Working Paper 2000-5/Document de travail 2000-5

Estimating the Fractional Order of Integration of Interest Rates Using a Wavelet OLS Estimator

by

Greg Tkacz

Bank of Canada Banque du Canada

ISSN 1192-5434
ISBN 0-662-28685-5

Printed in Canada on recycled paper

Bank of Canada Working Paper 2000-5

February 2000

Estimating the Fractional Order of Integration of Interest Rates Using a Wavelet OLS Estimator

by

Greg Tkacz

Department of Monetary and Financial Analysis
Bank of Canada
Ottawa, Canada K1A 0G9
gtkacz@bank-banque-canada.ca; 613-782-8591

The views expressed in this paper are those of the author.
No responsibility for them should be attributed to the Bank of Canada.

Contents

Acknowledgements

I would like to thank Joseph Atta-Mensah, Jean-Pierre Aubry, Ben Fung, John W. Galbraith, Jack Selody, and seminar participants at the Bank of Canada for several useful comments and suggestions. I would also like to thank Mark Jensen for providing the wavelet OLS estimator Matlab code.

Abstract

The debate on the order of integration of interest rates has long focused on the I(1) versus I(0) distinction. In this paper, we use instead the wavelet OLS estimator of Jensen (1999) to estimate the fractional integration parameters of several interest rates for the United States and Canada from 1948 to 1999. We find that most rates are mean-reverting in the very long run, with the fractional order of integration increasing with the term to maturity. The speeds of mean-reversion are lower in Canada, due likely to a positive country-specific risk premium. We also demonstrate that yield spreads contain noticeable persistence, indicating that these are also not strict I(0) processes. The consequences of these findings are that shocks to most interest rates and their spreads are very long-lasting, yet not necessarily infinite.

JEL classifications: C13, E43
Bank classifications: Econometric and statistical methods; Interest rates

Résumé

Le débat entourant l'ordre d'intégration des taux d'intérêt est depuis longtemps centré sur l'opposition I(1)-I(0). Dans cette étude, l'auteur a recours à l'estimateur des moindres carrés ordinaires à ondelettes de Jensen (1999) pour estimer le paramètre d'intégration fractionnaire de divers taux d'intérêt canadiens et américains sur la période allant de 1948 à 1999. Il constate que la plupart des taux reviennent à la moyenne à très long terme et que l'ordre d'intégration fractionnaire augmente avec l'éloignement de l'échéance. Le retour à la moyenne est plus lent au Canada, probablement en raison de la présence d'une prime de risque-pays positive. L'auteur démontre également que les écarts de rendement affichent une persistance notable, ce qui donne à penser qu'eux non plus ne constituent pas de véritables processus intégrés d'ordre zéro. Si l'on en croit ces résultats, les chocs imprimés à la plupart des taux d'intérêt de même qu'aux écarts entre ceux-ci ont des effets très persistants, mais pas forcément de durée illimitée.

Classifications JEL: C13, E43
Classifications de la Banque: Méthodes économétriques et statistiques; Taux d'intérêt

1. Introduction

Interest rates play important roles in both macroeconomics and finance. In macroeconomics, for example, they are crucial to the conduct of monetary policy, as policy is primarily implemented in most developed countries through the setting of short-term interest rates. Interest rate movements in turn have an impact on spending and saving decisions, thereby affecting macroeconomic activity. The finance literature is also prolific in models of, and uses for, interest rates, since their movements are crucial to investment and portfolio decisions.

In modern time-series econometrics, it has become standard practice to verify the order of integration of each variable entering a model. If variables are found to be integrated of order one, denoted I(1), then the focus shifts towards locating cointegrating relationships between these variables in order to exploit any long-run equilibrium properties of the data. The order of integration of each variable is usually determined using one or more of the countless unit root tests available, where the null of a unit root is tested against the alternative of (mean or trend) stationarity, denoted I(0), or vice-versa. Dolado, Jenkinson, and Sovilla-Rivero (1990), for example, present a comprehensive survey on unit roots.

The order of integration of nominal interest rates has long remained a contentious issue. In theory, it is impossible for interest rates to follow a unit root process without drift, since this would impose no bounds on the movements of such variables; in practice, however, they cannot be negative. If a drift term is included, it is also difficult to justify how interest rates can tend to infinity in the presence of a unit root. This would imply that expected inflation would also follow a random walk, with the consequence that its path cannot be influenced by monetary policy. In macroeconomics, one would also imagine that shocks to interest rates resulting from, for example, a currency crisis would eventually dissipate once the crisis subsides. Thus, country-specific risk premiums embedded within interest rates would have to be relatively constant, with pertubations being short-lived. Similarly, a stationary interest rate process is also a requirement of financial term structure models of the type of Cox, Ingersoll, and Ross (1985), since the short-term interest rate factor has to be mean-reverting in order to make the model tractable.

In practice, however, several authors have failed to reject the unit root hypothesis for interest rates. For example, in their tests for the presence of unit roots in several key macroeconomic variables, Nelson and Plosser (1982) find that the interest rate series is an I(1) process. Although reversing most of the unit root conclusions through the introduction of structural breaks in the unit root tests, Perron (1989) is unable to reverse the Nelson and Plosser conclusion for interest rates.

Because empirical studies are unable to reject the unit root hypothesis for interest rates, it is quite likely that the I(0) alternative is too stringent for the unit root tests used. Instead, some authors have suggested that interest rates may in fact be fractionally integrated, or I(d), where $0 < d < 1$. When d is estimated to lie between 0 and 0.5, the process is said to exhibit long memory in the sense that the autocorrelation function (ACF) decays at a much slower rate than the exponential decay of the ACF exhibited by stationary ARMA processes. In other words, the rate of interest at time t is correlated to the rate at t-k for some $k > 0$; this correlation diminishes, but is non-negligible, as k increases. When d equals 0.5 or greater but less than 1, the process will still return to its equilibrium in the long run but will also possess an infinite variance.

In previous work on fractional integration and interest rates, Backus and Zin (1993) show that there is some evidence of long memory in the 3-month zero-coupon rate, and that allowing for long memory in the short rate improves the fitted mean and volatility yield curves. Pfann, Schotman, and Tschernig (1996) also find that allowing for long memory in the short-term process improves the fit of term structure models.

