short-term rate at time $t+s$ as

$$r_{t+s} = \begin{cases} E(r^*_{t+s}|I_t) + u_{s,t+s}, & if \quad u_{s,t+s} > c_{t+s} \\ 0, & otherwise. \end{cases}$$

Then, the conditional expectation of r_{t+s} is the weighted average

$$E(r_{t+s}|I_t) = E(r_{t+s}|I_t, u_{s,t+s} > c_{t+s})Pr(u_{s,t+s} > c_{t+s}),$$

$$+ E(r_{t+s}|I_t, u_{s,t+s} \le c_{t+s})Pr(u_{s,t+s} \le c_{t+s}). \qquad (10)$$

Note that $E(r_{t+s}|I_t, u_{s,t+s} \le c_{t+s}) = 0$. Since the forecast $E(r^*_{t+s}|I_t)$ is known at time t, $E(r_{t+s}|I_t, u_{s,t+s} > c_{t+s}) = E(r^*_{t+s}|I_t) + E(u_{s,t+s}| I_t, u_{s,t+s} > c_{t+s})$. Plugging these intermediate results into equation (10), and using $Pr(u_{s,t+s} > c_{t+s}) = 1 - F_s(c_{t+s})$, the conditional expectation of the short-term rate at time $t+s$ is

$$E(r_{t+s}|I_t) = [E(r^*_{t+s}|I_t) + E(u_{s,t+s}|I_t, u_{s,t+s} > c_{t+s})]$$

$$[1 - F_s(c_{t+s})],$$

as claimed.

Although equation (9) is a mathematical description of $E(r_{t+s}|I_t)$ when r_t is subject to the non-negativity constraint, the expression is not operational because it is not clear how to compute $E(u_{s,t+s}|I_t, u_{s,t+s} > c_{t+s})$ and $F_s(c_{t+s})$ in the general case. Note in equation (6) that for horizons $s > 1$, $u_{s,t+s}$ includes interest-rate forecast errors. Due to the non-linear nature of r_t, these forecast errors do not follow a standard distribution. Unreported simulations indicate that at low interest rates, the density of the forecast errors depends on the level of the short-term interest rate, the forecast horizon, and the model parameters. Thus, in general, it is not possible to write analytically the probability density function of $u_{s,t+s}$, or closed-form expressions for the terms $E(u_{s,t+s}|I_t, u_{s,t+s} > c_{t+s})$ and $F_s(c_{t+s})$. In turn, this means that $E(r_{t+s}|I_t)$ does not have a closed form and it is not possible to derive general analytical results.

To address this difficulty, this paper follows a two-pronged approach. First, it focuses on the special case of a two-period bond with normally distributed shocks. For this case, it is possible to write a closed-form expression linking the short- and long-term interest rates and derive analytically the implications of the zero lower bound. Second, the paper uses the simulation procedure proposed in Ruge-Murcia (2002) to compute numerically the conditional forecasts $E(r_{t+s}|I_t)$ and examine empirically the Canadian term structure.

1.1 A special case

Consider the special case where the long-term bond is a two-period bond. The term-structure relation (equation 3) for the case $n = 2$ is

$$R_t^{(2)} = (1/2)[r_t + E(r_{t+1}|I_t)] + \theta_t, \tag{11}$$

where $R_t^{(2)}$ denotes the two-period bond return and the rest of the notation is as previously defined. In addition, specialize equation (4) to

$$r_t^* = \alpha + \psi r_{t-1} + \varepsilon_t, \tag{12}$$

where α is a non-negative intercept, $\psi > 0$, and the disturbance term, ε_t, is assumed to be serially uncorrelated and normally distributed with zero mean and variance σ_ε^2.

Although this specification is restrictive, it delivers the following closed-form expression for the conditional expectation, $E(r_{t+1}|I_t)$.

Proposition 2. Assume that the short-term interest rate follows the limited-dependent process (equation 2) where r_t^* is determined according to equation (12). Define the variable $c_{t+1} = -E(r_{t+1}^*|I_t)/\sigma_\varepsilon = -(\alpha + \psi r_t)/\sigma_\varepsilon$. Then, the conditional expectation of the short-term nominal interest rate at time $t+1$ constructed at time t is given by

$$E(r_{t+1}|I_t) = (\alpha + \psi r_t)(1 - \Phi(c_{t+1})) + \sigma_\varepsilon \phi(c_{t+1}), \tag{13}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cumulative and density functions of a standard normal variable, respectively.

