Endogenous Value and Financial Fragility

*Karine Gobert, Patrick González, Alexandra Lai, and Michel Poitevin**

Introduction

A system is financially fragile relative to another when its expected value in the steady state is lower owing to an inability to manage liquidity in a manner that is dynamically efficient. We show that a decentralized mechnism for allocating liquidity is more fragile than a centralized system, because of a divergence between social and market values of firms when there is a potential for aggregate liquidity constraints to bind in any period. A market mechanism is unable to correctly value firms in terms of their ability to provide liquidity in the future, and hence can allow a firm to go bankrupt even though it would be socially valuable to refinance it. This is because the potential to be a liquidity supplier increases the value of other firms, but this externality is not accounted for in the market value of firms.

Correctly valuing a firm (or a project) is a central issue in finance. The value of a firm is typically equal to the expected discounted value of its future benefits, conditioned on its survival. In the autarkic case where no refinancing is available, the firm will eventually go bankrupt when there is a positive probability of distress, and the computation of its value takes this probability into account. The probability of bankruptcy enters into the

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"effective" discount rate. The difficulty in the computation of the value arises when refinancing is potentially available but subject to an endogenous liquidity constraint. In a dynamic context, the flow of future benefits in the firm is conditioned by the possibility of financial distress and the firm's ability to obtain refinancing in future periods, should it become distressed. Bankruptcy is then endogenous to current and future refinancing possibilities, and the computation of the firm's value becomes a non-trivial exercise.

In an environment of perfect financial markets, there are no liquidity constraints facing the firm as long as its value, net of its liquidity needs, remains positive. Firms are easy to value in this world, and bankruptcy, when it occurs, is efficient. We present a model of firm valuation when financial markets are imperfect. We focus on a limited aggregate supply of liquidity as a source of market imperfection. A firm may not be able to obtain financing even though it would be profitable to do so because the aggregate supply of liquidity is bounded. This assumption can limit the extent of refinancing a firm can obtain and affect its current value. In addition, current and potential liquidity constraints create a divergence in a firm's social and market value, which causes a decentralized market for liquidity to be dynamically inefficient or financially fragile.

In this model, a firm is identified with an infinite random sequence of benefits, conditional on its survival. In each period, a firm realizes a net benefit. For example, this benefit represents its cash flow consisting of revenues minus costs net of any new investment requirement. If this benefit is below a threshold level (normalized to zero), the firm is in financial distress and needs refinancing to pursue its activities. Without refinancing, it must declare bankruptcy. If this benefit is positive, the firm can choose to either consume its benefit or use it to refinance a distressed firm.

We develop a procedure for valuing firms when there is a potential aggregate shortage of liquidity. We suppose that there is no deep-pocket financier that could refinance all firms whenever it is optimal to do so. Instead, we have a finite number of firms that can provide financing to one another when they have sufficient liquidity. As long as the value of a firm is greater than its liquidity needs, it is optimal to refinance it. This may not be possible, however, if the other firms do not generate enough liquidity. A firm may become financially vulnerable because the aggregate supply of liquidity in the economy is low, and not because its net value falls below zero.

Within this context, we study two environments. In the first, we assume that all firms are part of a coalition in which financing decisions are centralized to maximize the value of this coalition. In each period, the set of surviving firms is chosen to maximize the future value of the coalition of surviving firms. If there is an aggregate liquidity constraint, some firms cannot be

refinanced and must be shut down. The decision about which firms should survive in this case depends on the marginal contribution of firms to the future value of the coalition. This contribution depends on the ability of a given firm to "rescue" other firms in the future. We compute a two-firm example to illustrate our results.

In the second environment, we assume that, instead of a centralized decision-making mechanism, there exists a market for liquidity, and distressed firms must borrow on this market at the equilibrium rate of interest. For each period, we characterize the equilibrium interest rate that determines which firms are refinanced. These firms have the highest market value net of refinancing costs.

We then compare the efficiency of the two mechanisms. For each case, we show that the economy converges to a stable coalition of firms, a set in which no bankruptcies can occur. This limit set may be history-dependent. More interestingly, we show that the two mechanisms can produce different sets of stable coalitions. *Any stable coalition in a decentralized market is also stable in the centralized mechanism, but the converse is not true*. In a decentralized market, firms with the highest market value net of refinancing costs are refinanced. This value, however, does not include the impact that the firm may have on the future refinancing possibilities of other firms. When there is an aggregate liquidity constraint that may bind in some future period, each firm has a shadow value that depends on its potential for rescuing other firms in that period; that is, each firm has an externality on the value of other firms.[1] The market for liquidity cannot take this externality into account, but a centralized mechanism can. For example, suppose that firm A has a higher net market value than firm B today, but that firm B is more likely to "rescue" firm C from bankruptcy in the future (maybe because its returns are negatively correlated with those of firm C). Suppose there is an aggregate liquidity constraint that prevents the refinancing of firms A and B. A central planner may prefer to rescue firm B over firm A if this increases the value of firm C sufficiently. However, a decentralized market does consider this externality when computing the value of firms. In this sense, the market is not dynamically efficient, and that is why it is more fragile than a centralized mechanism. We use a simple numerical example to show how the market may fail to correctly compute the true value of firms, while a centralized coalitional organization would perform efficiently.

The issue of endogenous bankruptcy has already been studied in the literature on optimal capital structure. Using a no-arbitrage argument, Merton (1974) computes the value of a firm's equity when its benefits follow

1. This externality vanishes when there is no aggregate liquidity constraint.

a diffusion-type stochastic process. Merton assumes that the firm issues a zero-coupon bond with maturity at time *T*. If the value of assets is less than the face value of debt at *T*, the firm is bankrupt and the equity is worth zero. This makes the equity value resemble a European call option, which is valued using Black and Scholes' (1973) formula. Merton's formula per se does not consider bankruptcy as an endogenous event. It can be used, however, to price any claim on a firm whose benefits are described by a diffusion process.

Leland (1994) considers a more complex type of debt with a continuous coupon, and computes the equity value when bankruptcy is either exogenous or endogenous. Bankruptcy is exogenous when it is triggered by the assets' value falling below a predetermined exogenous target level. Bankruptcy is endogenous when it is triggered by the impossibility to pay the coupon by issuing additional equity. In this case, there is a minimum value V_B of the firm's assets below which equity is worth zero and the firm is bankrupt. The firm chooses this lower bound to maximize the total value of the firm. On the one hand, the lower bound V_B must be low enough to minimize the occurrence of bankruptcy; on the other hand, it cannot be too low, since equity must remain positive for a value of assets above the bound. Leland finds that the lower bound V_B on the value of assets that triggers bankruptcy is proportional to the debt coupon, independent of the current value of assets, increasing in the risk-free rate of interest and decreasing in the volatility of the assets' value process. Leland assumes that the firm can always refinance on the market as long as its equity value is positive. This translates into an environment of perfect financial markets. In this model, bankruptcy is said to be efficient.

In their paper, den Haan, Ramey, and Watson (1999) also study the fragility of an economic system in which there is an aggregate liquidity constraint. Borrowers and lenders are matched and, in each period, lenders receive a random liquidity endowment. The realized endowment affects the viability of a match. The main difference of this paper from our approach is that the authors assume there is no short-run market for liquidity. Assuming that liquidities can flow across agents is a main feature of our analysis. We show that an economy may still be fragile despite having a short-run competitive market for liquidities.

In section 1, we introduce the model and notation. We then compute the value of a firm in two benchmark cases: in autarky and when there is a deep-pocket financier who supplies liquidity in each period. In the following sections, we assume that the aggregate supply of liquidity is finite and given by the cash-flow realizations of all firms in the economy. In section 2, we develop our centralized coalitional model and illustrate our results with a

two-firm example. In section 3, we assume a decentralized market for liquidity in each period and characterize the market equilibrium. In section 4, we compare the efficiency of the two mechanisms and illustrate our results with an example. The conclusions follow.

1 The Model

Consider a multi-period, single-good economy where all consumers have (risk-neutral) linear preferences with respect to random consumption paths. They discount future consumption by a common factor δ. Consumers are assumed to have rational expectations; that is, they perfectly anticipate future prices contingent on available information and coordinate on the same equilibrium if many equilibria can exist.

There is an infinite random sequence of i.i.d. states $(s_t)_{t \in \mathrm{N}}$ where t is a time subscript. Each state s_n is drawn from $(S, \mathcal{S}, \mu)$, where S is a compact set of states, $\mathcal{S}$ is a σ-algebra on S, and μ is a probability measure. In what follows, the time subscript is dropped whenever this does not create any confusion. Hence, s usually refers to the current state.

There are many productive projects, owned by the consumers.[2] The number of projects can decrease over time with the occurrence of bankruptcy. We forbid the entry of new projects, however.[3] In each period, projects generate random benefits measured in units of the consumption good. A project is described by a measurable continuous function $y : S \to \mathrm{R}$, which relates each state, s, to the random benefit the project generates in that state, $y(s)$.