Intuitively, interest rates may be slowly mean-reverting even if few shocks tend to have long-lasting effects. Parke (1999) introduces error-duration models that can generate series displaying long memory. The basic idea is that shocks can be of a stochastic magnitude and of stochastic duration; an observed value of a variable at a point in time is essentially a sum of all shocks that survive up to that point. If only a few shocks are long-lasting, then a long-memory process can ensue. In the case of nominal interest rates, significant shocks to inflation expectations—such as those resulting from shifts in policy regimes—may indeed take a long time to dissipate, and thus nominal interest rates may take a long time prior to reverting to their respective means. If a number of notable long-lasting shocks occur within a short period, interest rates may indeed behave like non-stationary processes.

If interest rates are I(d), then this may help explain why the null of a unit root cannot be rejected for such variables. For example, Diebold and Rudebusch (1991) conclude that the standard Dickey-Fuller test has low power against I(d) alternatives. As such, outright estimation of the order of integration d may prove more useful than the use of low-power tests in determining whether interest rates possess a unit root.

Several methods have been proposed to estimate the fractional integration parameter d. Among the most popular is the frequency domain method of Geweke and Porter-Hudak (1983), henceforth GPH. Unfortunately, this estimator possesses no satisfactory asymptotic properties. Sowell (1990) proposes a maximum likelihood method to estimate d and the ARMA(p,q) parameters jointly; as with all maximum likelihood methods, it can perform poorly if the model is misspecified. Backus and Zin (1993) use Sowell's estimator in their empirical work, but the divergent estimates of d that they obtain for each ARFIMA specification reduce the usefulness of their results.

Jensen (1999) proposes a new estimator of d that is constructed using wavelets. Wavelets can be described most simply as functional transforms in the same spirit as Fourier transforms, but with properties that allow them to identify more effectively either long rhythmic behaviour or short-run phenomena. Jensen's wavelet ordinary least squares (WOLS) estimator of d is derived from the smooth decay of long-memory processes. From Jensen's simulations, the mean squared errors (MSEs) of the estimates of d are roughly four to six times smaller than the MSEs of the GPH estimator at the sample sizes that are of interest to us in this study (sample sizes T = 256 or 512 observations). This estimator and its properties are discussed more fully below.

Once a proper estimate of the fractional order of integration is obtained, more robust models can be constructed that exploit the properties of the data. For example, Cheung and Lai (1993) show how one can utilize the information content of fractionally integrated processes in an analysis of purchasing power parity. More generally, if we assume that two processes x_1 and x_2 are fractionally integrated such that x_1~I(d) and x_2~I(d) with $0 < d < 1$, then a long-run fractional cointegrating relationship may exist for $y = f(x_1,x_2)$ ~ I(d - b), where $b < d$, which may be short-run stationary if (d - b) = 0, or follow a long-memory process if (d - b) < 1.0. If the latter, then x_1 and x_2 are said to be fractionally cointegrated, and an error-correction term would be most useful in determining the equilibrium relationship to which the series will revert in the very long run. Our discussion on fractional cointegration is expanded in Section 4 within the context of yield spreads and risk premiums.

To motivate the important implications of fractional cointegration, consider for example the debate surrounding the empirical work of Baillie and Bollerslev (1989). These authors estimated a cointegrating vector for a group of seven exchange rates from industrial countries, assuming that each exchange rate was I(1). Diebold, Gardeazabal, and Yilmaz (1994) subsequently argued that the error-correction term arising from the Baillie and Bollerslev work failed to improve over a simple martingale in an ex ante forecasting experiment, casting doubt on whether the exchange rates were cointegrated at all. Reconsidering their original findings, Baillie and Bollerslev (1994) find that the error-correction term arising from their original model is not I(0) as originally believed, but I(0.89); that is, their original variables turn out to be fractionally cointegrated. This leads them to conclude that adjustments to equilibrium are likely to take several years to complete, and therefore their estimated error-correction term will yield improvements in forecast performance only several years into the future.

Estimates of the fractional order of integration of interest rates should be of interest to policy-makers for at least two reasons. First, knowledge of d will enable them to determine whether shocks to interest rates are short-lived, long-lived, or infinitely lived. Second, if $d < 1$, then one may suspect that cointegrating relationships involving interest rates may not be precisely I(0), with the consequence that adjustments to re-establish an equilibrium state may follow long-memory processes. As Baillie and Bollerslev have discovered, this implies that fractionally cointegrated relationships may yield noticeable gains in forecast accuracy only within the context of longer-term forecasts.

In the next section, we motivate the intuition underlying wavelets and discuss the WOLS estimator more fully. In Section 3, we use the WOLS estimator to obtain the fractional integration parameters for the zero-coupon term structure data of McCulloch and Kwon (1993) for the United States, and standard market rates for Canada. In Section 4, we investigate whether yield spreads and risk premiums are fractionally integrated. The final section concludes.

2. Methodology

2.1 Basic wavelet theory

In mathematics, it is often possible to approximate a complicated function as a linear combination of several simple expressions. One of the better-known examples is that of spectral, or Fourier, analysis where, by the spectral representation theorem, any covariance-stationary process x_t can be expressed as a linear combination of sine and cosine functions in the frequency domain. For example, the Fourier series of any real-valued function $f(x)$ on the [0,1] interval is expressed as

$$f(x) = b_0 + \sum_{k=1}^{\infty} [b_k \cos 2\pi kx + a_k \sin 2\pi kx], \tag{1}$$

where the parameters a_k, b_0, and b_k, for $\forall\, k$ can be solved using least squares.

However, few economic series follow the smooth cycles suggested by sine and cosine functions, thereby making Fourier analysis less appealing for economists. A recently developed alternative to Fourier transforms are *wavelet transforms*, where the same function $f(x)$ can be expressed in the wavelet domain in the following manner:

$$f(x) = c_0 + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j - 1} c_{jk} \psi(2^j x - k) \tag{2}$$

with $\psi(x)$ defined as

$$\psi(x) = \begin{cases} 1 \text{ if } 0 \le x < \frac{1}{2} \\ -1 \text{ if } \frac{1}{2} \le x < 1 \\ 0 \text{ otherwise} \end{cases} . \tag{3}$$

The group of functions $\psi_{jk}(x) = \psi(2^j x - k)$ for $j \ge 0$ and $0 \le k < 2^j$ are orthogonal and collectively form a basis in the space of all square-integrable functions $\mathbf{L}_2$ along the [0,1] interval. The index j is the dilation (or scaling) index, which compresses the function $\psi(x)$, and the index k is the transition index that shifts the function $\psi(x)$. More generally, any such basis in $\mathbf{L}_2(\mathbf{R})$ is known as a wavelet, and (3) is more commonly known as the Haar wavelet.