Proof. Define the standardized normal variable $\xi_t = \varepsilon_t/\sigma_\varepsilon$ and use the definition of c_{t+1} to write the process of the short-term rate at time $t+1$ as

$$r_{t+1} = \begin{cases} E(r_{t+1}^*|I_t) + \varepsilon_{t+1}, & if \quad \xi_{t+1} > c_{t+1} \\ 0, & otherwise. \end{cases}$$

Write the conditional expectation of r_{t+1} as the weighted average

$$E(r_{t+1}|I_t) = E(r_{t+1}|I_t, \xi_{t+1} > c_{t+1})Pr(\xi_{t+1} > c_{t+1})$$
$$+ E(r_{t+1}|I_t, \xi_{t+1} \le c_{t+1})Pr(\xi_{t+1} \le c_{t+1}).$$

Note that $E(r_{t+1}|I_t, \xi_{t+1} \le c_{t+1}) = 0$. Since the forecast $E(r^*_{t+1}|I_t) = \alpha + \psi r_t$ is known at time t,

$$E(r_{t+1}|I_t, \xi_{t+1} > c_{t+1}) = \alpha + \psi r_t + E(\varepsilon_{t+1}|I_t, \xi_{t+1} > c_{t+1}).$$

Use results for censored normal variables (for example, Maddala 1983, 366) to write $E(\varepsilon_{t+1}|I_t, \xi_{t+1} > c_{t+1}) = \sigma_\varepsilon \phi(c_{t+1})/(1-\Phi(c_{t+1}))$, where $1-\Phi(c_{t+1})$ stands for $Pr(\xi_{t+1} > c_{t+1})$. With these intermediate results,

$$E(r_{t+1}|I_t) = (\alpha + \psi r_t)(1-\Phi(c_{t+1})) + \sigma_\varepsilon \phi(c_{t+1}),$$

as claimed.

Equipped with this closed form for $E(r_{t+1}|I_t)$, we can proceed to examine the implications of the zero lower bound for the term structure. Substitute equation (13) into equation (11) to obtain

$$\begin{aligned} R_t^{(2)} = {} & (1/2)r_t + (1/2)[(\alpha + \psi r_t)(1-\Phi(c_{t+1})) \\ & + \sigma_\varepsilon \phi(c_{t+1})] + \theta_t. \end{aligned} \tag{14}$$

First, note that the return on the two-period bond is related non-linearly to the one-period return. More precisely, the long-term interest rate is convex in r_t for any $\psi \ne 0$. The easiest way to see this is to take the first and second derivatives of $R_t^{(2)}$ with respect to r_t

$$\partial R_t^{(2)}/\partial r_t = 1/2 + (\psi/2)(1-\Phi(c_{t+1})),$$

$$\partial^2 R_t^{(2)}/\partial r_t^2 = (\psi^2/2\sigma_\varepsilon)\phi(c_{t+1}) > 0.$$

Note that $\partial^2 R_t^{(2)}/\partial r_t^2$ is non-zero only as a result of the second term in the right-hand side of equation (14). Since this term stands for $(1/2)E(r_{t+1}|I_t)$, it is clear that this non-linear effect is due solely to the effect of the zero lower bound on expectations.

Second, because the relation between $R_t^{(2)}$ and r_t is non-linear, changes in r_t produce asymmetric movements in the long-term rate. In particular, provided $\psi > 0$, a decrease in the short-term rate produces a smaller response (in absolute value) in the long-term rate than an increase of the same magnitude. That is,

$$\left|R_t^{(2)}(r_t - \Delta) - R_t^{(2)}(r_t)\right| < \left|R_t^{(2)}(r_t - \Delta) - R_t^{(2)}(r_t)\right|,$$

where Δ is the change in the short-term interest rate. To verify this claim, use $\partial R_t^{(2)} / \partial r_t > 0$ (that is satisfied for $\psi > 0$) and the definition of absolute value to write $-(R_t^{(2)}(r_t - \Delta) - R_t^{(2)}(r_t)) < R_t^{(2)}(r_t + \Delta) - R_t^{(2)}(r_t)$. Rearranging delivers $(R_t^{(2)}(r_t + \Delta) + R_t^{(2)}(r_t - \Delta))/2 > R_t^{(2)}(r_t)$, which holds because the function $R_t^{(2)}(r_t)$ is convex in its argument.

Third, given the current short-term interest rate, the non-linear model predicts that the long-term rate is an increasing and convex function of the conditional standard deviation of r_t. To see this, take the first and second derivative of $R_t^{(2)}$ with respect to σ_ε to obtain

$$\partial R_t^{(2)} / \partial \sigma_\varepsilon = \phi(c_{t+1})/2 > 0,$$

$$\partial^2 R_t^{(2)} / \partial \sigma_\varepsilon^2 = c_{t+1}^2 \phi(c_{t+1})/2\sigma_\varepsilon > 0.$$

Under Black's interpretation of interest rates as options, this is the result that option-pricing theory would predict.