A negative benefit generated by a project represents a temporary shortage of liquidity that prevents the project from investing in its technology to continue to create value. A negative benefit that is not refinanced results in the bankruptcy of the project. We assume limited liability so that if a project has a negative benefit and declares bankruptcy, it forgoes its financial liabilities. A bankrupt project can never be reactivated so that if it goes bankrupt in period t, it brings a benefit of zero in period t and all subsequent periods. A positive benefit, on the other hand, creates excess liquidities that can be used to refinance other projects or be consumed by the owners of the project. There is no storage technology for transferring liquidities in the current period to a future period: all positive benefits created in the economy must be used in the same period.

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2. In this paper, we use the terms “project” and “firm” interchangeably.

3. Although this assumption simplifies the analysis, it is not crucial, in the sense that allowing the entry of new projects would not change the results qualitatively, as long as the entry of new projects does not eliminate the possibility of the aggregate liquidity constraint binding in some states.

A project is said to be in *financial distress* in state s if $y(s) < 0$. We say that the project is *solvent* in one period if its benefit is non-negative or if it can obtain refinancing to survive until next period. Since there is no storage technology, refinancing can be obtained only from positive benefits realized by other projects.[4]

Let us denote the current population of projects by $\mathbf{y}$. If $\mathbf{y}$ is small, we will denote it by its elements. For instance, if it contains two projects $\mathbf{y} = \{x, z\}$, we denote it simply by xz. For a subset $\mathbf{z}$ of the population $\mathbf{y}$, $\mathbf{z}(s)$ is the set of benefits generated by each project in $\mathbf{z}$ in state s. The sum of the elements of $\mathbf{z}(s)$ is denoted by $\Sigma\mathbf{z}(s)$. Furthermore, $\mathbf{z}(s)^+$ is the subset of those benefits that are non-negative, and $\mathbf{z}_s^+$ is the subset of $\mathbf{z}$ obtained using the labels associated with the values of $\mathbf{z}(s)^+$. $\mathbf{z}(s)^-$ and $\mathbf{z}_s^-$ are defined the same way.

1.1 Autarky

A project that lives in complete autarky has no access to refinancing. It is solvent if and only if its benefit is non-negative. The value of an autarkic project is then the expected discounted sum of its current and future benefits, taking into account that it goes bankrupt whenever its benefit $y(s)$ is negative. Up to a bankruptcy episode, benefits are stationary. Hence, the continuation value is either zero if the project is bankrupt or some constant non-negative expected discounted value if the project is solvent.

Let us denote by y^+ (y^-), the set of states in which $y(s) \geq 0$ $(y(s) < 0)$, that is,

$$y^+ \equiv \{s \in S | y(s) \geq 0\}, \quad \text{and} \quad y^- \equiv \{s \in S | y(s) < 0\}.$$

We will keep this notation for any other measurable function on S throughout the paper. Under the assumption of stationarity of the benefit

4. Like the no-entry assumption, the no-storage assumption is made for tractability. In a closed economy, saving does not take the form of storage but of investment that increases the productive capacity of the economy. Hence, we assume that the capacity of the economy is somewhat fixed and we focus on real shocks around a zero-growth trend. We conjecture that our results would be qualitatively unaffected if growth was taken into account as long as the magnitude of shocks is related to the size of the economy: one can reinterpret the (stationary) process of shocks *y* on the level of output of a given project as a (stationary) process of shocks on its rate of growth. A project is then in financial distress in state *s* if its rate of growth *y(s)* is smaller than –1. This approach needs to keep track of the various project sizes to account for the total amount of good produced in a single period-state, but, as the economy grows, the problem of financing distressed projects in states of nature where not all projects can be rescued remains acute.

function y, the value of the project depends only on the current state and may be defined as a random variable $\upsilon_0(y) : S \to \mathrm{R}$,

$$v_0(y)(s) = \begin{cases} y(s) + \delta V_0(y) & \text{if } s \in y^+, \\ 0 & \text{if } s \in y^-, \end{cases} \tag{1}$$

where $\delta \in (0, 1)$ is the discount rate. Let us denote $V_0(y)$ the expected value of $v_0(y)$. Because benefits are stationary, this expected continuation value is constant. Hence, taking the expectation on equation (1) yields

$$\begin{aligned} V_0(y) &= \mathrm{E}(v_0(y)) \\ &= \mu(y^+)\mathrm{E}(y + \delta V_0(y)|y^+) \\ &= \frac{1}{1 - \delta\mu(y^+)}\mu(y^+)\mathrm{E}(y|y^+), \end{aligned} \tag{2}$$

where $\mathrm{E}(y|y^+)$ is the conditional expectation of y given the event y^+. Equation (2) yields a formula for the valuation of a project that has a constant probability $\mu(y^-)$ of becoming bankrupt.

1.2 Unconstrained refinancing for a single project

Let us suppose that the project has access to refinancing in states where its current benefit is negative, $y(s) < 0$. Refinancing the project makes economic sense if its continuation value is greater than its current liquidity requirement $-y(s)$. Thus, current and future refinancing can increase the value of the project. This implies that the continuation value itself is affected by the availability of refinancing in the future. Hence, the probability that the project will become bankrupt again is not necessarily $\mu(y^-)$, and $V_0(y)$ is no longer the expected future value of the project.

Define by S^* the set of states in which the firm is either not distressed or is successfully refinanced and, therefore, solvent. Since the decision to refinance is independent of current financial liabilities and benefits are stationary, the set S^* is time-independent. Using similar computations as those in the previous section, the expected discounted value of all future benefits is given by

$$\frac{\delta}{1 - \delta\mu(S^*)}\mu(S^*)\mathrm{E}(y|S^*).$$

This is the maximum amount of financial capital the firm can raise. Hence, the firm is solvent in state s if and only if its net present value is non-negative, that is

$$y(s) + \frac{\delta}{1-\delta\mu(S^*)}\mu(S^*)\mathrm{E}(y|S^*) \geq 0\,. \tag{3}$$

The set S^* is the set of states s for which condition (3) is satisfied. It is easy to see that, if $s \in S^*$, then all states s', such that $y(s') \geq y(s)$, are also in S^*. This implies that there exists some lower bound y^* below which the firm is optimally bankrupt. The lower bound y must be negative, because it is never optimal to declare bankruptcy when the current benefit is non-negative. The set of solvency states is given by $S^* = \{s \in S | y((s) \geq y^*)\} = (y - y^*)^+$. The lower bound y^* solves

$$y^* + \frac{\delta}{1-\delta\mu((y-y^*)^+)}\mu((y-y^*)^+)\mathrm{E}(y|(y-y^*)^+) = 0\,. \tag{4}$$

This equality implicitly defines the set S^*.

We can now compute the expected value of the project, using $y(s) < y^*$ as the bankruptcy condition. In any period and state s, we have

$$v_{y^*}(y)(s) = \begin{cases} y(s) + \delta V_{y^*}(y) & \text{if } s \in S^*, \\ 0 & \text{if } s \in S \setminus S^*. \end{cases} \tag{5}$$

Taking expectations on equation (5) yields

$$\begin{aligned} V_{y^*}(y) &= \mathrm{E}(v_{y^*}(y))\,, \\ &= \mu(S^*)\mathrm{E}(y + \delta V_{y^*}(y)|S^*), \\ &= \frac{1}{1-\delta\mu(S^*)}\mu(S^*)\mathrm{E}(y|S^*)\,. \end{aligned} \tag{6}$$

Equation (6) gives the expected value of the project in an environment without liquidity constraints. For $y(s) \geq y^*$, it is profitable to keep the project operating. Bankrupting it would destroy value, since its future value is larger than the amount of liquidity required to keep it solvent. For $y(s) < y^*$, it is optimal to bankrupt the project since its future value is smaller than the amount of liquidity required to keep it solvent.

Without an aggregate liquidity constraint, a project can raise funds up to its discounted expected value taking into account the probability of bankruptcy. The value $V_{y^*}(y)$ can be compared to the autarkic value $V_0(y)$, which corresponds to the case $y^* = 0$. It is easily shown that $V_{y^*}(y) \geq V_0(y)$, and therefore, the availability of outside liquidity raises the value of the project.

1.3 Refinancing firms in the face of aggregate liquidity constraints

From now on, we relax the assumption that there is no aggregate liquidity constraint. We suppose instead that liquidities have to be supplied by existing projects and hence cannot exceed the sum of positive benefits in the economy, $\Sigma \mathbf{y}(s)^+$. Therefore, a project must rely on other projects' liquidities to refinance a negative benefit. The availability of refinancing for a project also depends on the demand for liquidity by other projects. This means that there might be some states where a given project should optimally be refinanced but may not be, owing to a shortage of aggregate liquidity. The survival of a project then depends on the aggregate liquidity of the economy. This means that the value of a project y is no longer equal to $V_{y^*}(y)$.

For example, there may be states s and s' such that $y(s) = y(s')$, but the project is solvent in state s and bankrupt in state s', although its current liquidity requirement and future expected value are the *same* in both states.[5] Liquidity constraints may bind at the aggregate level so that states s and s' differ in the sense that it is easier for the project to get refinancing in state s than in state s'. Hence, liquidity constraints increase the probability that a project becomes bankrupt and reduce its value.[6] This is important, since when a project goes bankrupt, the aggregate flow of liquidity in the future is reduced. Moreover, this could jeopardize the solvency of future projects.