Several different wavelets have been proposed, which usually involve smoothing the step function (3). The Daubechies (1988) wavelet is an example of such a smooth wavelet; it is used in our applications. Our choice of the Daubechies wavelet is motivated by two factors: first, by its common usage in many applications outside economics (especially signal processing); and second, because the desirable properties of the WOLS estimator were demonstrated with the use of this wavelet. Several alternative wavelets are presented in, for example, Vidakovic (1999).

As noted by Jensen (1999), the strengths of wavelets lie in their ability to simultaneously localize a process in time and scale. They can zoom in on a process's behaviour at a point in time, which is a distinct advantage over Fourier analysis. Alternatively, wavelets can also zoom out to reveal any long and smooth features of a series. The interested reader is referred to Strang (1993) and Strichartz (1993) for more extensive expositions on wavelets.

2.2 Fractional integration and the wavelet OLS estimator

Consider the random process x_t,

$$(1-L)^d x_t = \varepsilon_t , \tag{4}$$

where L is the lag operator, ε_t is i.i.d. normal with zero mean and constant variance σ^2, and d is a differencing parameter. When $d = 0$, the process x_t is simply equal to ε_t, so $x_t \sim N(0,\sigma^2)$, or $x_t \sim I(0)$. When $d = 1$, however, x_t follows a unit root process (without drift), implying it has a zero mean with infinite variance.

More generally, if we allow d to take non-integer values, the process x_t is said to be fractionally integrated, making (4) an ARFIMA process (i.e., fractionally integrated ARMA). As shown by Hosking (1981), when $0 < d < \frac{1}{2}$, the autocovariance function of x_t declines hyperbolically to zero, making x_t a long-memory process. If $\frac{1}{2} \le d < 1$, x_t has an infinite variance; however, it will still revert to its mean (or trend) in the very long run. Table A below summarizes the different values of d and the corresponding consequences for the mean (or trend), variance, and duration of a shock.

Jensen (1999) demonstrates that, for an I(d) process x_t with $|d| < \frac{1}{2}$, use of the autocovariance function implies that the wavelet coefficients c_{jk} in (2) are distributed as $N(0, \sigma^2 2^{-2jd})$. If $R(j)$ denotes the wavelet coefficient's variance at scale j, then after taking logarithms, an estimate of d can be obtained from

$$\ln R(j) = \ln \sigma^2 - d \ln 2^{2j}. \quad (5)$$

Thus, the wavelet transform is applied to the autocovariance function of a particular interest rate, not to the interest rate itself. The wavelets are used only in the estimation of the d consistent with the observed autocovariance function. Furthermore, because of the form of the wavelet expansion (2), it should be noted that the number of observations for the underlying process x_t must be a factor of 2.

Table A: Summary of fractional integration parameter values

d	Mean (or trend) and variance	Shock duration
$d = 0$	Short-run mean-reversion Finite variance	Short-lived
$0 < d < 0.5$	Long-run mean-reversion Finite Variance	Long-lived
$0.5 \le d < 1$	Long-run mean-reversion Infinite variance	Long-lived
$d = 1$	No mean-reversion Infinite variance	Infinite
$d > 1$	No mean-reversion Infinite variance	Infinite; effect increases as time moves forward

3. Estimates of *d* for interest rate levels

We consider several nominal interest rates for the United States and Canada. Given the sample size restrictions for the implementation of the estimation, we use monthly data such that we have common sample sizes of $T = 2^8 = 256$ and $T = 2^9 = 512$ observations. We use 1-, 3-, 6-, and 9-month T-bill rates, and 1-, 3-, 5-, and 10-year zero-coupon bond rates from 1948:7 to 1991:2, all obtained from McCulloch and Kwon (1993) for the United States. CANSIM data for commercial paper and government bond rates are used for Canada from 1956:11 to 1999:6. The computations are performed using the Matlab toolbox *Wavekit* of Ojanen (1998).

3.1 United States

In Table 1 we present the estimates of the fractional integration parameters over the full sample. To verify the robustness of our results, we consider the unsmooth Haar wavelet, and three different degrees of smoothing for the Daubechies wavelet, where the smoothness increases with the order of the wavelet. Three important findings emerge. First, all estimated parameters are at least two standard errors above 0.5, indicating that all rates have an infinite variance. Second, all rates on securities with maturities of one year or less are at least two standard errors below 1.0, implying that they are not strict unit root processes. The implication is that they have a tendency to revert back to their means in the very long run. Finally, the fractional integration parameters increase as the term to maturity increases. We therefore find that the longest rates, the 5- and 10-year rates, are the rates that display properties that most closely resemble unit processes. This evidence is strengthened by examining the autocorrelation functions plotted in Figure 1. We see that the decay of the autocorrelations of the 1-month rate is more pronounced than it is for the 10-year rate. However, both series display very slow decay overall, with the autocorrelations remaining positive even after five years.

The finding of increasing “non-stationariness” with increases in the term to maturity is probably one of the more interesting results. Since we are using zero-coupon rates, any differences we uncover are likely due to the effects of term premia. Thus, because the order of integration increases with the term, we suspect that term premia are non-constant and fluctuate enough to induce the longest rates to follow unit root processes.

In Table 2, we trim our sample to $2^8 = 256$ observations to examine whether the estimated parameters change in notable manners. This exercise is useful because researchers often do not use data from the 1940s and 1950s, since the thin bond markets in existence at that time did not cause interest rates to fluctuate much in response to market conditions, as they do today. The results are now less straightforward. We find that the standard errors of the estimates are noticeably larger, as we would expect given the smaller sample size. The implication is that there are now several estimated parameters within two standard errors of 0.5, indicating that some interest rates may exhibit properties of long-run mean-reversion with finite variance. This feature emerges for all rates using the Daubechies wavelets. The second result is that the longer-term rates are still within two standard errors of the unit root scenario. However, the significant uncertainty surrounding the estimates prevents us from making any firm conclusions other than we can still exclude zero as a possible order of integration.

Examining the autocorrelation functions for this sample (Figure 2), we now find that both short- and long-term rates decay more quickly, with autocorrelations reaching zero after 3.5 years for the 1-month rate and 4.5 years for the 10-year rate. Based on this evidence, we find that there is significant improvement in the rate of mean-reversions for all interest rates over the shorter sample, and this is reflected in the estimated parameters. We can also state that the shorter rates are less non-stationary than the longer rates, again due presumably to the effects of term premia.