Fourth, the response of long-term rates to changes in the short-term rate (whether an increase or a decrease) is smaller in the neighbourhood of the zero lower bound than the one predicted by the standard linear model. For the linear model that ignores the zero lower bound on interest rates, the counterpart of the process in equations (2) and (12) is $r_t = \alpha + \psi r_{t-1} + \varepsilon_t$. It is easy to prove that in this case, $E(r_{t+1} | I_t) = (\alpha + \psi r_t)$, $R_t^{(2)} = (1/2) r_t + (1/2)\ (\alpha + \psi r_t) + \theta_t$, and the derivative of $R_t^{(2)}$ with respect to r_t is $\partial R_t^{(2)} / \partial r_t = (1 + \psi)/2$. Recall that for the non-linear model, $\partial R_t^{(2)} / \partial r_t = 1/2 + (\psi/2)(1 - \Phi(c_{t+1}))$. Hence, this fourth implication is based on the observation that

$$1/2 + (\psi/2)(1 - \Phi(c_{t+1})) \leq (1 + \psi)/2.$$

At low interest rates, therefore, the impact of adjustments to the short-term rate on the long-term rate is dampened by the effect of the non-negativity constraint.

Notice that the effects just described disappear as the short-term interest rises well above zero. Then, c_{t+1} decreases and $\Phi(c_{t+1}) \to 0$. In this case, the standard linear model may be a good approximation of the time-series behaviour of interest rates. This observation holds not only for the special case in this section but also for the general case. Notice in equation (9) that as r_t rises, (c_{t+1}) decreases, $E(u_{s,t+s} | I_t, u_{s,t+s} > c_{t+s})$ converges to $E(u_{s,t+s} | I_t)$, and $F_s(c_{t+s}) \to 0$. Then, the conditional forecast $E(r_{t+s} | I_t)$ tends to the one obtained under the linear forecasting model that ignores (in this case, correctly) the effect of the zero lower bound on expectations.

A more subtle point concerns the notion of "distance" between the nominal interest rate and the zero lower bound. Note that the appropriate measure of distance involves a normalization by the (conditional) variance of the interest rate innovation. (See the definition of c_{t+1} in Proposition 2.) Hence, in considering whether the effects just described may be empirically relevant, it is not sufficient to focus only on the level of the current interest rate. This is especially true because of the empirical observation that the level and volatility of nominal interest rates are positively correlated. It is entirely possible, therefore, that one may find equally large non-linear effects for interest rates of, say, 1 and 0.1 per cent, because the conditional volatility is larger in the former case than in the latter.

1.2 Computation of the conditional expectations

For bonds with maturity longer than two periods, it is not possible to obtain a closed form for the conditional expectations. However, given a parametric process for the short-term interest rate, it is possible to compute numerically the conditional forecasts $E(r_{t+s}|I_t)$ by means of stochastic simulation. The procedure outlined below was proposed by Ruge-Murcia (2002), and it is basically an application of the frequency simulators by Lerman and Manski (1981) and McFadden (1989) to dynamic non-linear rational-expectations models.

The simulation procedure involves the following steps.

Step 1: Having found analytically or numerically (see below), the one-step-ahead conditional expectation of the nominal interest rate, $E(r_{t+1}|I_t)$, use the definitions (8) and (7) for $s = 2$ to obtain c_{t+2}.

Step 2: Simulate M observations of the short-term interest rate at time $t+1$ using equations (2) and (4). The non-negativity constraint can be enforced numerically by substituting a negative realization of r_{t+1} with zeros. Compute the M realizations of the forecast error, $\mu_{1,t+1} = r_{t+1} - E(r_{t+1}|I_t)$.

Step 3: Draw M realizations of ε_{t+2} and w_{t+2} and combine them with the μ's according to equation (6) to obtain M realizations of $u_{2,t+2}$.

Step 4: construct an estimate of $F_2(c_{t+2})$ as the proportion of observations of $u_{2,t+2}$ that are larger than c_{t+2}:

$$F_s(c_{t+s}) = (1/M)\sum_{j=1}^{M} \Im(u_{2,t+2} > c_{t+2}), \tag{15}$$

where $\Im(\cdot)$ is an indicator function that takes value 1 when its argument is true and 0 otherwise. Construct an estimate of $E(u_{2,t+2}|I_t, u_{2,t+2} > c_{t+2})$ by taking the arithmetic average of observations of $u_{2,t+2}$ that fall above c_{t+2}:

$$E(u_{s,t+s}|I_t, u_{s,t+s} > c_{t+s}) = (1/M)\sum_{j=1}^{M} u_{s,t+s}$$

$$\Im(u_{2,t+2} > c_{t+2}). \tag{16}$$

Step 5: Applying relation (9) for $s = 2$ delivers $E(r_{t+2}|I_t)$. Using $E(r_{t+2}|I_t)$, the procedure can then be repeated recursively for $s = 3, 4, \ldots, n-1$.