The determination of which distressed projects go bankrupt when there is not enough liquidity for all of them depends on the allocation mechanism. In the next section, we compute project values when aggregate liquidities are optimally allocated by a central planner. The optimal allocation maximizes the value of the group of projects surviving in each possible state of nature. In section 3, we decentralize the allocation of funds so that projects can obtain funds from a liquidity market at a competitive price.

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5. Since a state s is a description of the whole economy, it is conceivable that a project may have the same benefit in two different states, while benefits of other projects differ in these two states.

6. To a large extent, our model fits this story: exogenous shocks on the total supply of funds affect the effective discount rate different projects face since they affect their probability of bankruptcy. This can be contrasted with standard macroeconomic models where changes in the effective discount rate are driven by exogenous technological shocks.

2 A Centralized Model of Refinancing

The ability of projects to obtain refinancing is limited by the aggregate constraint on the supply of liquidity. We derive a recursive formula to compute the value of a *coalition* of projects. A *coalition* is a finite set of projects belonging to a network and providing each other with liquidities. Our approach is to maximize the current expected value of the coalition's liquidities. This is done through a complex financial "contract" that optimally assigns realized liquidities to a surviving coalition.

2.1 The coalition model

We take the convention that $\mathbf{y}$ denotes the current coalition before the realization of the state of nature in any period. Since there is no entry of new projects and not all projects survive from one period to the other, the existing population may decrease with time. A coalition $\mathbf{y}$ faces a liquidity constraint in a given state if the sum of all positive liquidities in the coalition is lower than the sum of requirements by distressed projects that are "worth" saving. In this case, only a subcoalition of $\mathbf{y}$ can survive and some projects must disappear. The coalition optimally designs a survival policy that determines which project should be refinanced and which should be bankrupted. The coalition $\mathbf{z}$ that survives after coalition $\mathbf{y}$, and realization of state s, is *feasible* if and only if it satisfies the following two properties.

Admissibility (AD): If a project y has a non-negative benefit in state s, then it must belong to the surviving coalition in state s. Equivalently, if $\mathbf{z}$ is the surviving subcoalition in state s, then $\mathbf{y}_s^+ \subseteq \mathbf{z}$.

Budget balance (BB): If coalition $\mathbf{z}$ survives in state s, then

$$\Sigma \mathbf{z}(s) \geq 0 .$$

In any given state s, admissibility requires that all projects in the set $\mathbf{y}_s^+$ survive. Budget balance ensures that the surviving coalition satisfies the aggregate liquidity constraint. This is possible if and only if the total liquidity requirement $-\Sigma z_s^-(s)$ of these distressed projects in the surviving coalition, $\mathbf{z}$, does not exceed the total liquidity, $\Sigma z_s^+(s)$, generated by the projects with positive benefits.

The optimal survival policy maximizes the value of the surviving coalition. It is thus necessary to compute the value of all possible coalitions of projects. Suppose that we know how to compute the expected value of an arbitrary coalition of projects $\mathbf{z}$ of size less than or equal to $M \geq 1$. Let $V(\mathbf{z})$ be this expected value. In what follows, we show how to compute the value of an arbitrary coalition $\mathbf{y}$ of $M+1$ projects. Let $2^{\mathbf{y}}$ be the power set of

subcoalitions of **y**. Assume that the current set of active projects is **y**. In state *s*, an optimal survival policy selects a coalition that solves

$$\text{Program 1:} \max_{\mathbf{z} \in 2^{\mathbf{y}}} \quad \Sigma \mathbf{z}(s) + \delta V(\mathbf{z}),$$

$$\text{s.t.} \quad \mathbf{y}_s^+ \subseteq \mathbf{z}, \tag{AD}$$

$$\Sigma \mathbf{z}(s) \geq 0. \tag{BB}$$

This problem is well defined by assumption, up to $V(\mathbf{y})$, which is unknown; that is, the expected value $V(\mathbf{z})$ of all subcoalitions **z** of no more than M projects is known by assumption, but the expected value of the (current) coalition **y** of $M + 1$ projects is unknown.

By admissibility (AD), for all states *s* such that $\mathbf{y}_s^-$ is empty, the set of instruments contains only **y**, and Program 1 reduces to

$$\Sigma \mathbf{y}(s) + \delta V(\mathbf{y}). \tag{7}$$

Consider now the states for which $\mathbf{y}_s^-$ is not empty. The following restricted program, for which **y** is not a solution, is well defined,

$$\text{Program 1a:} \max_{\mathbf{z} \in 2^{\mathbf{y}}} \quad \Sigma \mathbf{z}(s) + \delta V(\mathbf{z}),$$

$$\text{s.t.} \quad \mathbf{y}_s^+ \subseteq \mathbf{z}, \tag{AD}$$

$$\Sigma \mathbf{z}(s) \geq 0, \tag{BB}$$

$$\mathbf{z} \neq \mathbf{y}.$$

By construction, we know how to solve Program 1a, since $V(\mathbf{y})$ need not be evaluated.

Program 1 can be represented as a dynamic program where, if $\mathbf{y}_s^-$ is non-empty, one decides first if **y** should survive and, in the case where it should not, which coalition **z** should survive. Define the random variable υ: $S \rightarrow \mathrm{R}$ that takes the value of Program 1a. The value $v(s)$ of Program 1 then becomes

$$v(s) = \begin{cases} \Sigma \mathbf{y}(s) + \delta V(\mathbf{y}) & \text{if } y_s^- = \emptyset \\ \max\{\Sigma y(s) + \delta V(y), v(s)\}, & \text{if } y_s^- = \emptyset \text{ and } \Sigma \mathbf{y}(s) \geq 0 \\ \upsilon(s), & \text{otherwise.} \end{cases}$$

Since this is a stationary value, $V(y) = E(v)$.

Now let

$$S^* = \{s \in S | \Sigma(\mathbf{y}(s) + \delta V(\mathbf{y}) \geq \upsilon(s) \text{ and } \Sigma(\mathbf{y}(s) \geq 0))\}.$$

This is the set of states where the full coalition $\mathbf{y}$ survives, either because $\mathbf{y}_s^-$ is empty, or because it is feasible and profitable to refinance all distressed projects. In what follows, we assume that $\mu(S^*) \in (0, 1)$. The following lemmas describe the solution. All proofs are found in the appendix.

Lemma 1. $\{ s \in S / \mathbf{y}_s^- = \emptyset \} \subseteq S^* \subseteq \{s \in S | \Sigma y(s) \geq 0\}$.

Lemma 2. *Monotonicity. Let $s \in S^*$ and consider s'. If, for all projects, $y(s') \geq y(s)$, then $s' \in S^*$.*

For any given coalition $\mathbf{y}$, its value is the real number $V(\mathbf{y})$ that solves equation (8)

$$V(\mathbf{y}) = \max_{S^* \in S} \mu(S^*)(\mathrm{E}(\Sigma \mathbf{y} | S^*) + \delta V(\mathbf{y})) + (1 - \mu(S^*))\mathrm{E}(\upsilon | S \setminus S^*). \quad (8)$$

We have shown in section 1 that a coalition composed of a single project ($\mathbf{y} = y$) has an expected value of $V(\mathbf{y}) = V_0(y)$. We have shown that if we know how to compute the expected value of M projects or less, we may compute the value of $M + 1$ projects. By induction, we can therefore compute the expected value of an arbitrary but finite coalition of projects. In the next section, we do so explicitly for a coalition of two projects.

2.2 A two-project coalition

Let $\mathbf{y} = xz$ and let y refer to either x or z. We know that $V(y) = V_0(y)$. We want to compute $V(\mathbf{y})$. To do so, we need to identify S^*.

By Lemma 1, we need only to identify those states where only one project is distressed and it makes economic sense to refinance it. If $z(s) > 0 > x(s)$ and $x(s) + z(s) \geq 0$, then project x will be rescued if

$$x(s) + z(s) + \delta V(\mathbf{y}) \geq z(s) + \delta V_0(z),$$

that is, if

$$x(s) \geq \delta(V_0(z) - V(\mathbf{y})) \equiv x^{**}.$$

Hence, both projects remain solvent as long as $\Sigma\mathbf{y}(s) \geq 0$ and each $y(s)$ is at least equal to some endogenous stationary value y^{**} that depends on $V(\mathbf{y})$. $V(\mathbf{y})$ may be obtained as the solution to equation (8), where

$$S^* = \{s \in S | x(s) \geq x^{**},\ z(s) \geq z^{**} \text{ and } x(s) + z(s) \geq 0\}.$$

Notice that y^{**} being independent of s is an artifact of the two-project coalition. In general, this threshold value depends on the state s. For example, suppose there are three projects, *w, x*, and *z*. Further assume that only one project is solvent (say, project *w*) and that it can refinance only one of the two distressed projects. Whether, say, project *z* is refinanced or not depends not only on the net future payoff of doing so (as is the case with two projects), but also on the cost of bankrupting project *x*. This cost depends on the current amount of liquidity needed to refinance project *x*. Hence, survival rules may depend on the state s for a coalition of three or more projects.