3.2 Canada

In Table 3, we present the fractional integration parameters for the Canadian interest rates from 1956 to 1999. Using the Haar and Daubechies-20 wavelets, we find that the order of integration tends to rise as the term to maturity increases from one month to ten years. As with the U.S. results, we find that the long-term rate is most likely to follow a unit root process. Results using the two other wavelets are largely inconclusive, given the large standard errors associated with the estimates. The autocorrelation functions in Figure 3 demonstrate that short-term rates revert to their means more quickly, consistent with lower orders of integration.

When focusing on the second half of the sample (Table 4), we find that all wavelets yield very similar estimates. Unlike our previous results, we find that the orders of integration on the two shortest rates are somewhat higher than for the longer-term rates. Recalling that these are rates on commercial paper, which carry a greater default risk than government bonds, we can surmise that the relatively higher estimates on the short rates are due to the changing effects of such risks. The fractional integration parameters for bond rates from one year onwards display a similar increasing pattern that we have explained previously as likely due to varying term premiums.

The orders of integration on these variables estimated for data covering the last 21 years suggest that all rates follow near unit root processes, clearly are not I(0), and would appear to be long-run mean-reverting. Figure 4 suggests that after five years the autocorrelations of both short and long rates remain positive, thereby providing visual evidence that Canadian rates have longer memory than their U.S. counterparts.

4. Estimates of *d* for spreads and real rates

The purpose of this section is to estimate the orders of integration of standard transformations that are often applied to nominal interest rates. We consider both long-short yield spreads and adjustments for inflation.

We can decompose the long (10-year) and short (3-month) nominal interest rates in the following manner:

$$i^{10} = r^{10} + \pi^e + rpcan + term \tag{6}$$

$$i^3 = r^3 + \pi^e + rpcan \tag{7}$$

where r is the real rate, π^e is expected inflation, *rpcan* is a country-specific risk-premium (which is assumed to equal zero for the United States and to be greater than zero for Canada), and *term* is a positive term risk premium. Subtracting (7) from (6), we find that the yield spread can be expressed in two different manners:

$$(i^{10} - i^3) = (r^{10} + rpcan + term) - (r^3 + rpcan), \tag{8}$$

or

$$(i^{10} - i^3) = (i^{10} - \pi^e) - (i^3 - \pi^e). \tag{9}$$

Equation (8) states that the nominal yield spread equals the difference between risk-premium-adjusted real rates, while (9) states that it equals the inflation-adjusted nominal rates. The real rates, country-specific risk premium, and the term premium are all unobservable. As such, we cannot formally estimate the orders of integration of the components in (8). However, by using actual inflation as a proxy for expected inflation, we can at least estimate the fractional orders of integration of all terms entering (9). This allows us to determine whether our proxies for the real rates follow long-memory processes.

Recall that two I(d) variables are said to be fractionally cointegrated if a linear combination of these variables yields an I(d-b) series, where $d-b < d$. If the two variables are of different orders of integration, say d_1 and d_2, they are fractionally cointegrated if the resulting linear combination yields an I(d-b) series, where $d = \min(d_1,d_2)$. We are therefore interested in knowing whether the fractional order of integration of the yield spread is lower than the orders of integration of both nominal and real rates, which is evidence of fractional cointegration between the rates.

4.1 United States

In Table 5, we present the orders of integration of (expected) inflation, nominal rates, nominal rates less expected inflation, and the yield spread. The rates used here are obtained from the St. Louis Fed, and expected inflation is the year-over-year growth of total CPI. Beginning with the nominal rates, we find that the orders of integration are all above 0.86, while the order of integration of the spread is 0.66 or less. This implies that the simple difference of the rates, which is the equivalent of a [1, -1] cointegration vector, yields a process of lower order of integration. However, the order of integration of the yield spread is noticeably above 0.0, implying that it follows a long-memory process.

In the same table, we also present the orders of integration of the nominal rates less expected inflation, as denoted in (9). For each individual wavelet, we find that the orders of integration of the "expected-inflation-adjusted" rates are again greater than the order of integration of the yield spread. For example, for the Daubechies-12 wavelet, we find the order of the long and short rates to be 0.8830 and 0.7348 respectively, both larger than the spread order of 0.6625. This implies that both of these series are also fractionally cointegrated. That is, the difference between the estimates of long and short *real* rates yields a long-term relationship that is mean-reverting.

4.2 Canada

We present the Canadian results in Table 6. The short rate used here is a 90-day treasury bill rate. As with the United States, we find that the nominal rates are fractionally cointegrated, since the simple yield spread has an order of integration lower than the individual nominal rates. In fact, the order of integration is around 0.65, a similar order as for the United States.

The finding that the long-short yield spread is a long-memory process is somewhat surprising, since it has always been believed to be a stationary I(0) process. However, it is consistent with empirical work that has found it to be a good indicator of economic activity, peaking in explanatory power at the 4- to 6-quarter forecast horizon. As explained by Baillie and Bollerslev (1994), long-memory error-correction terms should possess adequate forecasting power only at longer horizons. The spread is, in fact, a [1,-1] cointegration vector; therefore, if it is a long-memory process, it should be most useful at longer horizons. Short-memory variables, such as the quarterly growth rate of money, should be superior short-run predictors.

Returning to Table 6, subtracting expected inflation from the nominal rates, we find that the orders of integration of either the short or long rate are below the order of integration of the yield spread. For example, with the Daubechies-12 wavelet, the order of integration of the inflation-adjusted short rate is 0.6798; for the long rate, it is 0.6904. The simple yield spread, however, has an order of integration of 0.6822. This implies that these rates are either not cointegrated, or that a [1, -1] cointegration vector is not appropriate. We may conjecture that the country-specific risk-premium terms in (8) are the cause of the additional non-stationarity. This would be consistent with our earlier observation that nominal Canadian rates have larger fractional orders of integration than their U.S. counterparts. The economic implication is that a world interest rate shock will be more rapidly absorbed by U.S. rates than Canadian rates, since the Canadian rates are also affected by a non-stationary risk premium.

5. Conclusion

This paper estimates the fractional integration parameters for several interest rates using a recent estimator with desirable properties. The purpose is to contribute to the debate on the order of integration of nominal interest rates. The estimated orders of integration may be of use to macroeconomic and financial modellers who seek more robust results.

Our findings differ somewhat over U.S. and Canadian rates. For the United States from 1948 to 1991, all short-term interest rates are long-run mean-reverting, while longer-term rates are most likely to follow unit root processes, as the estimated fractional integration parameters increase with the term to maturity. Non-constant term premia may be the cause of this last result. When restricting our attention to the latter half of the sample, we find that the evidence in favour of non-stationarity is diminished. This leaves open the possibility that some rates, especially at the shorter horizons, may be following stationary long-memory processes.