The one-step-ahead forecast, $E(r_{t+1}|I_t)$, required to start the recursion, can be found analytically in the special case where ε_t is normally distributed. More generally, $E(r_{t+1}|I_t)$ could be computed numerically using the same procedure above. In this case, the recursion would start with r_t rather than with $E(r_{t+1}|I_t)$, but one would omit Step 2. Step 2 constructs realizations of the forecast errors and it is not necessary in the case of $s = 1$ because $r_t - E(r_t|I_t) = 0$.[4]

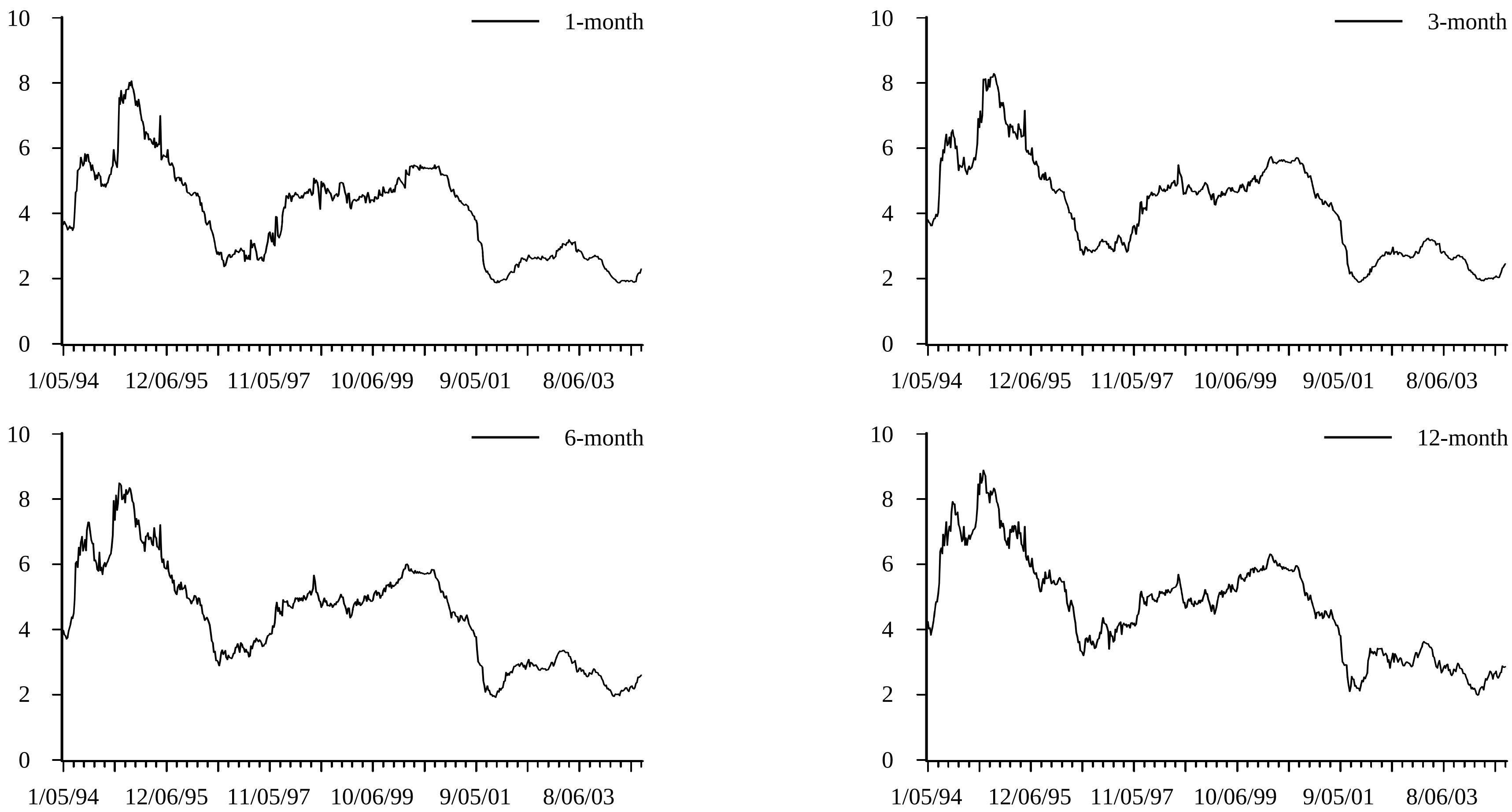
This simulator will be used to examine the implications of the zero lower bound for the Canadian term structure and the transmission of monetary policy via this channel.

