Continuous versus Discrete-time Modeling:

Does it make a Difference? *

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I. Introduction.

Whenever one writes down a model some fundamental choices have to be made. One of these is the question of discrete versus continuous time modeling. Some argue that discrete-time modeling should be used because decisions are made and data are released at discrete intervals. Others, however, argue that life unfolds continuously suggesting that a continuous time framework is more realistic. Until recently the choice seemed to be more stylistic since few believed that there were any important economic differences between the two. This paper challenges this commonly held assumption. At least when it comes to matters of local and global stability there can be important differences and hence different policy prescriptions depending on which modeling environment they adopted.

We start out by looking at a very simple but stark example. Is it desirable for the monetary authority to adopt an interest rate peg? A continuous time modeler might conclude yes but a modeler working with discrete time would almost certainly eschew such a proposal as it suffers from local indeterminacy. A key difference between a discrete-time and a continuous-time model is the no-arbitrage relationship between bonds and capital. In discrete time the *future* marginal productivity of capital equals the real interest rate; in continuous time *today's* marginal productivity of capital equals the real interest rate. This contemporaneous condition provides an extra restriction in continuous time. In a model in which the central bank conducts policy with an interest rate instrument, this extra restriction alters the determinacy conditions across the two models.

This difference only arises in a model with capital. Yet differences can also exist in labor only economies. In a recent provocative paper Benhabib, Schmitt-Grohe, and Uribe (2003) argue that if you relied solely on local analysis you would be led to believe that aggressive, backward-looking interest rate rules are sufficient for determinacy. But they argue that from the perspective of global analysis, backward-looking rules do not guarantee uniqueness of equilibrium and indeed may lead to cyclic and even chaotic equilibria.

This paper revisits their conclusions and argues that within the context of a discrete time model their policy prescription may be premature. First, compared to the corresponding continuous time model, cyclic equilibria are much *less* likely to arise in a discrete time model. Second, pure backward-looking rules are *less* likely to suffer from these global indeterminacy problems than rules that include current inflation. By "pure" we mean rules where *only* lagged inflation rates are in the interest-rate rule. This distinction between lagged and current inflation does not arise in Benhabib et al.

II. Local Indeterminacy and Interest Rate Rules

This section looks at a result that is pursued further in Carlstrom and Fuerst (2003). The economy consists of numerous households and firms each of which we will discuss in turn. We are concerned with issues of local determinacy. Hence, without loss of generality we limit the discussion to a deterministic model.

Households are identical and infinitely-lived with preferences over consumption, real money balances and leisure given by

$$\sum_{t=0}^{\infty} \beta^t U(c_t, M_{t+1}/P_t, 1-L_t),$$

where β is the personal discount rate, c_t is consumption, and $1-L_t$ is leisure. We utilize a MIUF environment because of its generality (see Feenstra (1986)).

Whenever one adopts a MIUF framework a decision has to be made whether to adopt in the words of Carlstrom and Fuerst (2001) cash-in-advance (CIA) timing or cash-when-I'm-done (CWID) timing. This amounts to answering the question whether end- or beginning-of-period money balances enter in the utility function. This choice can have important implications in determining whether an interest rate rule (or a money growth rule see Carlstrom and Fuerst (2003)) is determinate. Although we have argued that CIA-timing makes more economic sense, to be consistent with continuous time we adopt CWID-timing. As noted by Benhabib et al. (2001), the discrete time analog to a continuous time MIUF model is CWID timing. For the purposes of this paper this timing issue is of limited importance. We assume that utility is given by:

$$U(c,m,1-L) \equiv \ln(c_t) + V(M_{t+1}/P_t) - L_t.$$

The household begins the period with M_t cash balances and B_{t-1} holdings of nominal bonds. Before proceeding to the goods market, the household visits the financial market where it carries out bond trading and receives a cash transfer of $M_t^s(G_t - 1)$ from the monetary authority where M_t^s denotes the per capita money supply and G_t is the gross money growth rate. After engaging in goods trading, the household ends the period with cash balances given by the intertemporal budget constraint:

$$M_{t+1} = M_t + M_t^s (G_t - 1) + B_{t-1} R_{t-1} - B_t + P_t \{ w_t L_t + [r_t + (1-d)] K_t \} - P_t c_t - P_t K_{t+1} + \Pi_t \ .$$