Overall, we conclude that the unit root hypothesis is unduly harsh for the United States except for the longer-term interest rates. Furthermore, the assumption of short-run mean-reversion is also strongly rejected by the data. The hypothesis that nominal interest rates follow long-memory processes seems the most plausible.

Canadian rates also exhibit strong persistence over the full sample. Unlike the U.S. rates, however, this persistence remains even over the second half of the sample. This indicates that shocks to interest rates take longer to dissipate in Canada than the United States. As we noted above, long-term bonds may be more non-stationary than short-term bonds due to the additional risk captured in the term premia. Canadian bonds are usually riskier than their American counterparts due to, for example, political uncertainty and exchange rate movements. This additional element of risk may be reflected in the larger order of integration of Canadian bonds in the last 20 years.

For the applied researcher, the following conclusions emerge from our findings. First, the rate of mean-reversion decreases with the term to maturity, with longer-term rates reverting more slowly, if at all, to their means than short-term rates. Second, if these interest rates are used in cointegration analysis, then the underlying vectors may not be strict I(0) processes. In other words, if the cointegrating relationship is fractionally integrated, then adjustments to shocks may take a long time to be finalized. Finally, the fractional order of integration may indicate whether a given variable would be most adequate as a short-run or long-run indicator. A I(0.60) variable may be preferable for long-run forecasts, while an I(0) variable would be most appropriate for the short-run.

Table 1: Estimated fractional integration parameters (*d*), United States
1948:7 to 1991:2 (512 observations)

Rate	Wavelet			
	Haar	Daubechies-4	Daubechies-12	Daubechies-20
1-month	0.8127 (0.0334)	0.8479 (0.0346)	0.8608 (0.0318)	0.8550 (0.0324)
3-month	0.8444 (0.0329)	0.8806 (0.0387)	0.8891 (0.0318)	0.8866 (0.0331)
6-month	0.8533 (0.0325)	0.8909 (0.0402)	0.8950 (0.0285)	0.8928 (0.0281)
9-month	0.8570 (0.0324)	0.8930 (0.0387)	0.8976 (0.0258)	0.8930 (0.0246)
1-year	0.8701 (0.0319)	0.9010 (0.0365)	0.9089 (0.0231)	0.9007 (0.0214)
3-year	0.9267 (0.0336)	0.9414* (0.0337)	0.9547 (0.0185)	0.9424 (0.0152)
5-year	0.9550* (0.0395)	0.9640* (0.0346)	0.9791* (0.0202)	0.9672* (0.0185)
10-year	0.9885* (0.0462)	0.9951* (0.0327)	1.0098* (0.0215)	0.9948* (0.0228)

Note: Standard errors in parentheses. A (*) indicates that the unit root is within two standard errors of the estimated *d*.

Table 2: Estimated FI parameters (*d*), United States
1969:11 to 1991:2 (256 observations)

Rate	Wavelet			
	Haar	Daubechies-4	Daubechies-12	Daubechies-20
1-month	0.7168 (0.0491)	0.7075† (0.1066)	0.6756† (0.1335)	0.4706† (0.2474)
3-month	0.7604 (0.0521)	0.7491 (0.1151)	0.7007† (0.1425)	0.5667*† (0.2186)
6-month	0.7654 (0.0545)	0.7681 (0.1098)	0.7147† (0.1325)	0.5244*† (0.2404)
9-month	0.7681 (0.0524)	0.7648 (0.1103)	0.7086† (0.1338)	0.5527† (0.2205)
1-year	0.7845 (0.0492)	0.7609 (0.1167)	0.7025† (0.1441)	0.6176† (0.1880)
3-year	0.8464 (0.0415)	0.7118*† (0.1713)	0.5351*† (0.2692)	0.7800* (0.1181)
5-year	0.8737 (0.0480)	0.6820*† (0.2087)	0.5456*† (0.2854)	0.8389* (0.1059)
10-year	0.9024* (0.0558)	0.6327*† (0.2630)	0.7217*† (0.2115)	0.8998* (0.0974)

Note: Standard errors in parentheses. A (*) indicates that the unit root is within two standard errors of the estimated *d*, while (†) indicates that *d* is within two standard errors of 0.5, the threshold below which the series can have a finite variance.

Table 3: Estimated fractional integration parameters (*d*), Canada
1956:11 to 1999:6 (512 observations)

Rate	Wavelet			
	Haar	Daubechies-4	Daubechies-12	Daubechies-20
1-month	0.8157 (0.0281)	0.7042† (0.1397)	0.3892*† (0.3201)	0.7852 (0.0955)
3-month	0.8206 (0.0326)	0.7241† (0.1301)	0.5765*† (0.2190)	0.7887 (0.1024)
1- to 3-years	0.8611 (0.0214)	0.4918*† (0.2653)	0.7488*† (0.1295)	0.8559 (0.0703)
3- to 5-years	0.8762 (0.0246)	0.4588*† (0.2914)	0.7616*† (0.1309)	0.8671* (0.0687)
5- to 10-years	0.9019 (0.0286)	0.5226*† (0.2713)	0.7889* (0.1294)	0.8944* (0.0688)
10-years and over	0.9414 (0.0274)	0.4036*† (0.3624)	0.8242* (0.1288)	0.9267* (0.0686)

Note: Standard errors in parentheses. A (*) indicates that the unit root is within two standard errors of the estimated *d*, while (†) indicates that *d* is within two standard errors of 0.5, the threshold below which the series can have a finite variance.

Table 4: Estimated fractional integration parameters (*d*), Canada
1978:3 to 1999:6 (256 observations)

Rate	Wavelet			
	Haar	Daubechies-4	Daubechies-12	Daubechies-20
1-month	0.8598 (0.0272)	0.9123* (0.0444)	0.9412* (0.0385)	0.9326 (0.0243)
3-month	0.8752 (0.0285)	0.9202* (0.0454)	0.9529* (0.0362)	0.9520* (0.0266)
1- to 3-years	0.8751 (0.0267)	0.8983 (0.0426)	0.9378 (0.0259)	0.9299 (0.0116)
3- to 5-years	0.8841 (0.0307)	0.8997* (0.0515)	0.9438* (0.0310)	0.9303 (0.0147)
5- to 10-years	0.9008 (0.0359)	0.9059* (0.0644)	0.9511* (0.0337)	0.9438 (0.0170)
10-years and over	0.9226 (0.0384)	0.9184* (0.0705)	0.9603* (0.0332)	0.9572 (0.0209)

Note: Standard errors in parentheses. A (*) indicates that the unit root is within two standard errors of the estimated *d*.