Finally, it is now possible to isolate the individual value of a single project within coalition **y**. Denote the value of project $y \in \mathbf{y}$ by

$$V^y(\mathbf{y}) = \frac{\mu(S^* \cup S_y)\mathrm{E}(y|S^* \cup S_y) + \mu(S_y)\delta V_0(y)}{1 - \delta\mu(S^*)},$$

where S_y is the set of states for which only project y is solvent. This value is the discounted expected sum of returns from project y within the coalition **y**. It is bounded below by the value of the flow of returns that can be realized in autarky; that is, $V^y(\mathbf{y}) \geq V_0(y)$. It depends implicitly on the value of the whole coalition through its dependence on the set S^*. Individual values are such that $V^x(\mathbf{y}) + V^z(\mathbf{y}) = V(\mathbf{y})$.

The individual value of a project x must be distinguished from the *contributory* value of x to coalition **y**. The contributory value is the difference of values between the coalition **y** with the project x and the coalition without it; that is,

$$CV^x(\mathbf{y}) = V(\mathbf{y}) - V(\mathbf{y} / x) = V(\mathbf{y}) - V_0(z) = \delta^{-1},$$

where $\mathbf{y} / x$ is the remaining coalition after removing project x from the coalition **y**. The sum of the two contributory values in the coalition **y** exceeds the value of the coalition, or

$$CV^x(\mathbf{y}) + CV^z(\mathbf{y}) = 2V(\mathbf{y}) - V_0(x) - V_0(z) \geq V(\mathbf{y}).$$

The contributory value of a project exceeds its individual value: $CV(x) = V(\mathbf{y}) - V_0(z) \geq V(\mathbf{y}) - V^z(\mathbf{y}) = V^x(\mathbf{y})$, since $V_0(z) \leq V^z(\mathbf{y})$. Each project, therefore, has a shadow value that reflects its externality on the value of the other project.

3 Decentralization

We now decentralize our coalition economy to examine the characteristics of the surviving set of projects when refinancing can be obtained from other projects at a market price. We first propose a static general-equilibrium model with a liquidity market for an economy with an arbitrary number of projects. We then proceed to a dynamic general-equilibrium analysis in a four-project economy.

3.1 The liquidity market

A project may enter a period with an obligation to repay a debt or a claim on the debt repayment from its participation in the liquidity market in the previous period. Suppose that a project with a negative benefit today has lent the amount x in a previous period[7] that entitles it to receive Rx today. Suppose that $Rx > -y$. In this case, the project's net liquidity is $y(s) + Rx > 0$. Nevertheless, if the project's owners decide to use the amount Rx to rescue their project, they are lending the liquidity to themselves. An alternative option would be to let the project die and invest Rx on the liquidity market. Hence, whether it is used by the project to refinance itself or invested in another project, the amount Rx is part of the supply of funds, and the amount $-y(s)$ potentially becomes part of the demand for funds. Likewise, all debt repayments made in this period become part of the supply of funds while the demand for funds is driven by projects with negative realizations of y.

As we have argued above, we may assume without loss of generality that each project enters the liquidity market with either a non-negative supply of funds $y \geq 0$, or an input requirement $y < 0$ and a non-negative future (option) market value $V_m(y) \geq 0$ of keeping the project alive for the next period. We shall assume here, as a first step, that this value is independent of the current price of funds $R > 0$. We consider the relevant case where at least one project has a positive cash flow $y > 0$. The financial instrument exchanged by projects for current funds on this market is generic. It could be a share in the project or a promise of a future payment (we refer to it as

7. To unclutter the notation, the reference to the current state s is omitted in this section; hence, $y(s)$ is simply noted y unless it leads to confusion.

"future funds" below). Since all agents are risk-neutral, the equilibrium risk premium is necessarily zero. Consequently, the value of every financial instrument is equal to its expected discounted payoff measured in units of the good.

The market rate R, the price of current funds, determines the current solvency of projects. Consider Figure 1, where the positions for a seven-project economy are drawn (black dots) in (x, v) space, where x denotes a project's holdings of current funds, and v denotes holdings of future funds. We will refer to each project by its current cash flow $y(s)$; for instance, project –14 (point *a*) has the highest future expected value $V_m(y)$. There are five distressed projects (–14, –12, –8, –6, and –4), located in the negative quadrant of the graph, and two projects with positive cash flows (10 and 16), located in the positive quadrant. A distressed project (for which $y < 0$) is solvent if it has a non-negative value at the ongoing rate of return, R:

$$y + V_m(y)/R \geq 0,$$

or equivalently,

$$Ry + V_m(y) \geq 0.$$

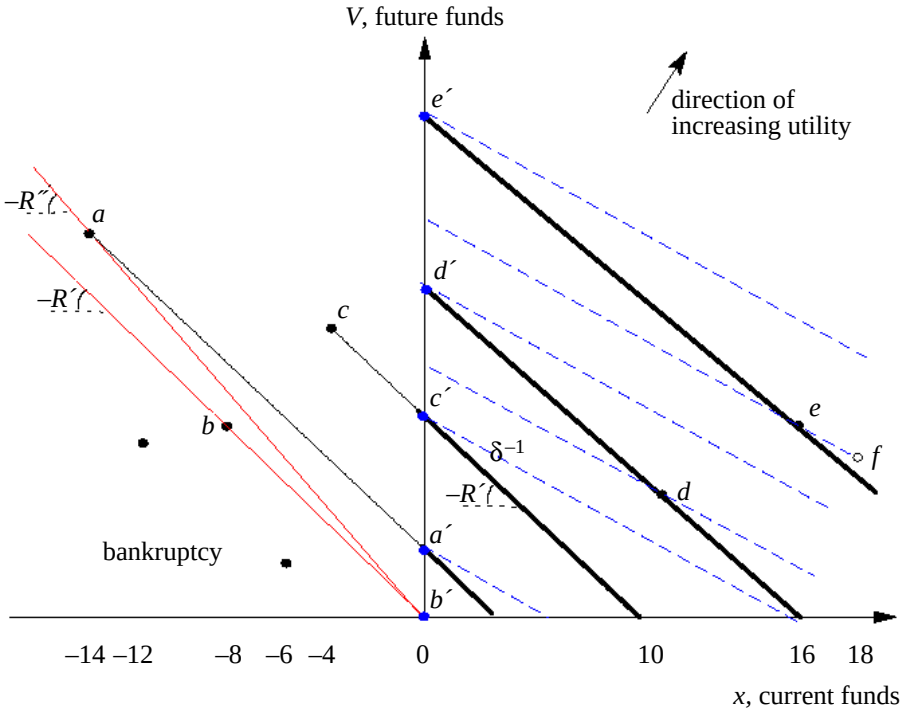
Hence, to be solvent, a project must belong to the half space above the line of slope $-R$ that goes through the origin. For example, we have drawn two thin solid lines from the origin representing the solvency condition for interest rates R' and R''. Whether or not a distressed project is solvent depends on R. At $R = R'$, project –8 is just solvent and projects –4 and –14 are solvent, while project –6 and –12 are bankrupt. But if $R = R'' > R'$, project –8 is no longer solvent and project –14 is just solvent.

To meet the input requirement, by borrowing current funds z such that $y + z \geq 0$, the manager of a solvent project can sell equity or borrow using the equity as collateral. The maximum amount of (current) funds that a solvent project can borrow is $V_m(y)/R$ (the discounted expected value of the option of having the project around at the beginning of next period). This financial operation is represented by a change from y to x in the manager's holdings of funds and a change from $V_m(y)$ to v in the value of the shareholders' portfolios (net of current dividends) such that the total wealth of the shareholders remains the same (no arbitrage):

$$Rx + v = Ry + V_m(y).$$

The manager of a solvent project can thus take any position (x, v) such that $x \in [0, y + V_m(y)/R]$ and $v = Ry + V_m(y) - Rx$. This yields a kind of

Figure 1
Equilibrium on the liquidity market



"budget line" of slope $-R$ in the positive quadrant for each solvent project (the bold lines in Figure 1).

The manager orders two positions (x, v) and (x', v') in the plane, where x are funds available today and v is an expected value that can be realized tomorrow, according to the shareholders' time preferences, which are parameterized by the discount rate δ. Hence, $(x, v) \geq (x', v')$ if $x + \delta v \geq x' + \delta v'$. These linear preferences yield a map of linear indifference curves of slope $-\delta^{-1}$ that are sketched in the figure with dashed lines.

Given these linear preferences, the demand for current funds of each manager is easy to compute. If $R = \delta^{-1}$, the demand correspondence of each project is confounded with its budget line. If $R > \delta^{-1}$, each manager wants all of the project's value in future funds. This implies that all projects end up at the intersection of their budget line and the vertical axis. If $R < \delta^{-1}$, they want all value in current funds; that is, they want to be at the intersection of their budget line and the horizontal axis. For instance, given $R > \delta^{-1}$ in Figure 1, project –14 wants to move from a to a', project –8 from b to b', project –4 from c to c', project 10 wants d to d', and project 16 from e to e'. Given the configuration we have drawn, $R = R'$ constitutes an equilibrium since the demand of current funds (4 + 8 + 14) equals the supply of current funds (10 + 16).