2 The Canadian Term Structure

This section reports preliminary results of the analysis of the Canadian term structure using the limited-dependent-variable model proposed above. The data are 561 observations of weekly (Wednesday) observations of the one-, three-, six-, and twelve-month nominal interest rates on treasury bills between 5 January 1994 and 29 September 2004. The data source is the Bank of Canada website (www.bank-banque-canada.ca). The sample starts with the first observation available at the source and ends with the latest observation available at the time that the data were collected. All data series are plotted in Figure 1. Note that Canadian interest rates are safely above the

4. Ruge-Murcia (2002) also discusses the case where ε_t is conditionally heteroscedastic and presents a kernel-smoothed version of this frequency simulator.

Figure 1
Canadian nominal interest rates



zero lower bound for most of the sample, but since early 2002, they are at historically low levels. In the period 2, January 2002 to 29 September 2004, the average one-, three-, six-, and twelve-month interest rates are only 2.44, 2.63, 2.86, and 2.53, respectively. To the extent that monetary policy affects long-term interest rates through the term structure, it is of practical interest to study whether the implications of the zero lower bound outlined above are empirically relevant for the recent Canadian experience.

2.1 The short-term interest rate

The following empirical analysis takes the one-month interest rate as the short-term interest rate, r_t.[5] The process of r_t^* is described in terms of past realizations of the short-term interest rate with the conditional variance of ε_t modelled using an autoregressive conditional heteroscedasticity (ARCH(2)) specification. That is, $\varepsilon_t = \sqrt{h_t}\upsilon_t$, where υ_t is i.i.d. N(0,1) and $h_t = \zeta + \delta_1\varepsilon_{t-1}^2 + \delta_2\varepsilon_{t-2}^2$.[6] Thus, the estimated process is:

$$r_t = \begin{cases} r_t^*, & if \quad r_t^* > 0 \\ 0, & otherwise, \end{cases}$$

with

$$\underset{}{r_t^*} = \underset{(0.016)}{0.0068} + \underset{(0.077)}{0.977}r_{t-1} - \underset{(0.145)}{0.053}r_{t-2} + \underset{(0.086)}{0.074}r_{t-3} + \varepsilon_t,$$

and

$$h_t = \underset{(0.001)}{0.010} + \underset{(0.087)}{0.495}\varepsilon_{t-1}^2 + \underset{(0.090)}{0.397}\varepsilon_{t-2}^2.$$

To examine whether the parsimonious ARCH(2) model captures the volatility changes in the short-term interest rate, Lagrange Multiplier tests for neglected ARCH were applied to the standardized squared residuals of

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5. The overnight money market rate would be a more natural choice as a short-term interest rate, for two reasons. First, it is the shortest maturity available in the market. Second, it is directly under the control of the Bank of Canada. However, a complete model of the overnight interest rate should also incorporate the effect of the operating band on the expectations of market participants and the adoption of the Large Value Transfer System in February 1999. I plan to undertake this generalization of the model in future work.

6. In preliminary work, I also considered using a GARCH(1,1) model for the conditional variance, but the results are very similar to those reported here.

the estimated model. If the ARCH model is correctly specified, then the residuals corrected for heteroscedasticity and squared should be serially uncorrelated. Under the null hypothesis of no autocorrelation, the test statistic is distributed chi-square with degrees of freedom equal to the number of autocorrelations tested for. The statistics for up to five autocorrelations are 0.67, 1.48, 1.71, 1.85, and 3.66, respectively. Since all statistics are below the 5 per cent critical value of their appropriate distributions, the null hypothesis cannot be rejected at the 5 per cent level. These results suggest that an ARCH(2) process adequately captures the conditional heteroscedasticity in the Canadian one-month interest rate.