Table 5: Estimated FI parameters (*d*)

United States spread, real and nominal rates

1978:3 to 1999:6 (256 observations)

Rate or Spread	Wavelet			
	Haar	Daubechies-4	Daubechies-12	Daubechies-20
π^e	1.0142* (0.0682)	0.9282* (0.1567)	0.9421* (0.1608)	1.0131* (0.1135)
i^3	0.8645 (0.0410)	0.8814* (0.0634)	0.9394* (0.0592)	0.9423* (0.0414)
i^{10}	0.9226* (0.0611)	0.9502* (0.0871)	1.0099* (0.0542)	1.0122* (0.0275)
$(i^3 - \pi^e)$	0.6402† (0.1112)	0.7104 (0.0809)	0.7348 (0.0966)	0.7530 (0.0716)
$(i^{10} - \pi^e)$	0.7560*† (0.1378)	0.6807*† (0.2555)	0.8830* (0.1234)	0.9383* (0.0906)
$(i^{10} - i^3)$	0.3457† (0.2433)	0.6549† (0.0969)	0.6625† (0.1054)	0.5948† (0.1536)

Note: Standard errors in parentheses. A (*) indicates that the unit root is within two standard errors of the estimated *d*, while (†) indicates that *d* is within two standard errors of 0.5, the threshold below which the series can have a finite variance.

Table 6: Estimated FI parameters (*d*)

Canada, spread, real and nominal rates

1978:3 to 1999:6 (256 observations)

Rate or spread	Wavelet			
	Haar	Daubechies-4	Daubechies-12	Daubechies-20
π^e	1.0085* (0.0563)	0.9693* (0.0716)	1.0095* (0.0714)	1.0180* (0.0565)
i^3	0.8843 (0.0305)	0.9367* (0.0480)	0.9714* (0.0364)	0.9732* (0.0329)
i^{10}	0.9226 (0.0384)	0.9184* (0.0705)	0.9603* (0.0332)	0.9572 (0.0209)
$(i^3 - \pi^e)$	0.4514† (0.1642)	0.6378† (0.0823)	0.6798 (0.0845)	0.6688† (0.0900)
$(i^{10} - \pi^e)$	0.7333 (0.0816)	0.6984† (0.1052)	0.6904† (0.1311)	0.3536† (0.3288)
$(i^{10} - i^3)$	0.6259† (0.0879)	0.6394† (0.1202)	0.6822† (0.1120)	0.7142 (0.0875)

Note: Standard errors in parentheses. A (*) indicates that the unit root is within two standard errors of the estimated *d*, while (†) indicates that *d* is within two standard errors of 0.5, the threshold below which the series can have a finite variance.

Figure 1: Autocorrelation functions, U.S. 1-month and 10-year rates, 1948:7 to 1991:2

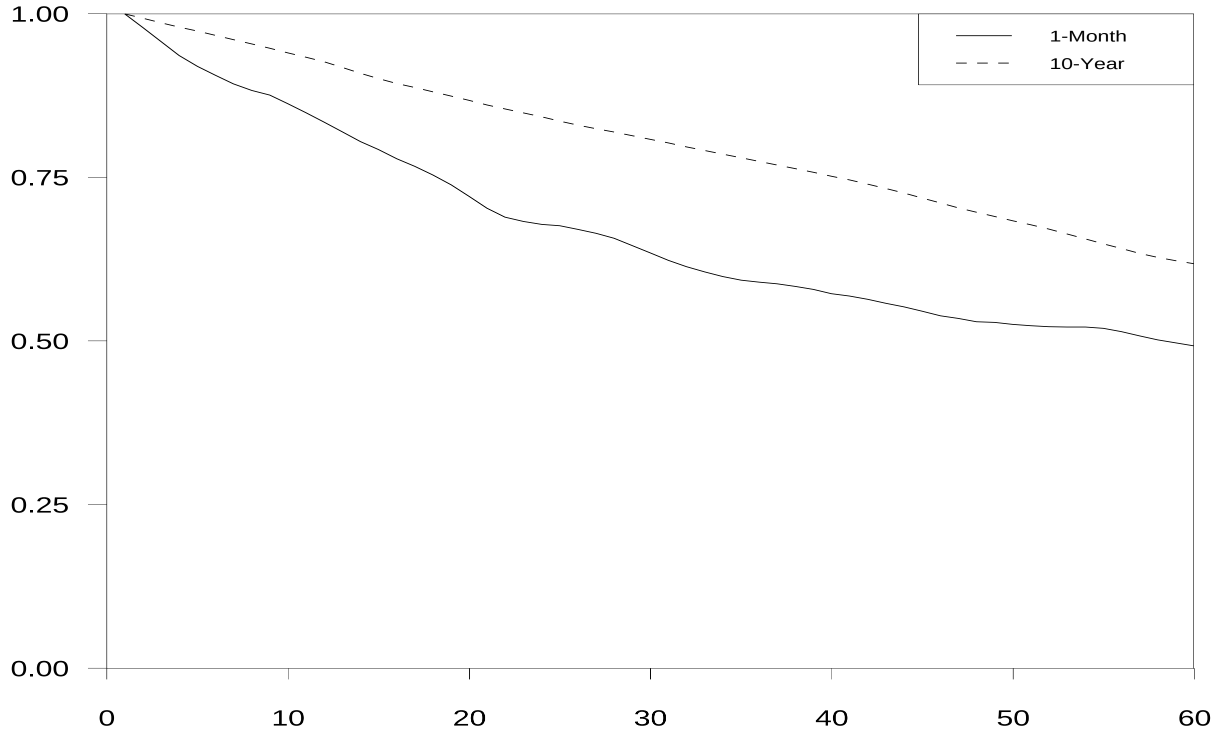


Figure 2: Autocorrelation functions, U.S. 1-month and 10-year rates, 1969:11 to 1991:2