K_t denoes the households accumulated capital stock that earns rental rate r_t and depreciates at rate δ. The real wage is given by w_t while Π_t denotes the profit flow from firms. The first order conditions to the household's problem include the following:

$$\frac{U_L(t)}{U_c(t)} = w_t \tag{1}$$

$$U_c(t) = b\{U_c(t+1)[r_{t+1} + (1-d)]\} \tag{2}$$

$$\frac{U_c(t)}{P_t} = b\ R_t \left\{ \frac{U_c(t+1)}{P_{t+1}} \right\} \tag{3}$$

$$\frac{U_m(t)}{U_c(t)} = \frac{R_t - 1}{R_t} \tag{4}$$

Equation (1) is the familiar labor supply equation, while (2) is the asset accumulation margin. Equation (3) is the Fisherian interest rate determination in which the nominal rate varies with expected inflation and the real rate of interest on bonds. Equation (4) is the model's money demand function.

As for firm behavior, we assume prices are sticky and adopt the assumption of staggered pricing as in Calvo (1983). The exact formulation is due to Yun (1996). This gives rise to an endogenous mark-up in the following equations. A recursive competitive equilibrium is given by stationary decision rules that satisfy (3), (4), (5), (6), and the following:

$$\frac{U_L(t)}{U_c(t)} = z_t\, f_L(t) \tag{7}$$

$$U_c(t) = b\{U_c(t+1)[z_{t+1} f_K(K_{t+1}, L_{t+1}) + (1-d)]\} \tag{8}$$

$$c_t + K_{t+1} - (1-d)K_t = f(K_t, L_t) \equiv Y_t \tag{9}$$

As for marginal cost we have the following log-linearized "Phillips curve"

$$\tilde{p}_t = K\tilde{z}_t + b\tilde{p}_{t+1}, \tag{10}$$

where $\pi_t = (P_t/P_{t-1}) - 1$, denotes the inflation rate. To close the model we need to specify the central bank reaction function. In this section we adopt a simple interest rate peg $R_t = \overline{R}$.

Using the Fisher equation (3) the capital accumulation equation is

$$\frac{\overline{R}}{p_{t+1}} = b\{z_{t+1} f_K(K_{t+1}, L_{t+1}) + (1-d)\} \tag{11}$$

Equation (11) is key in what follows. This is a no-arbitrage relationship: the real return on bonds must be equal to the real return on capital accumulation. This expression is entirely in terms of time t+1 variables.

Note that since cash balances are separable, the money demand curve (4) is irrelevant for determinacy issues. Money is the residual that can be backed out at the end. Log-linearizing the continuous time counterpart to this discrete-time system is given by

$$\begin{pmatrix} \dot{\tilde{c}}_t \\ \dot{\tilde{p}}_t \\ \dot{\tilde{K}}_t \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -l(1-a) & r + la/a_2 & 0 \\ -\left(\dfrac{(1-a)Y_{ss} + c_{ss}}{K_{ss}}\right) & \dfrac{(1-a)Y_{ss}(t-1)}{a_2 K_{ss}} & \dfrac{c_{ss}}{K_{ss}} \end{pmatrix} \begin{pmatrix} \tilde{c}_t \\ \tilde{p}_t \\ \tilde{K}_t \end{pmatrix}$$

where $a_2 = 1 - b(1-d)$. For determinacy, we need exactly one negative eigenvalue. By inspection, we have one eigenvalue equal to $c_{ss}/K_{ss} > 0$. The remaining two are the solution to the following

quadratic equation:

$$h(q) \equiv a_2 q^2 - [\,l\,a + r a_2]q - l(1-a)a_2 \,.$$

In continuous time determinacy requires that one root be negative and one be positive. Inspection reveals h → ∞ as either q → -∞ or q → ∞ and h(0) < 0. Thus the system is locally determinate.

