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Uncovering Inflation Expectations and Risk Premiums from Internationally Integrated Financial Markets

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Abstract

Theory and empirical evidence suggest that the term structure of interest rates reflects risk premiums as well as market expectations about future inflation and real interest rates. We propose an approach to extracting such premiums and expectations by exploiting both the comovements among interest rates across the yield curve and between two countries, Canada and the United States. This approach involves estimating a multi-factor affine-yield model jointly for the two countries, in which we identify a common factor as representing real rate expectations and two other factors as representing two separate inflation expectations for the two countries. To estimate the model, we apply a Kalman filter to monthly data on zero-coupon bond yields for 2-year, 5-year and 10-year maturities as well as inflation. Our estimates suggest that Canadian inflation expectations were slow to adjust to a new inflation-targeting regime. We also find inflation-risk premiums that vary between 10 and 90 basis points in the two countries, with U.S. bonds commanding smaller premiums.

Résumé

Les études théoriques et empiriques donnent à penser que la structure à terme des taux d'intérêt reflète tant les primes de risque que les attentes du marché à l'égard de l'évolution future de l'inflation et des taux d'intérêt réels. Pour extraire l'information relative à ces primes et à ces attentes, les auteurs proposent d'exploiter à la fois les covariations des taux d'intérêt le long de la courbe de rendement et leurs covariations entre deux pays, soit le Canada et les États-Unis. Cette méthode nécessite l'estimation d'un modèle affine de taux de rendement à plusieurs facteurs et à deux pays, dans lequel interviennent un facteur commun, qui représente les attentes en matière de taux d'intérêt réels dans les deux pays, et deux autres facteurs reflétant des attentes d'inflation distinctes, propres à chacun des pays. Pour estimer le modèle, les auteurs appliquent un filtre de Kalman aux données mensuelles relatives à l'inflation et aux rendements d'obligations coupon zéro échéant dans deux, cinq et dix ans. D'après les estimations des auteurs, les attentes relatives à l'inflation ont été lentes au Canada à s'ajuster à un nouveau régime axé sur la poursuite de cibles de maîtrise de l'inflation. Les auteurs constatent également que les primes liées au risque d'inflation oscillent entre 10 et 70 points de base dans les deux pays, mais qu'elles sont plus faibles dans le cas des obligations américaines.

1. Introduction

It is well established that the term structure of interest rates reflects market expectations about future inflation, real interest rates and the underlying risk premiums. The extraction of such information from the term structure is important for the implementation of monetary policy and the examination of central bank credibility. In this paper, we propose an approach to extracting information about inflation expectations and inflation-risk premiums by exploiting both the comovements among interest rates across the yield curve and the comovements among those interest rates between two countries, Canada and the United States.

The most difficult challenge in modelling the yield curve has been taking account of time-varying risk premiums. Attempts to extract expectations from the yield curve, e.g. Fama (1990) and Mishkin (1990), find that variations in term premiums obscure those expectations. Shiller, Campbell, and Schoenholtz (1983), Fama (1984), and Keim and Stambaugh (1986) establish the presence of such premiums in U.S. bond returns. Engle, Lilien and Robins (1987) fit an ARCH-M model to interest rate data and find a highly significant risk premium associated with conditional volatility. Tzavalis and Wickens (1997) show that allowing for such risk premiums can help reconcile the expectations hypothesis with the data. These studies, however, do not distinguish between real and inflation-risk premiums.

In modelling these risk premiums we would also like them to meet the following considerations: that they arise from the pricing of an explicitly specified risk; that they satisfy the equilibrium condition of no arbitrage; and that they are related to expectations about fundamentals. The challenge is to take account of such risk premiums and to estimate inflation expectations by means of the simplest possible term-structure model. Gong and Remolona (1997b) construct a two-factor affine term-structure model to estimate the inflation-risk premium in the United States that satisfies the above-mentioned considerations. In the model, risks arise because of revisions in expectations and the model assumes that these risks are priced by the bond market. By identifying the two factors as relating to inflation and real-rate expectations, they obtain separate estimates of the inflation and real-risk premiums that are time-varying because of square-root heteroscedastic shocks to the factors. The model has some success in capturing inflation expectations and producing reasonable risk premiums.

For a small, open economy like Canada, it is important to take into account the external economy because its bond yields are strongly influenced by world financial markets, in particular, the U.S. bond market. As a result, one would like to consider explicitly the close link between Canadian and U.S. financial markets. In this paper, we extend the two-factor affine-yield model in Gong and Remolona to a two-country setting by estimating the model jointly for Canada and the

United States. In the model, yields in each country are determined by two unobserved (latent) factors. We attempt to identify one of the factors as an inflation factor that represents inflation expectations and the other as a real factor representing expectations about real fundamentals. Since the factors are unobserved, an important question is, How to identify the factors?