2.2 Predictions for long-term interest rates

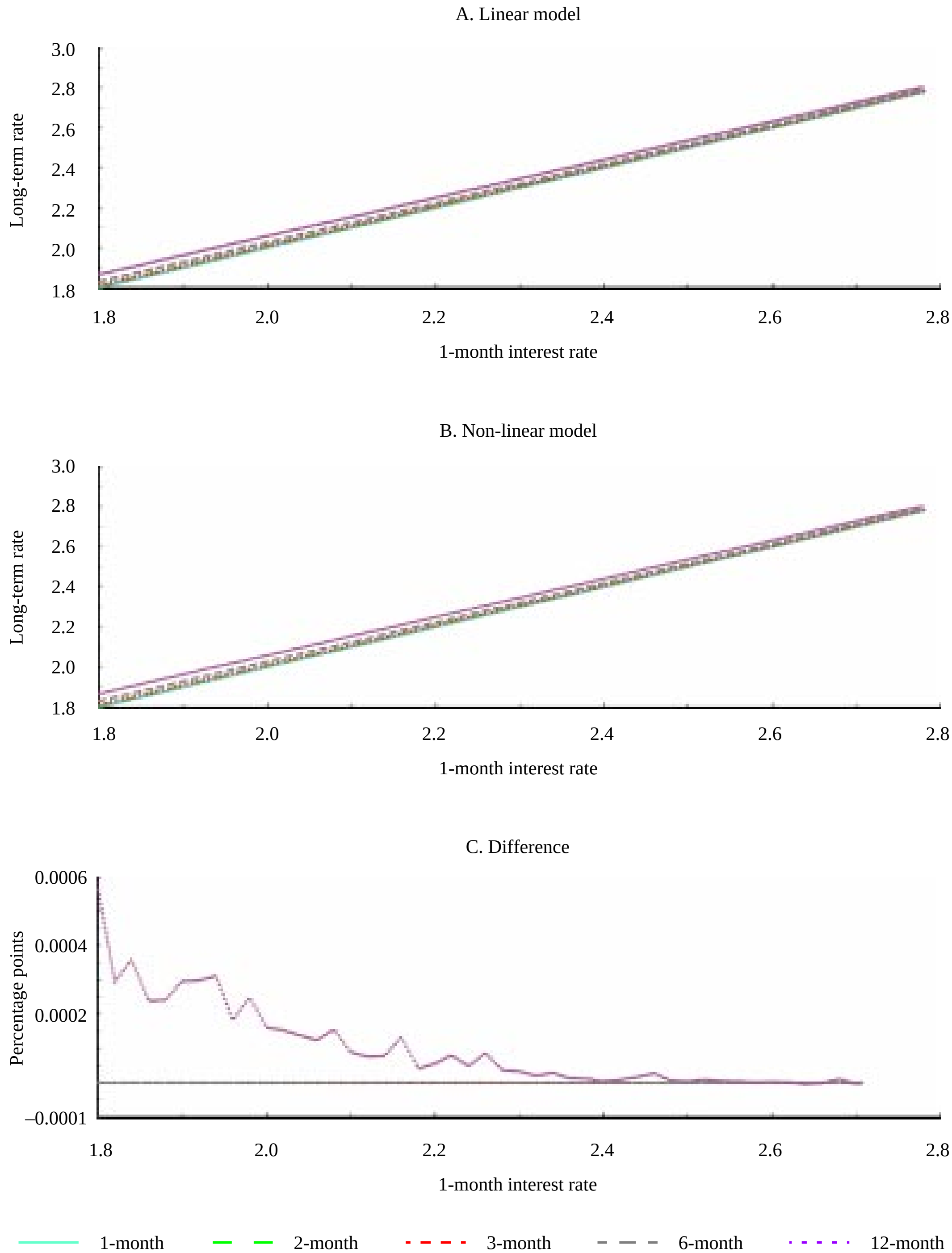
This section derives the implications of the model for the Canadian term structure, taking as given the estimated process for the short-term interest rate. The focus is on predictions regarding the level of the long-term interest rate and the response of the long-term interest rate to changes in the short-term rate. Predictions are derived under the non-linear model that takes into account—and the linear model that ignores—the effect of the zero lower bound on expectations. In the case of the former, the conditional expectations of r_t are computed using the frequency simulator proposed in section 1.2, and the conditional variance of ε_t is set to its sample median.[7]

In interpreting the results and conclusions of this paper, it is useful to remember that the predictions of the linear and non-linear models coincide when the current short-term interest rate is sufficiently high above the zero lower bound. Hence, comparing the predictions of both models sheds light on whether the non-linear effects implied by the LD-RE model are likely to be empirically relevant at the current Canadian short-term interest rates. If both models generate the same predictions, then incorporating the effect of the zero lower bound on expectations does not change the predictions of the linear forecasting model of the term structure. One must conclude that in this case the current short-term interest rates are sufficiently far from the non-negativity constraint on interest rates.

Figure 2 plots the two-, three-, six-, and twelve-month interest rates predicted by the linear and non-linear models in panels A and B,

7. In preliminary work, I also considered other values for the conditional variance of the innovation. Except in the cases where the conditional variance of ε_t was implausibly high, results are similar to the ones reported here.

Figure 2
Predicted long-term interest rate



respectively, and their difference in panel C.[8] The range of the one-month interest rate in this figure corresponds roughly to that observed in Canada in the past four years. Recall that, in the neighbourhood of the zero lower bound, the non-linear model predicts that long-term rates are non-linear and convex in the current short-term rate, and higher than predicted by the linear forecasting model. Three observations follow from Figure 2. First, the relation between the predicted long-term rates and the current short-term rates is well approximated by a straight line. Second, the two-, three-, and six-month interest rates predicted by both models are identical. Third, the non-linear model predicts a higher twelve-month interest rate than the linear model, but the difference is quantitatively small.[9] In particular, the difference between the non-linear and linear models is only 0.04 basis points when the current short-term interest rate is 1.8 per cent per year and drops rapidly with r_t.