How does the analysis differ in discrete time? The key difference is that the no-arbitrage equation (11) is entirely in time t+1 variables so that it does not provide any restriction on time-t behavior. To analyze the system we scroll (3), (7), (9) and 10 forward one period. Scrolling everything but the capital accumulation forward implies that the system is given by (where x ≡ L/K).

$$\tilde{z}_{t+1} = a\tilde{x}_{t+1} + \tilde{c}_{t+1} \tag{12}$$

$$\tilde{p}_{t+1} = -a_2 \tilde{z}_{t+1} - a_2(1-a)\tilde{x}_{t+1} \tag{13}$$

$$\tilde{c}_{t+2} - \tilde{c}_{t+1} = -\tilde{p}_{t+2} \tag{14}$$

$$\tilde{K}_{t+2} = \left(1 + \frac{c_{ss}}{K_{ss}}\right)\tilde{K}_{t+1} + \left(\frac{(1-a)Y_{ss}}{K_{ss}}\right)\tilde{x}_{t+1} - \left(\frac{c_{ss}}{K_{ss}}\right)\tilde{c}_{t+1} \tag{15}$$

$$\tilde{p}_{t+1} = l\tilde{z}_{t+1} + b\tilde{p}_{t+2}\,, \tag{16}$$

plus the following time t restrictions

$$\tilde{z}_t = a\tilde{x}_t + \tilde{c}_t \tag{17}$$

$$\tilde{c}_{t+1} - \tilde{c}_t = -\tilde{p}_{t+1} \tag{18}$$

$$\tilde{K}_{t+1} = \left(1 + \frac{c_{ss}}{K_{ss}}\right)\tilde{K}_t + \left(\frac{(1-a)Y_{ss}}{K_{ss}}\right)\tilde{x}_t - \left(\frac{c_{ss}}{K_{ss}}\right)\tilde{c}_t \tag{19}$$

$$\tilde{p}_t = l\tilde{z}_t + b\tilde{p}_{t+1}. \tag{20}$$

Using (12) and (13) to eliminate z_{t+1} and x_{t+1} from the system, we can write the dynamic part of the system in matrix form as

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \tilde{c}_{t+2} \\ \tilde{p}_{t+2} \\ \tilde{K}_{t+2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -l(1-a)/b & [a_2 + la]/ba_2 & 0 \\ -\left(\frac{(1-a)Y_{ss} + c_{ss}}{K_{ss}}\right) & \frac{-(1-a)Y_{ss}}{a_2 K_{ss}} & 1 + \frac{c_{ss}}{K_{ss}} \end{pmatrix}\begin{pmatrix} \tilde{c}_{t+1} \\ \tilde{p}_{t+1} \\ \tilde{K}_{t+1} \end{pmatrix}$$

Or

$$Ay_{t+2} = By_{t+1}$$

Notice the similarity in the discrete time versus continuous time matrices. This system has three eigenvalues. The time t restrictions (17)-(20) provide four restrictions on the equilibria. But we have seven unknowns. Hence, for determinacy, we need all three of the eigenvalues to lie outside the unit circle. By inspection, one of the eigenvalues is given by $1+c_{ss}/K_{ss}$. The remaining two are the solution to the following quadratic equation:

$$g(q) \equiv ba_2 q^2 - \{l[a + (1-a)a_2] + (1+b)a_2\}q + a_2 + la .$$

This quadratic is very similar to the h(q) function that arose with continuous time but there we needed only one explosive root now we need two explosive roots. Again inspection reveals that $g(0) > 0$ $g(1) < 0$ and $g \to \infty$ as either $q \to -\infty$ or $q \to \infty$. Thus the system has one eigenvalue inside the unit circle and one outside and is thus indeterminate.

The idea that an interest rate peg is indeterminate is of course not surprising. This is basically the analogue to the standard result that there is nominal indeterminacy with purely flexible prices. What is surprising is that there is determinacy in continuous time. A modeler could come to much different conclusions regarding the desirability of an interest rate peg depending on whether he uses a discrete versus a continuous time model.

The key difference is that in continuous time there is an additional restriction: at time t the marginal productivity of capital equals the real interest rate while in discrete time model this relationship is in terms of the future realizations of returns. Therefore the key difference is in the no-arbitrage relationship between bonds and capital.