In Gong and Remolona (1997b) and Jegadeesh and Pennacchi (1996), the inflation process is used to identify the inflation factor by empirically implementing a link between the term structure and observed inflation rates. In Remolona, Wickens and Gong (1998), index-linked zero-coupon bond yields for the United Kingdom are used to identify the perceived real-rate process, thus allowing them to extract the perceived inflation process from the nominal yields. In Fung and Remolona (1998), the factors are identified by the assumption that the inflation factor is specific to each country, representing independent inflation expectations for the two countries, while the real factor is common to both countries, representing common real-rate expectations.[1] The intuition is that a real shock originating in the United States also will affect Canada or that some real shocks originating from outside the two countries will affect both Canada and the United States in a similar way, because of their similar economies and close economic links. However, inflation shocks in Canada may differ from those of the United States because, with a floating exchange rate, Canada can pursue an independent monetary policy. However, the assumption of a common real factor in a two-factor model may not be adequate to properly identify the underlying factors, especially the U.S. inflation factor because of its dominant size. As a result, in this paper we also use the actual U.S. inflation process to identify the U.S. inflation factor, and thus the U.S. real factor. Then, the assumption of a common real factor in the two countries allows us to identify the Canadian inflation factor.

To estimate the model, we apply a Kalman filter to monthly data on the annualized one-month-ahead inflation rate and zero-coupon bond yields for 2-year, 5-year and 10-year maturities. The model’s arbitrage conditions allow us to focus on interest rate movements that can be accounted for by consistent expectations processes. Because the model assumes no correlation between inflation and real-rate expectations, we estimate the model only for longer-term yields where such an assumption can be reasonably justified. The estimation procedure allows us to exploit the conditional density of bond yields without imposing special assumptions on measurement errors. The model’s arbitrage conditions also serve as over-identifying restrictions. We estimate the model over the period January 1984 to December 1998. The sample starts after 1983 to avoid a likely change in monetary regime in the United States in October 1982. Once we

1. The idea of having a factor that is common to two countries in an affine model can also be found in Backus, Foreski, and Telmer (1998) and Ahn (1997). However, neither model attempts to identify the common factor as a real factor and neither uses the Kalman filter to recover the underlying factors.

obtain the parameter estimates of the model, we can back out from the model conditional forecasts of the unobserved factors, thus allowing us to conditionally decompose nominal bond yields into four components: expectations of real rates, the real-term premium, expectations of inflation, and the inflation-risk premium.

In evaluating the model, we rely on the implications of the parameters for inflation expectations and risk premiums. Our estimates suggest that Canadian inflation expectations were slow to adjust to a new inflation-targeting regime. We also find inflation-risk premiums that vary between 10 and 90 basis points in the two countries, with U.S. bonds commanding smaller premiums. The results show that the model is capable of extracting useful information from the yield curves. This suggests that it is important to exploit additional information contained in internationally integrated financial markets to study the term structure, and that the assumption of a common factor and country-specific factors is plausible.

The rest of the paper is organized as follows. Section 2 presents the two-country two-factor model. Section 3 discusses the data and estimation. Section 4 reports and discusses the empirical results. Section 5 concludes and provides suggestions for future research.

2. An affine-yield two-country two-risk two-factor model

2.1 The affine class of term-structure models

The term-structure model that we construct in this paper is a two-country two-factor affine model based on the class of term-structure models proposed by Duffie and Kan (1996). In this class of models, the interest rates and prices of bonds are linear (affine) functions of a small number of factors. The dynamics of these factors are described by a generalized square-root diffusion process. The major advantage of working with this class of models is that such models are tractable yet capable of capturing many shapes of the yield curve. The affine term-structure model nests many well-known models, such as the one-factor Vasicek (1977) and Cox, Ingersoll, and Ross (1985), as well as the two-factor model of Longstaff and Schwartz (1992).

Since we focus on the econometric testing of the model and its empirical implications, we follow Campbell, Lo, and MacKinlay (1997) and Gong and Remolona (1997a) by specifying the model in terms of a discrete-time stochastic-discount process or pricing kernel, which also avoids the pitfalls of estimating a continuous-time model with discrete-time data.[2] These models specify the stochastic processes of the factors and derive the bond prices (or yields) as functions of the factors and the time to maturity. Thus, these models make an explicit link between the time-series

2. See, for example, Aït-Sahalia (1996).

dimension and the cross-section dimension. This allows one to fully exploit the cross-sectional restrictions imposed by the term structure model and allows for the identification of the market price of risk. The basic two-factor model is similar to the one in Gong and Remolona (1997b).