Another dimension in which both forecasting models are similar is in the predicted response of the long-term interest rates to a change in the short-term interest rate. Figure 3 plots the responses of the three-, six-, and twelve-month interest rates to an increase and to a decrease of 25 basis points in the short-term rate under the linear (dotted line) and non-linear (continuous line) models. In constructing these responses, the current short-term interest rate is set to 2 per cent. Recall that for linear models of the term structure, an innovation to the short-term rate yields movements in the long-term rate that are symmetric, proportional, and history-independent. That is, the impulse response associated with a shock of size 1 (standard deviation) would be the mirror image of the response to a shock of size –1, one-half the response of shock size 2, and independent of the moment the shock is assumed to take place. In contrast, under the non-linear LD-RE model, the response of the long-term rate to an innovation in the short-term rate is asymmetric. This reflects the more general proposition that in non-linear systems, impulse responses can vary with the size and sign of the shock and the initial conditions (see Koop, Pesaran, and Potter 1996). From Figure 3, however, it is clear that responses predicted by both models are similar enough to be indistinguishable in the plots. Using other plausible values of the current short-term interest rate and the conditional variance of the innovation does not change this result.

8. Since the one-month rate is forecasted using weekly (rather than monthly) realizations of the variable, I construct the long-term rates by selecting the forecast of r_t at the horizons closest to the date the one-month bond would have been rolled over. For example, for the three-month rate, I use the four-week-ahead and nine-week-ahead forecasts. Using interpolation yields the same results as reported, but it is computationally more burdensome.

9. The difference between the models "wiggles" and appears to be non-monotonic, because the interest rate predicted by the non-linear model is obtained using simulation.