Although this was done in the context of a pure interest rate peg, Carlstrom and Fuerst (2003) examine this model with different policy reaction functions. They show that in a model with capital all forward-looking or current-looking rules are indeterminate. To guarantee determinacy the monetary authority must respond aggressively to yesterday's inflation.

This analysis suggests that there is only a difference between a discrete-time and a continuous-time model is because of capital. The next sections investigates a labor only model and shows that even in this environment there can be important differences in whether or not there is the possibility of cycling and or chaos in a global setup.

III. Bifurcations and Interest Rate Rules

It is well-known that a dynamical system that is locally determinate may still be subject to cycles and even chaos. With discrete time the textbook example of this arises when there is an eigenvalue equal to minus one. Even if a parameter is changed pushing the eigenvalue outside the unit circle the cycles associated with the minus one eigenvalue can survive. The issue of whether or not they survive is one of supercriticality – namely whether the higher order effect of the system tends to push the system back toward the steady state while the first order linear effects push the system away from the steady state. Usually these cycles are short-lived, however. This is not the case with another type of bifurcation. In discrete time, a Hopf bifurcation arises when two of the eigenvalues are complex conjugates with a norm equal to one. In continuous time a Hopf-bifurcation arises when the real part of two complex roots vanish while the imaginary part does not.

In what follows the analysis is indebted to the recent paper by Benhabib et al (2003). While Carlstrom and Fuerst argue that policy must react to past inflation to guarantee determinacy, they demonstrate that backward-looking rules may be subject to cycles and even chaos. They analyze a continuous time, money in the production function (MIPF) model and show that if policymakers look backward, cycles and chaos are likely to arise. Here we challenge this conclusion and suggest that a modeler working with a discrete-time model would conclude that the conditions necessary for cycles are less than what they conclude and that it is the current-looking elements in their rule – not the backward elements that even make cycles a theoretical possibility.

In what follows we change their analysis and utilize the MIUF framework as analyzed above. As shown by Feenstra (1986) MIPF is equivalent to a money in the utility function (MIUF) framework with a negative cross partial between consumption and real money balances ($U_{mc} < 0$). The existence of cycles arises because the cross partial is significantly negative ($U_{mc} << 0$). We show in what follows that with discrete time it has to be even more negative than it does for continuous time.

We start with the discrete time analogue to Benhabib et al and analyze the conditions necessary for the existence of a Hopf-bifurcation. We are interested in situations in which the system is locally determinate, but is globally indeterminate because of a supercritical Hopf bifurcation. In this paper we will not consider the issue of supercriticality, but leave this for future work. (In their continuous time model Benhabib et al. analyze whether their equilibria are super critical – that is, does the economy converge to an attracting cycle if the equilibriaium is pushed away from the Hopf-bifurcation point to a region which is locally determinate. For present purposes we simply assume that any bifurcation is supercritical. We use the terminology "Locally Determinate, Globally Indeterminate" (LDGI) for situations in which we have local determinacy, but a Hopf bifurcation exists.

A Discrete Time Model.

Preferences are the same with two modifications. Money is no longer assumed to be separable from consumption and utility is not linear in labor (L). Assume that preferences are separable in the consumption-money composite and in labor (L):

$$U(c, m, 1-L) \equiv V(c, m) - B\frac{L^{1+g}}{1+g},$$

and that production is linear in labor $y = L$. Along with the Phillips curve (10), the equilibrium conditions are given by

$$\frac{BL_t^g}{U_c(t)} = z_t$$

$$\frac{U_c(t)}{P_t} = b\,\frac{U_c(t+1)}{P_{t+1}}R_t$$

$$\frac{U_m(t)}{U_c(t)} = \frac{R_t - 1}{R_t}$$

Finally to close the model we adopt either a *current-looking, backward-looking, or a forward-looking rule:*

$$\tilde{R}_t = \left(\frac{1}{b}\right)\tilde{R}_{t-1} + t\tilde{p}_{t+i}\,,$$

where b > 1 and i = 0 for current-looking, i = -1 for backward-looking, and i = 1 for forward-looking interest rate rules.