2.1.1 The pricing kernel

The pricing kernel approach relies on a no-arbitrage condition. In the case of zero-coupon bonds, the real price of an n-period bond is given by[3]

$$P_{nt} = E_t[P_{n-1,t+1}M_{t+1}], \tag{1}$$

where M_{t+1} is the stochastic discount factor. This pricing equation says that the price of the n-period bond is equal to the expected discount value of the bond's next-period price. It rules out arbitrage opportunities by applying the same discount factor to all bonds.[4] In what follows, we will model $P_{n,t}$ by modelling the stochastic process of M_{t+1}.

To derive an affine-yield model, the distribution of the stochastic discount factor M_{t+1} is assumed to be conditionally lognormal. In addition to providing model tractability, this assumption keeps the discount factor positive and unique. Taking logs of (1)(1), we get:

$$p_{nt} = E_t[m_{t+1} + p_{n-1,t+1}] + \frac{1}{2}Var_t[m_{t+1} + p_{n-1,t+1}], \tag{2}$$

where lower-case letters denote the logs of the corresponding upper-case letters, for example, $p_{t+1} = \log(P_{t+1})$.

Since there are two factors, $x_{1,t}$ and $x_{2,t}$, that forecast m_{t+1}, an affine-yield model that satisfies the Duffie-Kan (1996) conditions can be written as:

$$-p_{nt} = A_n + B_{1n}x_{1t} + B_{2n}x_{2t}. \tag{3}$$

which is a linear function of the factors.[5] Because the n-period bond yield is $y_{nt} = -\frac{p_{nt}}{n}$, yields will also be linear in the factors. Note that both the intercept (A_n) and factor loadings (B_{1n}, B_{2n}) are time-invariant functions of the time to maturity (n). The basic problem here is to specify the coefficients A_n, B_{1n}, B_{2n} by solving (3) based on the stochastic processes of $x_{1,t}$ and $x_{2,t}$ and verify that (2) holds.

3. The pricing equation can be derived either by considering the intertemporal choice problem of an investor who maximizes the expectation of a time-separable utility function, or merely from the absence of arbitrage; see Campbell, Lo, and MacKinlay (1997).
4. Essentially, there exists a positive random variable, m, satisfying the pricing equation (1) on all traded bonds if the economy permits no pure arbitrage opportunities.
5. Duffie and Kan (1996) provide the necessary and sufficient conditions for the existence and uniqueness of a solution to the affine specification. See also Campbell et al (1997) and Backus, Foresi, and Telmer (1998).

We will consider two similar affine-yield two-factor models, one for Canada and one for the United States, that satisfy the Duffie–Kan conditions.

2.2 The U.S. model

The pricing kernel in this model is assumed to be driven by two factors: one reflects the expectations of inflation that are specific to the United States; the other is a real factor that is common to both the United States and Canada, representing real-rate expectations. Without loss of generality, we can specify the first factor to be the inflation factor and the second factor to be the real factor. We will show how we identify these factors later. The negative of the log-stochastic discount factor is forecast by the two factors that enter into the forecasting relationship additively:

$$-m_{t+1} = x_{1t} + x_{2t} + w_{t+1} \tag{4}$$

where w_{t+1} represents the unexpected change in the log-stochastic discount factor and will be related to risk.[6] The shock has a mean of zero and a variance that will be specified to depend on the stochastic processes of the two factors x_{1t} and x_{2t}. Each of these factors follows a univariate AR(1) process with heteroskedasticity shocks (depending on its own level) described by a square-root process

$$x_{1t+1} = (1-\phi_1)\mu_1 + \phi_1 x_{1t} + x_{1t}^{1/2} u_{1t+1} \tag{5}$$

$$x_{2t+1} = (1-\phi_2)\mu_2 + \phi_2 x_{2t} + x_{2t}^{1/2} u_{2t+1}, \tag{6}$$

where $(1–\phi_1)$ and $(1–\phi_2)$ are the rates of mean reversion $(0<\phi_1,\phi_2<1)$, μ_1 and μ_2 are the long-run means to which the factors revert, $u_{1,t+1}$ and $u_{2,t+1}$ are shocks with zero means and volatilities σ_1^2 and σ_2^2, and the shocks are assumed to be uncorrelated.[7]