All bankrupt projects have a negative current market value and are thus those that have the lowest current market value. This does not imply that a bankrupt project necessarily has a negative social value; for instance, if there was another project at location *f* (empty dot) in this economy, then equilibrium R would be lower (because of an increased supply of funds), and project –12 would have been saved. However, the fact that project –12 goes bankrupt in the absence of a project 18 is not per se a case of economic inefficiency, because the liquidity constraint is real so that the real social value should take into account the feasibility of keeping projects –12 together with –14, –8, and –4.

Define $R_y(s) = -V_m(y)/y(s)$ as the highest price at which the project is solvent with the convention that y is solvent at any price if $R_y(s) \leq 0$. At that price, the expected value of project x equals the value of its current input requirement.

We now derive the equilibrium on the liquidity market. Let B denote the bankruptcy event; that is, B is true if $0 < R_y < R$. Let $Z_y : \mathrm{R}_+ \to \mathrm{R}$ be the excess-demand correspondence for current funds of project y:

$$Z_y(R) = \begin{cases} 0 & \text{if } B, \\ -y & \text{if not } B \text{ and } R > \delta^{-1}, \\ [-y,\, V_m(y)/R] & \text{if not } B \text{ and } R = \delta^{-1}, \\ V_m(y)/R & \text{if not } B \text{ and } R < \delta^{-1}. \end{cases}$$

Aggregate excess demand is $Z(R) = \Sigma_{\mathbf{y}} Z_y(R)$. Figure 2 illustrates such a correspondence. At small R, every project is solvent and every project is on the demand side of the market. As R increases, the discounted expected value of each project decreases so that, by a simple wealth effect, the demand for funds decreases. A downward jump marks the bankruptcy of some project. The size of the jump matches the input requirement of the bankrupt project. The vertical section at price δ^{-1} marks the indifference of allocating funds at that price that matches exactly the consumers' preferences. The top extremity does represent the maximum demand of all solvent projects, constrained by their future value; and the bottom part represents the maximum supply of all solvent projects given that some of these projects do consume funds to meet their input requirement. If $R > \delta^{-1}$, all projects want to be on the supply side of the market if they can. When R increases, some projects eventually become bankrupt and the demand decreases since they cease to demand their input requirement. If the price is high enough, all distressed projects are bankrupt and the excess demand is a

horizontal straight line at a level matching the total amount of funds in the economy.

The excess demand of each project y can be unambiguously (if $R \neq \delta^{-1}$) decomposed in two parts, demand $X_y^D(R)$ and supply $X_y^S(R)$, one of these being zero, so that $Z_y(R) = X_y^D(R) - X_y^S(R)$. Formally, define

$$X_y^D(R) = \max\{0, Z_y(R)\},$$

$$X_y^S(R) = -\min\{0, Z_y(R)\},$$

where we implicitly assume that a given value has been selected in $Z_y(R)$ if $R = \delta^{-1}$. Aggregate demand and supply for current funds follow readily:

$$X^D(R) = \Sigma_{\mathbf{y}} X_y^D(R),$$

$$X^S(R) = \Sigma_{\mathbf{y}} X_y^D(R).$$

Let $\mathbf{y}(R)$ be the subset of distressed but solvent projects. Notice that when $R > \delta^{-1}$

$$X^D(R) = -\Sigma_{\mathbf{y}}(R), \tag{9}$$

$$X^S(R) = \Sigma \mathbf{y}^+. \tag{10}$$

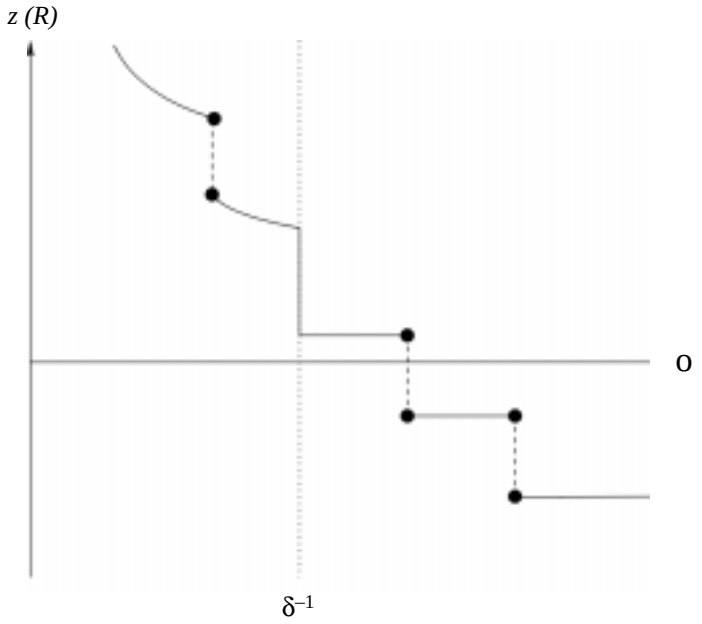
Furthermore, these values certainly belong to $X^D(R)$ and $X^S(R)$ when $R = \delta^{-1}$. Hence, for simplicity, we shall assume that, unless otherwise specified, these shall be the value of aggregate demand and supply at that price.

3.2 Equilibrium

By Walras Law, it is sufficient to find an equilibrium in the market for current funds to obtain a general equilibrium. Hence, we are looking for an equilibrium price R that equalizes the aggregate market demand and supply for current funds.

Since demand may be discontinuous (see Figure 2), our notion of an equilibrium must account for an excess supply at the "equilibrium" price R. As we shall see, the behaviour of the model depends crucially on the total demand that can be accommodated by the market. An excess supply of funds affects only the timing of consumption but has no effect on the overall performance of the economy. Consequently, we devise a rationing

Figure 2
Aggregate excess demand



mechanism that regulates an excess supply to yield an equilibrium on the market.

The device works as follows. Potential suppliers in the market are told that if there is a strictly positive demand, an unspecified fraction $\alpha \in [0, 1]$ of their supply will be channelled through the market. The rest of their supply will be returned for consumption. Notice that the supply of funds is unaffected by this device: if supplying X was optimal when the price is R and $\alpha = 1$, then supplying X is still optimal when $\alpha > 0$, and is of no consequence if $\alpha = 0$. Once $X^D(R)$ and $X^S(R)$ have been expressed at an equilibrium price (to be defined below), the parameter α is set by the market operator to a value that clears the market:[8] $\alpha = X^D(R)/X^S(R)$.

The set of equilibrium prices is defined to be

$$\text{argmax}_R Z(R) \quad \text{s.t.} \quad Z(R) \leq 0 .$$

Up to this point, our analysis is of a purely static nature, because it is assumed that the future market values $V_m(y)$ are independent of R. This is not generally true, since when R is raised, the set of (surviving) solvent projects shrinks, and that may affect the value of these projects in the future.

8. When $R > \delta^{-1}$, aggregate supply is positive, $X^S(R) = \Sigma \mathbf{y}^+ > 0$. When $R = \delta^{-1}$, individual supplies may be selected so that $X^S(R) > 0$.

To tackle this question in a satisfactory manner, we need a full dynamic analysis. This is done in the next section with a four-project economy.

4 Static and Dynamic Efficiency

Since the value of projects, and hence the survival rule, depends on whether liquidities are allocated by a central planner (centralized mechanism) or through a decentralized liquidity market, the allocation mechanism can condition the fragility of the system. To compare the performance of both mechanisms, we need to be able to compare the set of existing projects in each case, after a given history of realization of the states of nature. A natural point of comparison is the coalitions arrived at in the steady state under the two mechanisms. We will explain the notion of fragility more clearly when we have defined the concept of a steady state.

Let $\mathbf{y}_t$ be the coalition of projects in period *t*. As the history of shocks evolves, this coalition shrinks if some projects become bankrupt. Hence, the number of surviving projects weakly decreases through time until a stationary state is reached. Let us define this stationary state with the notion of a stable coalition.[9]

Definition 1 (stable coalitions). *Let* $\mathbf{y}_t$ *be the existing coalition in the beginning of period t. The coalition* $\mathbf{y}_t$ *is stable if and only if it is the surviving coalition after any realization of the state of nature in this period.*

A stable coalition defines the stationary state, because states of nature are drawn from identical and independent distributions in every period. If a coalition survives through all states in one period, it must survive in any state in the future.

There are two necessary conditions for a coalition **y** to be stable. One is that budget balance holds in every state of nature; that is, there is no state in which the aggregate liquidity constraint is binding. The other condition is that no project has to be bankrupted in any state of nature. This latter condition differs according to whether the mechanism is centralized or not.