Letting $t = q\left(\frac{b-1}{b}\right)$ we solve these rules backwards to yield the equivalent current-

$$\tilde{R}_t = q\tilde{p}_t^{\,p} = q\left(\frac{b-1}{b}\right)\sum_{j=0}^{\infty}\left(\frac{1}{b}\right)^j\tilde{p}_{t+i-j}\,.$$

Notice that if i = Δ or i = -Δ and Δ→0 all three rules converge to the same continuous time rule. Both the current rule and the backward rule are in the spirit of Benhabib et al. The forward case is included for completeness sake.

The key difference between the current rule and the backward rule is whether current inflation is part of the rule. In the case of the backward rule it is not, so the nominal interest rate is predetermined. In the case of the current rule, the nominal interest rate is not predetermined. In the forward case both tomorrow’s inflation and today’s inflation is part of the rule.

Current-looking Interest-Rate Rules.

We first analyze the model if the policy rule is current-looking. Log linearizing the equilibrium conditions for the current looking rule we have

$$\begin{bmatrix} -1 & 0 & q & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ b & 0 & -qA_{34} & 0 \end{bmatrix} \begin{bmatrix} p_{t+1} \\ p_t \\ p_t^p \\ l_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \frac{b-1}{b} & 0 & \frac{1}{b} & 0 \\ 1 & 0 & 0 & A_{44} \end{bmatrix} \begin{bmatrix} p_t \\ p_{t-1} \\ p_{t-1}^p \\ l_t \end{bmatrix}$$

where

$$A_{34} = \frac{KgU_{mc}}{DR_{ss}}$$

$$A_{44} = K\left(1 + \frac{gU_{mc}(R_{ss}-1)}{DR_{ss}} - \frac{gU_{mm}R_{ss}}{DR_{ss}}\right)$$

$$D = \left(U_{cc}U_{mm} - U_{mc}^2\right)c_{ss} > 0$$

The characteristic equation for the above system is

$$J(q) = q^4 - \frac{\{b - qA_{34} + b(1 + b + qA_{34} + A_{44})\}}{bb} q^3 + \frac{\{1 + b + A_{44} + b + (b-1)q(A_{34} + A_{44}))\}}{bb} q^2 - \frac{1}{bb} q = 0$$

We are interested in situations in which the system is locally determinate, but is globally indeterminate because of a supercritical Hopf bifurcation. In this note we will not consider the issue of supercriticality, but leave this for future work. For present purposes we simply assume that any bifurcation is supercritical. We use the terminology "Locally Determinate, Globally Indeterminate" (LDGI) for situations in which we have local determinacy, but a Hopf bifurcation exists. Following Benhabib et. al. we ask whether there exists a LDGI equilibrium as we vary b. Even when there cannot exist a LDGI with respect to b our analysis does not completely rule out the possibility of cycles that may emerge from varying other deep structural parameters such as β and γ.

The above system is locally determinate if two of the eigenvalues lie outside the unit circle while two lie inside the unit circle. By inspection, one root of J is zero, so that we are left with the following cubic:

$$J(q) = q^3 - \frac{\{b - qA_{34} + b(1 + b + qA_{34} + A_{44})\}}{bb} q^2 + \frac{\{1 + b + A_{44} + b + (b-1)q\,(A_{34} + A_{44}))\}}{bb} q \; - \frac{1}{bb} = 0$$

With discrete time a LDGI exists when one root of J is inside the unit circle and the other two are complex conjugates with a norm equal to one. We will henceforth make use of the following assumptions:

Assumptions A1: A_{44} > 0 and J(-1) < 0.

Note the assumption that $A_{44} > 0$ is the analogue of Benhabib et al.'s $A_{21} > 0$. The assumption that $J(-1) < 0$ is extremely weak and is equivalent to

$$b\big(2(1+b) + A_{44}(1+q) + 2qA_{34}\big) + 2(1+b) - 2qA_{34} + (1-q)A_{44} > 0\,.$$

Lemma 1: Suppose that assumptions A1 are satisfied. If $q > 1$, then one or three real roots of J lie in the unit circle. Thus, under A1, a necessary condition for determinacy is $q > 1$.