To model both inflation-risk and real-term premiums, the shock to m_{t+1} is specified to be proportional to the shocks to $x_{1,t+1}$ and $x_{2,t+1}$:

$$w_{t+1} = \lambda_1 x_{1t}^{1/2} u_{1,t+1} + \lambda_2 x_{2t}^{1/2} u_{2,t+1}, \tag{7}$$

where λ_1 and λ_2 represent market prices of risks. Here risks arise from revisions in expectations and the model assumes that these risks are priced by the bond market. Following Cox et al (1985) and Campbell et al (1997), we specify the volatilities of the shocks to be proportional to the

6. In other words, the minus-log pricing kernel is equal to the sum of two factors, adjusted for their risks.
7. The assumption that shocks to the expectation of real return are orthogonal to those of inflation expectations is not unreasonable because the expectation of real return is likely to be driven by real activity. Fama and Gibbons (1982) employ a similar orthogonality assumption to extract estimates of expected real returns from ex post inflation and short-term rates.

square root of the respective factors. Such square-root diffusions have several advantages; in particular, they induce time-varying risk premiums while keeping yields linear in the factors so that the model remains tractable.

Since a bond trades at par at maturity, normalization gives $\log(P_{0,t}) \equiv p_{0,t} = 0$. Thus the one-period yield is:

$$y_{1,t} = -p_{1,t} = \left(1 - \frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1t} + \left(1 - \frac{1}{2}\lambda_2^2\sigma_2^2\right)x_{2t}, \tag{8}$$

which is also linear in the factors, with the coefficients $A_1 = 0$, $B_{1,1} = 1 - \frac{1}{2}\lambda_1^2\sigma_1^2$ and $B_{2,1} = 1 - \frac{1}{2}\lambda_2^2\sigma_2^2$.

We can also verify that the price of an n-period bond is linear in the factors with the coefficients given by:[8]

$$A_n = A_{n-1} + (1-\phi_1)\mu_1 B_{1,n-1} + (1-\phi_2)\mu_2 B_{2,n-1}. \tag{9}$$

$$B_{1,n} = 1 + \phi_1 B_{1,n-1} - \frac{1}{2}(\lambda_1 + B_{1,n-1})^2{\sigma_1}^2, \tag{10}$$

$$B_{2,n} = 1 + \phi_2 B_{2,n-1} - \frac{1}{2}(\lambda_2 + B_{2,n-1})^2\sigma_2^2. \tag{11}$$

The coefficients B_{1n} and B_{2n} are factor loadings; the coefficient A_n represents the pull of the factors to their long-run means. Equations (9) to (11) impose cross-sectional restrictions to be satisfied by eight parameters: the rates of mean reversion $1-\phi_1$ and $1-\phi_2$, the long-run means μ_1 and μ_2, the prices of risks λ_1 and λ_2, and the volatilities σ_1 and σ_2.

2.3 The Canadian model

The Canadian model follows the same set-up as the U.S. model, except that those variables and coefficients that are specific to the Canadian model are denoted with an asterisk (*). Thus, the negative of the log stochastic discount factor is:

$$-m_{t+1}^* = x_{1t}^* + x_{2t} + w_{t+1}^*, \tag{12}$$

where w_{t+1}^* represents the unexpected change in the log-stochastic discount factor and will be related to risk. The shock has a mean of zero and a variance that will be specified to depend on the stochastic processes of the two factors x_{1t}^* and x_{2t}. Since the second factor is common to both countries, we only have to specify the process for the first factor:

8. See Appendix I for the derivations of these coefficients.

$$x_{1,t+1}^* = (1-\phi_1^*)\mu_1^* + \phi_1^* x_{1t}^* + (x_{1t}^*)^{1/2} u_{1t}^*, \tag{13}$$

where all the variables are defined similarly to those in the U.S. model.

The shock to m_{t+1}^* is specified to be proportional to the shock to $x_{1,t+1}^*$ and $x_{2,t+1}$:

$$w_{t+1}^* = \lambda_1^* x_{1t}^{*1/2} u_{1,t+1}^* + \lambda_2^* x_{2t}^{1/2} u_{2,t+1}. \tag{14}$$

Here, the price of risk of the common factor, λ_2^*, is specified to be different than that of the U.S. model.