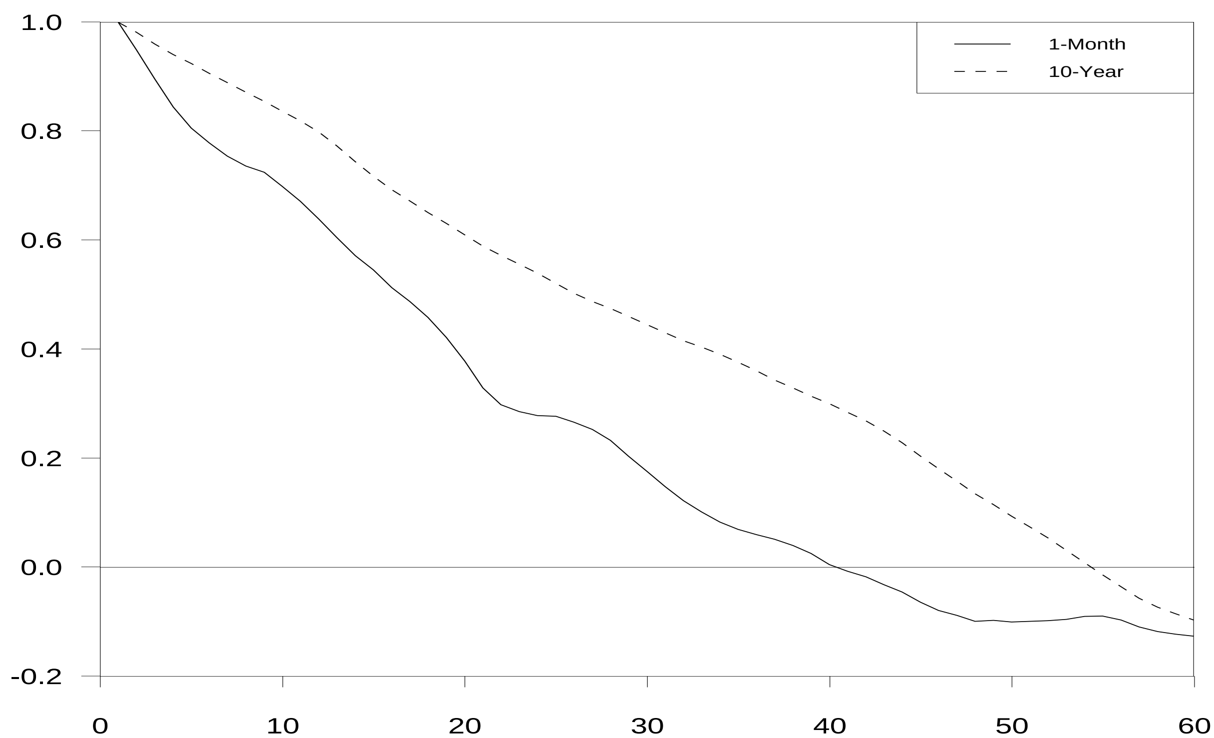


Figure 3: Autocorrelation functions, Cdn. 1-month and 10-year rates, 1956:11 to 1999:6

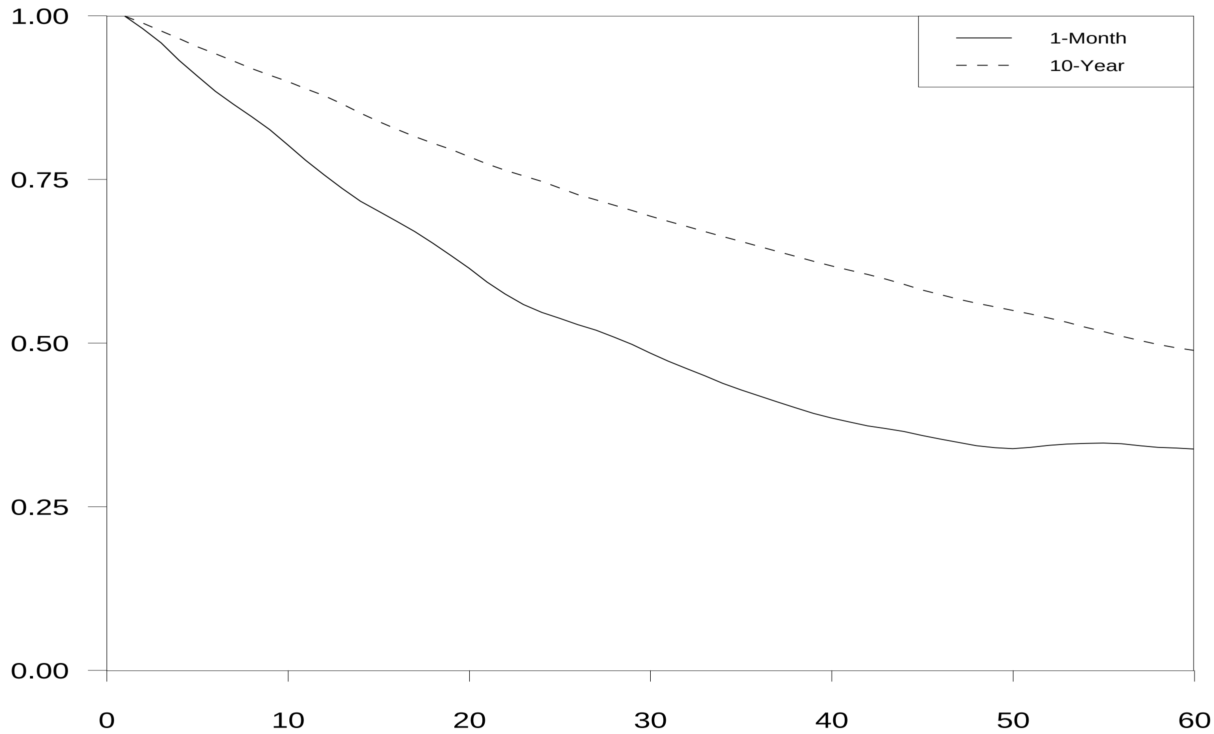
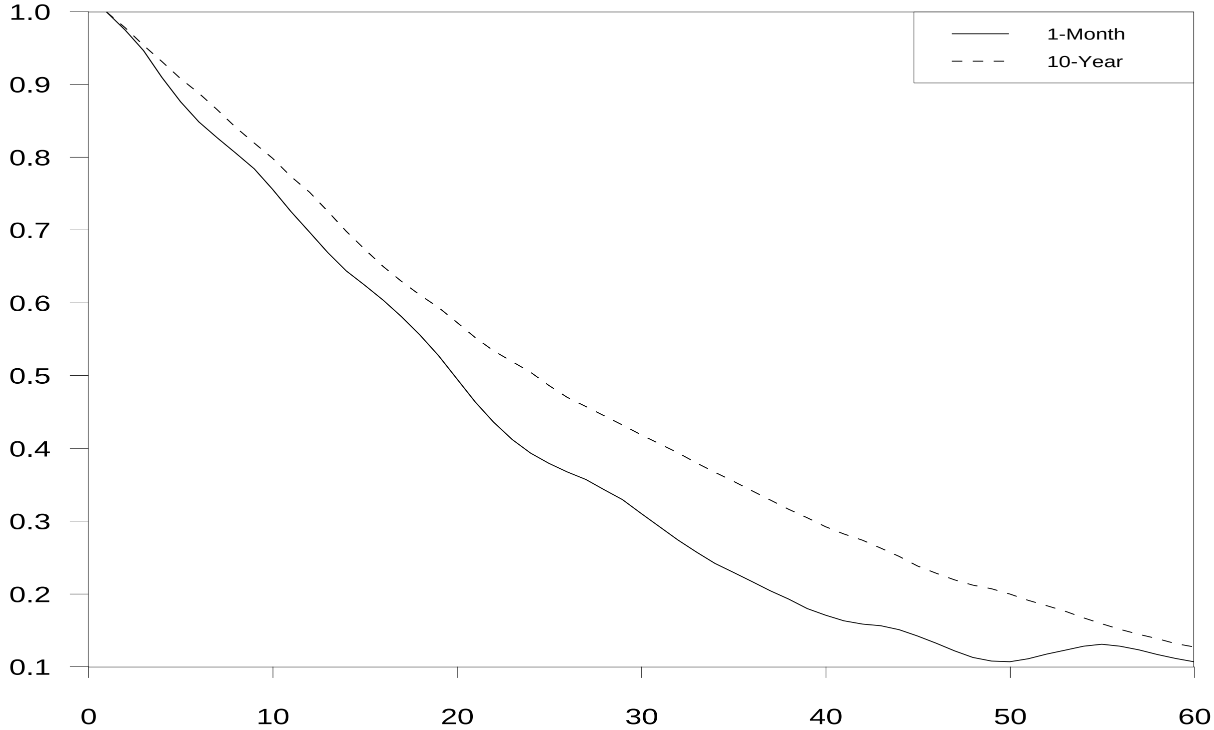