Proof: The above implies

$$J(1) = \frac{A_{44}\big((b-1)(q-1)\big)}{bb} > 0\,.$$

Since by assumption J(-1) < 0, there are an odd number of real roots in (-1,1). QED

Corollary 1: If J(-1) > 0 and $q > 1$ *then the system is either over-determined or under-determined so that there cannot exist a LDGI.*

Note that we follow Benhabib et al. and consider variations in the coefficient on the weighted average of current and past inflations. The condition for determinacy if we varied the coefficient on current inflation, τ, in the current-looking Taylor rule is

τ + 1/b > 1.

Lemma 2: Under the assumptions A1, $J(\frac{1}{bb}) = 0$ *and b > 1/β is a necessary and sufficient condition for a LDGI with respect to b.*

Proof:

Recall that the product of the three roots of J is equal to 1/bβ. This immediately establishes the necessity of this condition. As for sufficiency, suppose that $J(\frac{1}{bb}) = 0$.

The other two roots have a norm of unity. If they are real, one must be within the unit circle and one must be outside the unit circle. But from Lemma 1, there cannot be an even number of roots in the unit circle. Hence, the remaining two roots must be complex.

QED

Proposition 1: Assume A1 and q *> 1. A necessary and sufficient condition for a LDGI with respect to b is* $(1 - b + q(A_{34} + A_{44})) < 0$.

Proof: From Lemma 2, we need to analyze J(1/bβ).

$$J(1/bb) \equiv b^3 b^3 j(b) =$$

$$b(1-b+qA_{34}+A_{44}q)b^2-[b^2-1+A_{44}(b-1)-bq(A_{34}+A_{44})-qA_{34}]b+(1-b)+qA_{34}$$

A bifurcation exists if j(b) = 0 for some b > 1/β. Note that

$$j(1/b)=\frac{A_{44}\left((1-b)(q-1)\right)}{bb}>0.$$

Suppose that $(1-b+q(A_{34}+A_{44}))<0$. Then as b goes to infinity, j becomes negative.

Hence, there exists a b such that j(b) = 0. Suppose instead that $(1-b+q(A_{34}+A_{44}))>0$.

Since j is convex and j′(1/β) > 0, there does not exist a b > 1/β such that j(b) = 0.

QED.

Corollary 2: A necessary condition for a LDGI with respect to b is that

$$A_{34}+A_{44}=K\left(1+\frac{g(U_{mc}-U_{mm})}{D}\right)<0.$$

Thus a further necessary condition is $U_{mc}<U_{mm}<0$.

Note the assumption that J(-1) < 0 is necessarily true when U_{mc} < 0.

Backward-looking Interest-Rate Rules.

We next show that a LDGI is less likely if the policy rule looks purely backward.

Log linearizing the equilibrium conditions for the backward-looking rule yields

$$\begin{bmatrix}-1 & 0 & q & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ b & 0 & -qA_{34} & 0\end{bmatrix}\begin{bmatrix}p_{t+1}\\ p_t\\ p_t^p\\ l_{t+1}\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & \frac{b-1}{b} & \frac{1}{b} & 0\\ 1 & 0 & 0 & A_{44}\end{bmatrix}\begin{bmatrix}p_t\\ p_{t-1}\\ p_{t-1}^p\\ l_t\end{bmatrix}$$

where A_{34} and A_{44} are as defined earlier. Once again one eigenvalue of the system is zero so that we are left with the following cubic:

$$H(q) = q^3 - \frac{\{b + b(1+b+A_{44})\}}{bb} q^2 + \frac{\{1 + b + A_{44} + qA_{34} + b(1-qA_{34})\}}{bb} q + \frac{q(A_{34} + A_{44})(b-1) - 1}{bb} = 0$$

As before, we will utilize assumptions A1: $A_{44} > 0$ and H(-1) < 0. Note that H(-1) = J(-1). Since H(-1) < 0, H(1) > 0 is necessary for determinacy (as this implies an odd number of roots in the unit circle). Therefore a necessary condition for determinacy is H(1) > 0. Since

$$H(1) = \frac{A_{44}(b-1)(q-1)}{bb}$$

and b > 1 we conclude that θ > 1 (or τ + 1/b > 1) is necessary for local determinacy.