Figure 3
Response of the long-term rate to a change in the short-term rate

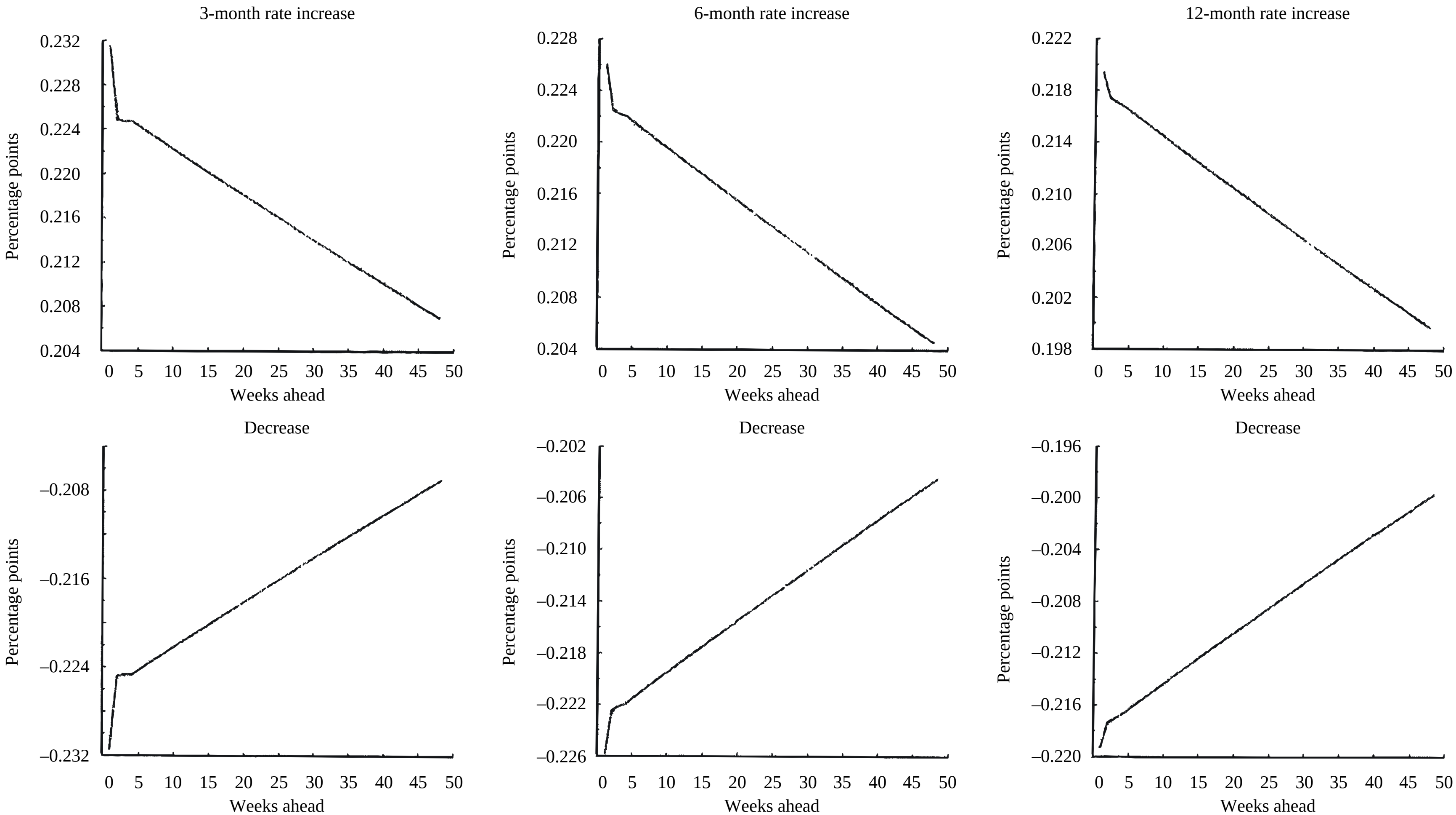


Table 1
Comparison of linear and non-linear forecasting models

Maturity	Model		
	Linear	Non-linear I	Non-linear II
A. In-sample RMSE			
3 months	55.073	55.067	55.069
6 months	88.058	88.007	87.917
12 months	32.477	32.179	32.172
B. In-sample MAE			
3 months	41.510	41.506	41.503
6 months	69.119	69.072	68.935
12 months	23.568	23.364	23.404
C. Out-of-sample RMSE			
3 months	21.661	21.660	21.568
6 months	45.265	45.075	44.169
12 months	15.468	14.728	14.435
D. Out-of-sample MAE			
3 months	17.314	17.314	17.249
6 months	35.123	34.993	34.418
12 months	13.699	12.420	11.846

Notes: This table reports RMSEs and MAEs in basis points. The linear model forecasts the one-month interest rate linearly. The non-linear model I takes into account the effect of the zero lower bound and the conditional standard deviation of ε_t. The non-linear model II takes into account the effect of the zero lower bound but fixes the conditional standard deviation of ε_t to its unconditional mean.

These results compare the predictions of both models for specific values of the current level and conditional variance of the short-term interest rate. A more complete strategy to compare empirically the relative merits of the linear and non-linear models is to compute forecast-error statistics. To that effect, I constructed the in-sample and the one-step-ahead, out-of-sample root-mean-squared error (RMSE) and mean absolute error (MAE) for the models. The out-of-sample measures are computed for the last 50 observations in the sample by recursively estimating the model and constructing the forecasts. The linear model constructs the long-term interest rate using linear forecasts of the one-month interest rate. The non-linear models take into account the effect of the zero lower bound on expectations, but differ in their treatment of the conditional standard deviation of ε_t. The non-linear model I computes the forecasts of the one-month interest rates using the ARCH(2) estimates of the conditional standard deviation of ε_t. The non-linear model II fixes the conditional standard deviation of ε_t to its unconditional mean. Statistics are reported in Table 1.

Notice that the non-linear models deliver smaller in-sample and out-of-sample RMSEs and MAEs than the linear model for all maturities. However, the gains from using the non-linear forecasting model are extremely small in all cases, though they tend to be larger in the longer-term maturities. These results are in line with results that indicate only small differences between the linear and non-linear models at the current short-term interest rates.

All of these results suggest that at the current interest rate levels in Canada, the non-linear and linear models yield roughly the same predictions regarding the long-term interest rate and fit the data equally well. Since both models coincide only when interest rates are sufficiently high above the zero lower bound, we must conclude that the predictions of the non-linear model do not appear to be verified in the data because they are still safely above the non-negativity constraint, in spite of the fact that Canadian interest rates are at historically low levels.

Conclusion

This paper was motivated by the observation that Canadian interest rates are at historically low levels. To the extent that monetary policy affects long-term interest rates through the term structure, it is useful to determine whether the implications of the zero lower bound outlined in Ruge-Murcia (2002) are empirically relevant for the recent Canadian experience. The results reported here indicate that the linear and non-linear models generate basically the same predictions and, consequently, there is no significant difference in forecasting power. To understand this finding, it is helpful to remember that the predictions of both models coincide only when the current short-term interest rate is sufficiently high above the zero lower bound. Since incorporating the effect of the zero lower bound on expectations does not change the predictions of the linear forecasting model, one must conclude that current Canadian short-term interest rates are sufficiently high above the non-negativity constraint that the predictions of the LD-RE model are not verified.

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