Definition 2 (stable coalition with a centralized mechanism). *With a centralized allocation mechanism,* **y** *is a stable coalition if and only if* **y** *is feasible, and there is no smaller coalition that would have a greater value in any state of the world. Formally,* **y** *is stable if and only if, for all* $s \in S$*,*

9. Note that the term coalition in this context does not imply that the allocation mechanism is centralized.

$$\mathbf{y} \in \textit{argmax}_{\mathbf{z} \in 2^{\mathbf{y}}} \sum z(s) + \delta \mathrm{E}(V(\mathbf{z})), \textit{ s.t. } \mathbf{y}_s^+ \subseteq \mathbf{z}, \textit{ and } \Sigma \mathbf{z}(s) \geq 0 .$$

Definition 3 (stable coalition on a decentralized market). *With a decentralized allocation mechanism,* $\mathbf{y}$ *is stable if and only if there exists a set of stationary contingent prices* $R(s)$ *such that the market for funds clears at these prices and every project in* $\mathbf{y}$ *is solvent. Formally, there exists R such that for all* $s \in S$,

$$R(s) \in \textit{argmax}_R Z(R) \quad \textit{s.t.} \quad Z(R) \leq 0 ,$$

and

$$R(s) \leq R_y(s) \quad \forall y \in \mathbf{y}^- .$$

This definition is tricky, because the critical prices $R_y(s)$ for project *y* are endogenously derived from its expected discounted market value. Nevertheless, if the existing coalition of projects is stable, the aggregate liquidity constraint never binds and, thus, the market gross rate of return can be set as low as δ^{-1} in every state. Hence, a necessary condition for a coalition $\mathbf{y}$ to be stable in a market equilibrium is that every distressed project is always refinanced at that price.

In accordance with both these definitions, we can say that the empty set is stable. This means that at least one stable coalition exists.

Since there is no entry, the number of projects in the economy can only weakly decrease in time. However, the rate at which projects disappear and the characterization of the stable coalition depend on the history of states of nature. This means that project failures that follow temporary liquidity shocks may have permanent effects. With no entry of projects in the system, a failure in period *t* may trigger further failures. Suppose that the set of stable coalitions achievable by a given allocation mechanism includes sets other than the empty set. We can say that a system is *fragile*, because the history of realized states can force the system towards a less valuable stable coalition. In the extreme, a system can be forced towards the empty set. Furthermore, since all firms that belong to any stable coalition would have had an episode of distress but were refinanced, they all have positive value, both in the individual and contributory sense. Hence, the stable coalition with the larger number of projects is more valuable than one with a smaller number of projects.

The set of states in which all projects survive in a stable coalition is the set *S* itself, since there are no bankruptcies. The value of a coalition is then simply

equal to the discounted expected value of the cash flows of all remaining projects in the centralized as well as in the decentralized mechanism,

$$V(\mathbf{y}) = \frac{\mathrm{E}(\Sigma \mathbf{y})}{1-\delta}.$$

As previously stated, projects are also easy to value in a decentralized market when we have a stable coalition, since their individual values can also be expressed as the expected discounted value of their cash flows at price δ^{-1}. Hence,

$$V_m(y) = \frac{\mathrm{E}(y)}{1-\delta}$$

for all projects y in a stable coalition.

Result 1. *If* **y** *is a stable coalition in a decentralized market, it is stable in a centralized mechanism.*

Proof for this Result is provided in the appendix.

Result 2. *If* **y** *is a stable coalition in a centralized environment, it may not be stable in a decentralized one.*

This result is illustrated by an example. Consider a simple economy with four projects $\{w, x, y, z\}$ and three equi-probable states of nature, 1, 2, and 3. Table 1 shows the benefits of each project and their sum Σ in each state of nature s. Assume that the discount rate is $1 \geq \delta \geq 1/2$ so that $\delta^{-1} \leq 2$ and that $X \geq 10$ and $Y \geq 5$. These bounds ensure that refinancing both projects x and y is an efficient option (see below). We shall consider two cases: case 1, where $3Y > X + 5$, and case 2, where $3Y \leq X + 5$.

Since the sum of the returns is always positive, the coalition *wxyz* can survive in every state. If X and Y are high enough, it is efficient to rescue project y in state 1 and projects x and y in state 3. Rescuing project x in state 3 can be done only by rescuing project z in state 1. If X is large enough, it is socially efficient to rescue project z in state 1 so that project x can be rescued in state 3 in a future period. We then conclude that an efficient allocation of funds must manage to have *wxyz* in all events. This is the stable coalition that would prevail in a centralized mechanism.

We show that, in this economy, the First Welfare Theorem does not hold: project z will not be refinanced in state 1, although it would be socially efficient to do so in order to refinance x in state 3.

Depending on the relative values of X and Y, we obtain two possible equilibrium allocations following the bankruptcy of project z in state 1.

Table 1
A four-project economy

s **:**	**1**	**2**	**3**
w :	3	0	2
x :	2	*X*	–3
y :	–1	*Y*	–1
z :	–2	0	2
Σ :	2	*X* + *Y*	0

To resorb the liquidity shortage in state 3, the price of funds will rise so that refinancing *x* is not an attractive option. In case 1, the expected value of project *x* is small so that a small rise in the market price will be sufficient. With such a small rise, refinancing *y* is still an attractive option. Hence, the initial demise of project *z* implies that only project *x* goes bankrupt, leaving *wy* as a stable market coalition in the long run. In case 2, the demise of project *z* will again eventually cause the bankruptcy of project *x,* but the market price necessary to drive *x* out of the market will also leave project *y* bankrupt, leaving *w* as the long-run stable market coalition.

Rational-expectations market prices clear current and future time/state contingent markets given the coalition of projects that pertains to these contingent markets. Notice that project *w* is always solvent so that it belongs to any surviving coalition of projects and forms by itself a stable market coalition. Coalition *w* should not be interpreted as a monopoly, since it could be the sum of returns of a large group of projects each having a non-negative return in every state.

To construct an equilibrium, we specify a price for funds in every contingent market such that the decision to refinance or not each distressed project is rational given these expected prices and such that each contingent market for funds is in a (possibly rationed) equilibrium. For instance, note that whatever the short-run composition of the market (with project *w* present), the market for funds is in equilibrium at price $R = \delta^{-1}$ in states 1 and 2, since there is no shortage of funds, and suppliers of funds are ready to supply any amount that they own at that price.

A shortage of funds will occur in state 3 once project *z* has been dropped in state 1. Project *z* is dropped in state 1 because if the future is discounted, it is not rational to spend two units of consumption today to refinance a project that has a zero expected value. Hence, to have an equilibrium in state 3, the price of funds will rise to some price $\rho > \delta^{-1}$.

Since there is a finite number of projects, there is a finite number of stable market coalitions we may end up with. Given the (long-run) stationary nature of our economy and starting with a stable market coalition, setting $R = \delta^{-1}$ in every event yields an equilibrium. Hence, we need only analyze the transition from the starting coalition, *wxyz*, to some stable coalition. We proceed with backward induction, starting with the smallest possible candidates. Except for *w*, these must be multi-project coalitions, since all other projects need refinancing in some state of the world and would disappear if they were to operate on their own.

The possible stable market coalitions are *wxyz*, *wxy*, *wxz*, *wyz*, *wx*, *wy*, *wz*, and *w*. We construct the equilibrium using Figures 3, 4, 5, and 6. These trees provide a complete description of the equilibrium. Starting with coalition *wxyz* in Figure 3, nature selects one of the three possible branches (states) to reach some node associated to a contingent market. The equilibrium market price in that state is written above the node (those will be established later). The market structure then evolves by possibly dropping some project along the way. The actual equilibrium selection of projects is indicated by the straight line. For instance, starting with *wxyz* in state 1 (Figure 3), the market price is δ^{-1}, project *z* is dropped, and we end up at the node *wxy*. To follow the rest of the event tree, we then switch to Figure 4, and so on. A stable market coalition is one where we always end up with the same coalition. Coalitions *w* in Figure 3 and *wy* in Figure 6 are stable in that sense.

In case 1, the left tree of Figure 4 is used for coalition *wxy*. In case 2, Figure 7 will be used. They differ only with respect to state 3.

Since we must eventually end up with either coalition *w* or *wy*, we start by solving for the equilibrium prices in these events. We then proceed backwards to compute the equilibrium prices, starting with the other (non-stable) market coalitions. For instance, once the stationary price that prevails with coalition *w* has been identified in Figure 3, we can compute the prices that prevail starting with coalition *wz* in Figure 6, etc. All equilibrium prices are chosen according to the definition given in section 3.2. The market price in states 1 and 2 is assumed to be δ^{-1} so that we only discuss the price in state 3.

Coalition *w*: The market price is δ^{-1}.