Proposition 2: If q > 1 then there exists a LDGI with respect to b if and only if there exists an a in (-1,1) and a b > 1, such that the following conditions are satisfied:

$$f(a,b) \equiv q(b-1)(1-2a)(A_{44} + A_{34}) - (1 + q(b-1))A_{44} + 2(a-1) + (b-1)(b-1) = 0$$

$$g(a,b) \equiv (q(A_{34} + A_{44}) + (1-b))(b-1) + bA_{44} + 2bb(1-a) = 0$$

$$b(b + q(A_{34} + A_{44})) > 1 + q(A_{34} + A_{44})$$

This implies that a necessary condition for a LDGI with respect to b is that

$(A_{44} + A_{34}) + (1-b) < 0$.

Proof:

We can construct the following polynomial that has real root r and complex roots on the unit circle with a real part equal to a:

$$h(q) = q^3 - (2a + r)q^2 + (1 + 2ra)q \ - r = 0 .$$

Comparing h and H, there exists a LDGI if and only if there exists an a in the unit circle and a b > 1 such that the quadratic and linear coefficients coincide. This provides two linear restrictions:

$$f(a,b) \equiv q(b-1)(1-2a)(A_{44} + A_{34}) - (1 + q(b-1))A_{44} + 2(a-1) + (b-1)(b-1) = 0$$

$$g(a,b) \equiv (q(A_{34} + A_{44}) + (1-b))(b-1) + bA_{44} + 2bb(1-a) = 0 .$$

Since b > 1 g(a,b) = 0 can only be satisfied if $(A_{44} + A_{34}) + (1-b) < 0$. For it to be an LDGI the real root of H must be in the unit circle. As in Proposition 1, the real root must be the negative of the constant term in H:

$$0 < \frac{1 - q(A_{34} + A_{44})(b-1)}{bb} < 1 .$$

Given that S < 0 the lower bound is always satisfied. The upper bound is equivalent to the last condition in the proposition.

QED

Under the case of β = 1, we can prove a stronger result:

Corollary 1: Assume b = 1. There exists an LDGI with respect to b if and only if

$$1 < q < \frac{-\{A_{44}(S+1) + S\}}{S^2}$$

where $S \equiv (A_{44} + A_{34}) < 0$. *Hence, for any S, a sufficiently large q eliminates the LDGI with respect to b.*

Proof:

With β = 1, we can sum f(a,b) = 0 and g(a,b) = 0 and find

$$(b-1)\{2(1-a)(qS+1)+A_{44}(1-q)\}=0\,.$$

Since b > 1, we can use this to solve for (1-a) and substitute this back into g(a,b) which will now be a function only of b:

$$g(b)\equiv bq\{qS^2+S(1+A_{44})+A_{44}\}-qS(qS+1)=0\,.$$

Since g(1) > 0, there exists a LDGI if and only if g′(b) < 0. This is only possible if

$$(qS^2+S(1+A_{44})+A_{44})<0\,.$$

QED

Corollary 2: Assume b = 1. A further necessary condition for an LDGI with respect to b is

$$1<q<\frac{(1+A_{44})^2}{4A_{44}}.$$

Hence a necessary condition for an LDGI to exist is if A_{44} > 1.

Proof:

From the previous corollary a necessary and sufficient condition for an LDGI is

$$f(S)=(qS^2+S(1+A_{44})+A_{44})<0\,.$$

This reaches a minimum at

$$f'(S^*)=(1+A_{44})+2qS^*=0\,.$$

Solving for S and plugging this value into f(S) yields the following necessary condition for an LDGI

$$f(S^*)=\frac{-(1+A_{44})^2}{4q}+A_{44}<0 \text{ or}$$

$$1<q<\frac{(1+A_{44})^2}{4A_{44}}.$$

QED

This condition is in contrast to the conditions for a LDGI when policy includes the current inflation rate. With the current rule, β = 1, and S < 0, there exists a LDGI with respect to b *for all* values of θ > 1. In the case of the backward rule, enough weight on past values of inflation rule out the existence of a LDGI. Thus backward-looking rules make these cycles less likely.