Figure 4: Autocorrelation functions, Cdn. 1-month and 10-year rates, 1978:3 to 1999:6



References

Backus, D. K. and S. E. Zin. 1993. “Long-Memory Inflation Uncertainty: Evidence from the Term Structure of Interest Rates.” *Journal of Money, Credit and Banking* 25: 681–700.

Baillie, R. T. and T. Bollerslev. 1989. “Common Stochastic Trends in a System of Exchange Rates.” *Journal of Finance* 44: 167–181.

———. 1994. “Cointegration, Fractional Cointegration, and Exchange Rate Dynamics.” *Journal of Finance* 49: 737–745.

Cheung, Y.-W. and K. S. Lai. 1993. “A Fractional Cointegration Analysis of Purchasing Power Parity.” *Journal of Business and Economic Statistics* 11: 103–112.

Cox, J., J. Ingersoll, and S. Ross. 1985. “A Theory of the Term Structure of Interest Rates.” *Econometrica* 53: 385–408.

Daubechies, I. 1988. “Orthonormal Bases of Compactly Supported Wavelets.” *Communications on Pure and Applied Mathematics* 41: 909–996.

Diebold, F. X., J. Gardeazabal, and K. Yilmaz. 1994. “On Cointegration and Exchange Rate Dynamics.” *Journal of Finance* 49: 727–735.

Diebold, F. X. and G. D. Rudebusch. 1991. “On the Power of Dickey Fuller Test Against Fractional Alternatives.” *Economics Letters* 35: 155–160.

Dolado, J. J., T. Jenkinson, and S. Sovilla-Rivero. 1990. “Cointegration and Unit Roots.” *Journal of Economic Surveys* 4: 249–273.

Geweke, J. and S. Porter-Hudak. 1983. “The Estimation and Application of Long Memory Time Series Models.” *Journal of Time Series Analysis* 4: 221–238.

Hosking, J. R. 1981. “Fractional Differencing.” *Biometrika* 68: 165–176.

Jensen, M. J. 1999. “Using Wavelets to Obtain a Consistent Ordinary Least Squares Estimator of the Long-memory Parameter.” *Journal of Forecasting* 18: 17–32.

McCulloch, J. H. and H.-C. Kwon. 1993. “U.S. Term Structure Data, 1947-1991.” Ohio State University Working Paper No. 93-6.

Nelson, C. R. and C. I. Plosser. 1982. “Trends and Random Walks in Macroeconomic Time Series: Some Evidence and Implications.” *Journal of Monetary Economics* 10: 139–162.

Ojanen, H. 1998. “WAVEKIT: A Wavelet Toolbox for Matlab.” Department of Mathematics, Rutgers University.

Parke, W. R. 1999. “What is Fractional Integration?” *The Review of Economics and Statistics* 81: 632–638.

Perron, P. 1989. “The Great Crash, the Oil Price Shock and the Unit Root Hypothesis.” *Econometrica* 57: 1361–1401.

Pfann, G. A., P. C. Schotman, and R. Tschernig. 1996. “Nonlinear Interest Rate Dynamics and Implications for the Term Structure.” *Journal of Econometrics* 74: 149–176.

Sowell, F. 1990. “The Fractional Unit Root Distribution.” *Econometrica* 58: 495–505.

Strang, G. 1993. “Wavelet Transforms Versus Fourier Transforms.” *Bulletin of the American Mathematical Society* 28: 288–305.

Strichartz, R. S. 1993. “How to Make Wavelets.” *American Mathematical Monthly* 100: 539–556.

Vidakovic, B. 1999. *Statistical Modeling by Wavelets*. New York: John Wiley and Sons.

Bank of Canada Working Papers
Documents de travail de la Banque du Canada

Working papers are generally published in the language of the author, with an abstract in both official languages. *Les documents de travail sont publiés généralement dans la langue utilisée par les auteurs; ils sont cependant précédés d'un résumé bilingue.*

Copies and a complete list of working papers are available from:
Pour obtenir des exemplaires et une liste complète des documents de travail, prière de s'adresser à:

Publications Distribution, Bank of Canada
234 Wellington Street Ottawa, Ontario K1A 0G9

Diffusion des publications, Banque du Canada
234, rue Wellington, Ottawa (Ontario) K1A 0G9

E-mail / Adresse électronique: publications@bank-banque-canada.ca
WWW: http://www.bank-banque-canada.ca/