Coalition *wx*: Suppose that we start in state 3 with coalition *wx*. Depending on whether project *x* is rescued or not, there are two possibilities for the next period: one that starts again with coalition *wx* and one that starts with coalition *w*. We know that the future market price in the latter branch will be δ^{-1}, and we want to compute the price ρ that will clear the current market and that will prevail again in the (zero-probability) event that the economy

Figure 3
Coalitions *wxyz* and *w*

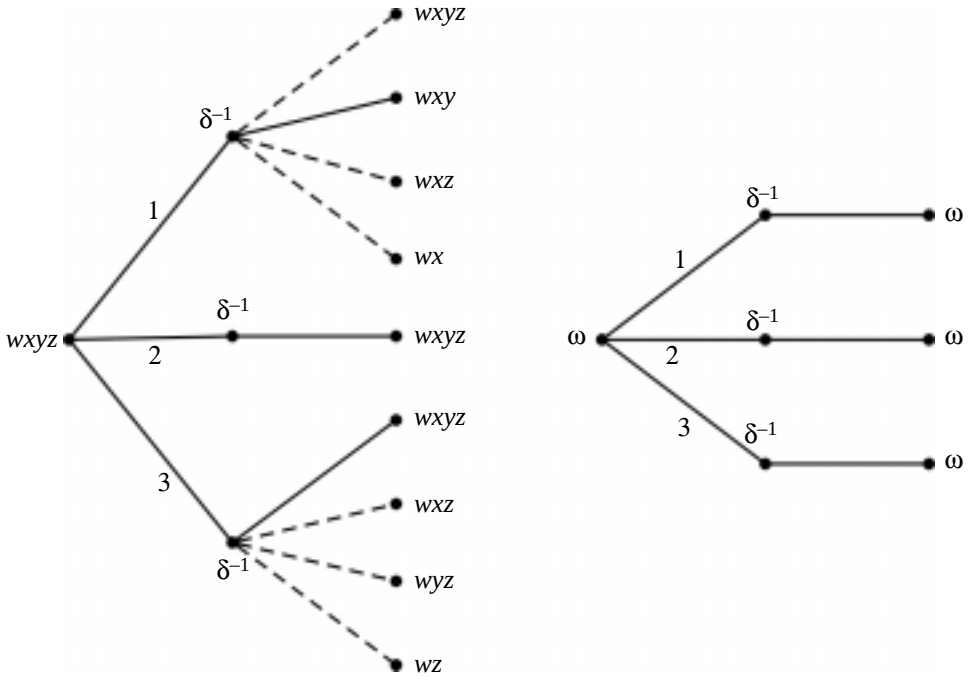


Figure 4
Coalitions *wxy* and *wxz*

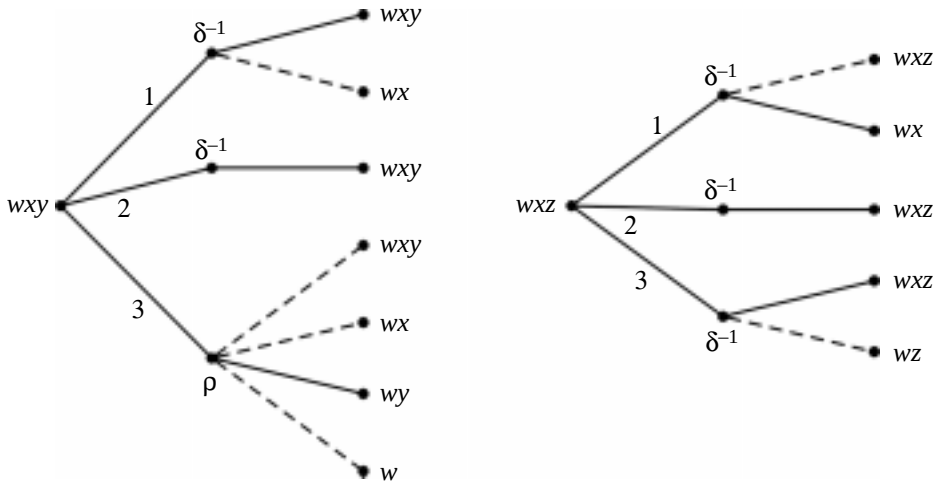


Figure 5
Coalitions *wyz* and *wx*

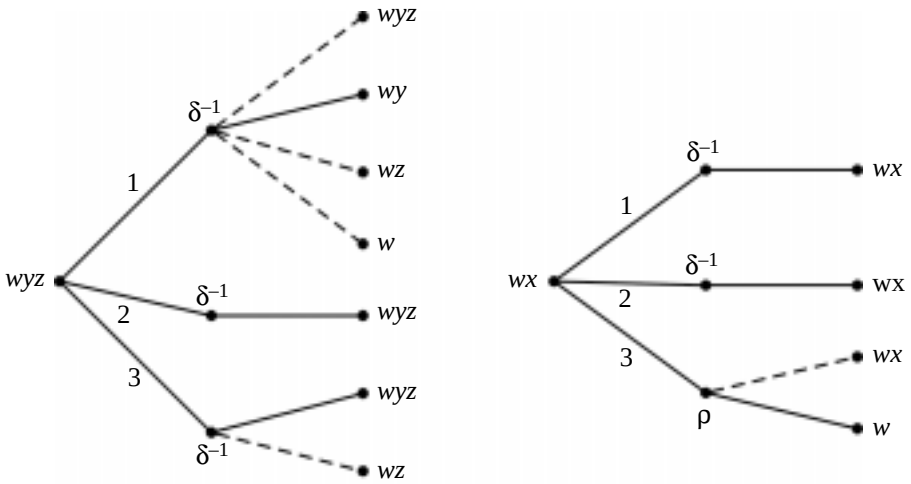


Figure 6
Coalitions *wy* and *wz*

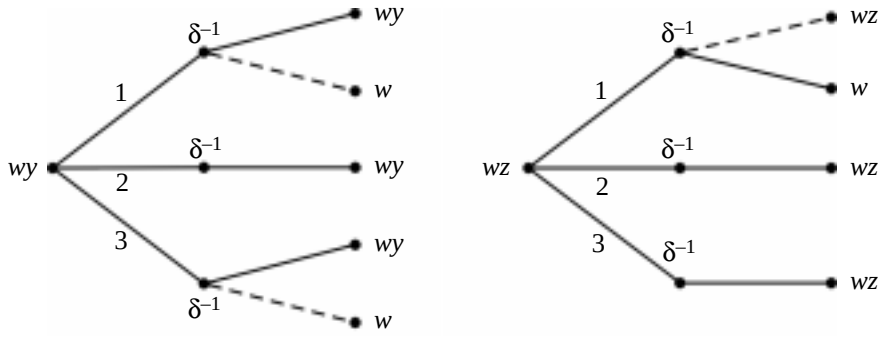
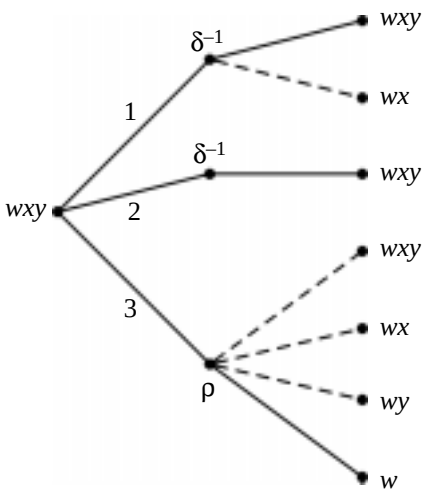


Figure 7
Coalition *wxy* when *X* is large



would follow the first branch with *wx*. Since project *x* must be dropped for the market to clear (with our rationing device), that price must be high enough to make refinancing project *x* an unattractive option.

At price $R_x = -V_m(x)/x(3) = V_m/3$, project *x* would be refinanced at the margin for a net gain of zero and the coalition would be stable. An investor is ready to pay $R_x 3$ today in the hope of ending in state 1 or 2 tomorrow: being stationary, the value of ending again in state 3 tomorrow would also be zero. Hence,

$$R_x 3 = V_m(x) = \frac{2 + \delta V_m(x)}{3} + \frac{X + \delta V_m(x)}{3} + \frac{0}{3},$$

$$= \frac{2 + X}{3 - 2\delta},$$

so that,

$$R_x = \frac{2 + X}{3(3 - 2\delta)} \geq 2 \geq \delta^{-1}.$$

To conclude, we may arbitrarily state that $\rho = R_x + \varepsilon$, with $\varepsilon > 0$, to ensure that project *x* is dropped. At that price, the owners of project *w* would like to lend all their funds but they can only lend to themselves. In short, our rationing device is at work and the market clears at price ρ.

Coalition *wy*: Suppose that we start in state 3 with coalition *wy.* Since there is no shortage of funds, setting the price at δ^{-1} equilibrates the market. As above, we verify that *y* is solvent at that price by computing the limit price R_y that would make it barely solvent:

$$R_y 1 = V_m(y) = \frac{-1 + \delta V_m(y)}{3} + \frac{Y + \delta V_m(y)}{3} + \frac{0}{3},$$

$$= \frac{Y-1}{3-2\delta} \geq 2 \geq \delta^{-1}.$$

Since $R_y \geq \delta^{-1}$, we conclude that *y* is refinanced at price δ^{-1} in state 3.

Coalition *wz*: Again the market price is $\delta^{-1} > 1$ in all events. At this price, refinancing *z* is never an attractive option, since one finances today 2 in the hope of recouping at most 2 with probability one-third tomorrow.

Coalition *wxy*: It is impossible to refinance project *x* in state 3 so that the market price will be set high enough to make that an unattractive option. The question is whether the required high price will push project *y* into bankruptcy. Consider case 1 in Figure 4 where it does. Notice that the condition $3Y > X + 5$ implies that $R_y > R_x$. Then in the event where project *x* stays solvent at the margin, so will project *y,* and the future market price in state 3 will be the same. As above, the limit price that makes project *x* solvent is R_x. Hence, we set the price again at $\rho = R_x + \varepsilon$ with ε sufficiently small so that $R_y \geq \rho > R_x$. Notice that project *w* gathers a rent in that case from project *y.*

We add that a price $R > R_y$ would not yield an equilibrium, since the aggregate excess demand would not be maximized under the non-positivity constraint.[10]

Consider now case 2 where project *y* is also dropped (Figure 7); that is, when $R_y \leq R_x$. Then, in the (zero-probability) event where project *x* stays solvent at the margin, the surviving coalition will be *wx* and the market price will be ρ (see Figure 5) in the future. Again, we may assume that the market price is $\rho > R_x \geq R_y$ so that both *x* and *y* are dropped.