Forward-looking Interest-Rate Rules.

We next show that a LDGI is not possible if the policy rule is forward looking and β=1. Log linearizing the equilibrium conditions for the forward-looking rule yields

$$\begin{bmatrix} -1 & q & 1 \\ \frac{1-b}{b} & 1 & 0 \\ b & -qA_{34} & 0 \end{bmatrix}\begin{bmatrix} p_{t+1} \\ p_t^p \\ l_{t+1} \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{b} & 0 \\ 1 & 0 & A_{44} \end{bmatrix}\begin{bmatrix} p_t \\ p_{t-1}^p \\ l_t \end{bmatrix}$$

where A_{34} and A_{44} are as defined earlier. The characteristic equation is given by the following cubic:

$$H(q)=q^3-\frac{\{b+q(A_{34}+A_{44})+b(1+b+A_{44}-q(A_{34}+A_{44}))\}}{bb-(b-1)qA_{34}}q^2+\frac{\{1+b+A_{44}+b\}}{bb-(b-1)qA_{34}}q-\frac{1}{bb-(b-1)qA_{34}}=$$

As before, we will utilize assumptions A1: $A_{44} > 0$ and H(-1) < 0. Once again H(-1) =

J(-1). All three variables above are jump variables, therefore determinacy requires that all three roots of H lie outside the unit circle. Since H(-1) < 0, H(0) < 0, H(1) < 0 is necessary for determinacy. Since

$$H(1) = \frac{A_{44}(b-1)(q-1)}{bb}$$

and b > 1 we conclude that θ < 1 (or τ + 1/b < 1) is necessary for local determinacy.

We next show that for β close to one there can never be an LDGI for this economy.

Proposition 3: If $b=1$ there can never be a LDGI with respect to b.

Proof:

Since all three are jump variables following our previous proofs a necessary condition for an LDGI is

$$\frac{1}{b-(b-1)qA_{34}} > 1 \text{ or}$$

$$1 - qA_{34} > b(1 - qA_{34})$$

which can never hold since b > 1. QED

The Continuous Time Model

Our focus has been on the discrete time model, but for completeness we note how this MIUF model matches up with the MIPF model in Benbabib et al. In the case of continuous time, the log-linearized system is given by

$$\begin{pmatrix} \dot{l} \\ \dot{p} \\ \dot{p}^{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -q \\ A_{44} & r & qA_{34} \\ 0 & b & -b \end{pmatrix} \begin{pmatrix} l \\ p \\ p^{p} \end{pmatrix}$$

Note that this system matches up identically with Benhabib et al, equations (29)-(30). Our θA_{34} (A_{44}) corresponds to their A_{23} (A_{21}). Following their results we have that a necessary condition for determinacy is $\theta > 1$.

As for bifurcations, in the continuous time model, there exists a LDGI as we vary b if and only if $r+\theta A_{34} < 0$. This is in contrast to either discrete time model. In the case of a current rule, there exists a LDGI if and only if $(1-b+q(A_{34}+A_{44}))<0$. This latter condition is stronger by the term $A_{44} > 0$. As prices approach perfect flexibility ($K\rightarrow\infty$), an LDGI arises in the continuous model whenever $U_{cm} < 0$. But for the discrete-time-current rule, this condition is

$$\left(1+\frac{g}{D}(U_{mc}-U_{mm})\right)<0,$$

a noticeably stronger condition. In summary the LDGI problem is most severe for continuous time, least severe for the discrete-time-backward-rule, with the discrete-time-current rule somewhere in between. A LDGI also does not arise for the forward-rule, however, determinacy is also much more difficult to achieve with a forward-rule likely making this point moot.

IV. Conclusions.

This paper demonstrates that for issues of both local and global equilibrium determinacy, the choice of timing is very important. We have illustrated this with two examples. First, in a continuous time model, an interest rate peg produces equilibrium determinacy while it generates indeterminacy in the corresponding discrete time model. Second, LDGI are much less likely in a discrete time model than in a continuous time model. These differing results are quite troubling. Resolving these timing issues is an important avenue for future research.

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