Coalition *wxz*: Assume that the price is δ^{-1} in every state and that *z* is dropped in state 1. Then it is optimal to drop *z* in state 1. In state 3, the supply of funds strictly exceeds demand at price δ^{-1} so that we have a (rationed) equilibrium at that price. We need to show that *x* is refinanced at that price.

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10. That is, if a price $R \leq R_y$ is expected in state 3 in the future, then a price $R > R_y$ is not a current equilibrium price.

Let $V_m'(x)$ be the current continuation value of project *x*, that is, the value of *x* if we end up in states 2 or 3 tomorrow with the same coalition and the same price δ^{-1}. In state 1, the price is also δ^{-1}, but *z* is dropped so that the coalition reduces to *wx* and we have shown above that *x* is worth $V_m(x) = 3R_x$ with this coalition. It follows that

$$V_m'(x) = \frac{2 + \delta V_m(x)}{3} + \frac{X + \delta V_m'(x)}{3} + \frac{-3 + \delta V_m'(x)}{3},$$

$$= \frac{X - 1 + 3\delta R_x}{3 - 2\delta},$$

$$\geq \frac{X + 2}{3 - 2\delta} \geq \delta^{-1}3,$$

and project *x* is refinanced in state 3.

Coalition *wyz*: The reasoning is similar to the one used above for coalition *wxz*: assume that the price is δ^{-1} in every state; then *z* is dropped in state 1. There is no shortage of funds at price δ^{-1} in any state. We need to show that *y* is refinanced at that price in states 1 and 3. If *z* is dropped, we end up with the stable market coalition *wy*, and the future prices are δ^{-1} in every state. It follows that *y* is evaluated at price δ^{-1} in every state. Since $\delta^{-1} \leq R_y$, *y* is refinanced.

Coalition *wxyz*: There is no shortage of funds at price δ^{-1} in any state. Similar arguments as above establish that all projects are refinanced except *z* in state 1. First, one can establish that *x* is worth $V_m(x)$ with coalition *wxy* once *z* is dropped in state 1. We then obtain that *x* is refinanced in state 3 since its continuation value is

$$\frac{X + 2}{3 - 2\delta} \geq \delta^{-1}3.$$

Likewise, *y* is worth $V_m(y) = R_y$ with coalition *wxy* so that its current continuation value $V_m'(y)$ is

$$V_m'(y) = \frac{-1 + \delta V_m(y)}{3} + \frac{Y + \delta V_m'(y)}{3} + \frac{-1 + \delta V_m'(y)}{3},$$

$$= \frac{Y - 2 + \delta R_y}{3 - 2\delta},$$

$$\geq \frac{Y-1}{3-2\delta} \geq \delta^{-1},$$

which warrants its refinancing in both states 1 and 3.

This completes the description of the equilibrium.

One may argue that the owner of project *x* suffers from myopia by not refinancing project *y* in state 1, since the demise of that project directly implies an even greater loss of value for project *x* in the future. But to make that argument, one must relax the assumption of a competitive equilibrium where agents react to current and "rationally" expected future prices. In short, the external effect of the demise of project *z* on the fate of project *x* is of a pecuniary nature and cannot be coherently "expected" in a "rational"-expectations equilibrium. Obviously, if the owner of project *x* could grasp this external effect, he would simply horizontally integrate with project *z*.

The introduction in state 1 of a current market for future funds in state 3 (so that the balance sheet of project *z* would turn positive in state 1) does not help. Such a market is implicitly present in our set-up, and project *z* is not financially viable despite the presence of that market. The problem is not that project *z*'s future funds are not negotiable in state 1 but that they are undervalued in a competitive equilibrium.

Project *z* gathers no rent from being the "white knight" in state 3 whose presence is necessary to save project *x*. Note that the presence of project *w* is no less necessary in that state for that purpose. Hence, if project *z* would receive a rent, then project *w* would receive it as well, since the funds of both projects are perfect substitutes. The fact that one's presence prevents the instance of a crisis is not sufficient to ensure a rent in a competitive equilibrium. In fact, there is something of a paradox here: the rent associated to funds in state 3 accrues (in case 1) to project *w* only when project *z* has been dropped, thus creating the crisis that rationalizes the increase in the interest rate. Hence, project *z* is dropped because only a rent-generating crisis could justify its current refinancing, and such a rent-generating crisis can occur only if project *z* is not refinanced.

Conclusions

We show in this paper that the efficiency of a liquidity-allocation mechanism depends on its ability to measure the value of a project, taking into account its contribution to the liquidity of the economy in future periods. This contribution is not taken into account by decentralized markets, because it represents an externality that cannot be priced on competitive liquidity markets. Our main result is given in Result 2 and states that a competitive

liquidity market may be more fragile than a centralized mechanism. This has implications on how a public authority could supervise financial markets to ensure that liquidities are properly allocated among productive projects. The existence of a competitive financing rate for liquidity exchanges is necessary to signal the opportunity cost of liquidities and drive the price of capital in the economy. However, intervention by a market regulator to rescue a distressed project that cannot find refinancing on the liquidity market may help ensure that this liquidity market remains sound.

The dynamic inefficiency due to the externality may be alleviated by contracts that could lead firms to integrate. Horizontal integration of the two firms that are affected by the externality could eliminate the externality and restore efficiency. Such integration, however, could be non-desirable for other motives such as antitrust. Furthermore, some externalities may not be internalized if firms cannot perfectly anticipate all future contingencies. For example, the economy may be subject to an unanticipated shock that could put in financial distress a firm that would have liked to merge with another in a previous period had it been able to foresee this shock. So, the dynamic ineffiency identified in this article should be of real concern.

It is interesting to consider the following interpretation to our model. The coalitional model can be related to a financial market with a financial intermediary.[11] The intermediary allocates financing among its firms to maximize the value of its portfolio of firms. A long-lived financial intermediary can therefore endogenize the type of externalities that prevent the market from being efficient; that is, it can take into account the potential future contribution of a financially distressed firm when deciding to re-finance it or not.

The only source of financial imperfection we consider is a potential shortage of liquidity at the aggregate level. If markets cannot decentralize the optimal allocation, firms may have to use complicated long-term contracts that would depend on all realized shocks in the economy. It would then be interesting to characterize the nature of these contracts when they suffer from this and other market imperfections such as non-commitment.

11. See Dolar and Meh (2002) for a non-technical survey of the literature on intermediary-based and market-based views of financial structure.

Appendix

Proof of Lemma 1

The first part comes directly from admissibility (AD). The second part, directly from the budget balance (BB) condition.

Proof of Lemma 2

If $\mathbf{y}_s^- = \emptyset$, the result is obvious. If $\mathbf{y}_s^- \neq \emptyset$, then the question becomes: *given that we manage to keep all projects solvent, would we want to drop a project now that aggregate liquidity has risen?* The answer is "no." Suppose that in state $s \in S^*$ the coalition $\mathbf{z}$ survives, and that projects $\mathbf{w} \subset \mathbf{z}$ are bankrupt in state $s' \in S^*$. This implies that

$$\Sigma\mathbf{z}(s) + \delta V(\mathbf{z}) \geq \Sigma\mathbf{z}\setminus\mathbf{w}(s) + \delta V(\mathbf{z}\setminus\mathbf{w}). \tag{A1.1}$$

In state s', y increases for all projects. Given stationarity, this affects only the first term on each side of condition (A1.1). Since there are more projects in $\mathbf{z}$ than in $\mathbf{z}\setminus\mathbf{w}$, this condition must also be satisfied in state s'. Hence, it is not optimal to bankrupt more projects in s' than in s.

Proof of Result 1

Since budget balance holds, the market rate of return has to be equal to δ^{-1}. The stability of $\mathbf{y}$ implies that all y in $\mathbf{y}$ are such that $y(s) + \delta V_m(y) \geq 0$ for any possible s. Suppose $\mathbf{y}$ is not stable with a centralized institution; then there is a state s in which subcoalition $\mathbf{y}\setminus\mathbf{z}$ must optimally be bankrupt. This also writes

$$\Sigma\mathbf{y}(s) + \delta V(\mathbf{y}) < \Sigma\mathbf{z}(s) + \delta V(\mathbf{z}), \tag{A1.2}$$

where $\mathbf{z}$ is the value-maximizing coalition in state s. This implies

$$\Sigma(\mathbf{y}\setminus\mathbf{z})(s) + \delta(V(\mathbf{y}) - V(\mathbf{z})) < 0. \tag{A1.3}$$

For stability on the decentralized market, we must have

$$\Sigma(\mathbf{y}\setminus\mathbf{z})(s) + \delta\Sigma_{y \in \mathbf{y}\setminus\mathbf{z}} V_m(y) \geq 0. \tag{A1.4}$$

This means that the contribution $V(\mathbf{y}) - V(\mathbf{z})$ of $\mathbf{y}\setminus\mathbf{z}$ to the centralized value of $\mathbf{y}$ is smaller than its market value, that is, a contradiction.

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