Since the Canadian model shares a common factor with the U.S. model, the price of an *n*-period bond is given by:

$$-p_{nt}^* = A_n^* + B_{1n}^* x_{1t}^* + B_{2n}^* x_{2t}. \tag{15}$$

Note that we allow the loading of the real factor, B_{2n}^*, to be different between the two countries because the prices of risk of the common factor are allowed to be different. We will let the data determine whether financial markets in the two countries price this common-source risk in the same way, given the assumption of a common real shock.

The one-period yield is:

$$y_{1t}^* = -p_{1t}^* = \left(1-\frac{1}{2}\lambda_1^{*2}\sigma_1^{*2}\right)x_{1t}^* + \left(1-\frac{1}{2}\lambda_2^{*2}\sigma_2^2\right)x_{2t}, \tag{16}$$

which is also linear in the factors, with the coefficients $A_1^* = 0$, $B_{1,1}^* = 1-\frac{1}{2}\lambda_1^{*2}\sigma_1^{*2}$ and $B_{2,1}^* = 1-\frac{1}{2}\lambda_2^{*2}\sigma_2^2$.

We can also verify that the price of an *n*-period bond is linear in the factors with the coefficients given by:

$$A_n^* = A_{n-1}^* + (1-\phi_1^*)\mu_1^* B_{1,n-1}^* + (1-\phi_2)\mu_2 B_{2,n-1}^*, \tag{17}$$

$$B_{1,n}^* = 1 + \phi_1^* B_{1,n-1}^* - \frac{1}{2}(\lambda_1^* + B_{1,n-1}^*)^2 \sigma_1^{*2}, \tag{18}$$

$$B_{2,n}^* = 1 + \phi_2 B_{2,n-1}^* - \frac{1}{2}(\lambda_2^* + B_{2,n-1}^*)^2 \sigma_2^2. \tag{19}$$

Again, the coefficients $B_{1,n}^*$ and $B_{2,n}^*$ are factor loadings, while the coefficient A_n^* represents the pull of the factors to their long-run means. Equations (17) to (19) impose cross-sectional restrictions to be satisfied by eight parameters: the rates of mean reversion $1-\phi_1^*$ and

$1-\phi_2$, the long-run means μ_1^* and μ_2, the prices of risks λ_1^* and λ_2^*, and the volatilities σ_1^* and σ_2.

2.4 The inflation process and the inflation factor

In order to identify the inflation factor in the U.S. model, we need to model the market's perception of the inflation process. Here, the identification relies on the assumption of rational expectations and a fairly simple inflation process perceived by market participants. Suppose the CPI inflation rate follows a stationary AR(1) process:

$$\pi_{t+1} = (1-\theta)\eta + \theta\pi_t + \varepsilon_{t+1}, \qquad (20)$$

where θ is the rate of inflation persistence, η is a fixed long-run mean, and ε_{t+1} is an unanticipated shock with a mean of zero.

Note that the short rate in (8) is a risk-free rate because there is no need for revisions in expectations in one period. Hence, we can decompose the short rate into the inflation expectation and the expectation of real return according to the Fisher equation. By specifying $x_{1,t}$ to be the inflation factor, the first term on the right-hand side of (8) is thus the inflation expectation. We have:

$$\left(1-\frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1t} \equiv E_t(\pi_{t+1}) = (1-\theta)\eta + \theta\pi_t. \qquad (21)$$

We then update (21) by one period to have:

$$\left(1-\frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1,t+1} \equiv E_{t+1}(\pi_{t+2}) = (1-\theta)\eta + \theta\pi_{t+1}, \qquad (22)$$

substitute (20) and (21) into (22), and compare to (5). Under rational expectations, the expectations process inherits the parameters of the true process, so that:

$$\theta = \phi_1,\ \eta = \left(1-\frac{1}{2}\lambda_1^2\sigma_1^2\right)\mu_1, \text{ and } \theta\varepsilon_{t+1} = \left(1-\frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1t}^{1/2}u_{1,t+1}.$$

For subsequent estimation purposes, it will be useful to write (21) as:

$$\pi_t = A_\pi + B_\pi x_{1t}, \qquad (23)$$

where

$$A_\pi = -\frac{(1-\phi_1)}{\phi_1}\left(1-\frac{1}{2}\lambda_1^2\sigma_1^2\right)\mu_1, \text{ and} \qquad (24)$$

$$B_\pi = \frac{1}{\phi_1}\left(1-\frac{1}{2}\lambda_1^2\sigma_1^2\right). \qquad (25)$$

Here, we can derive an explicit link between observed inflation and the unobserved inflation factor x_{1t}. In the estimation procedure, this equation serves to identify x_{1t} as the factor driven by the expectation of inflation.[9]

2.5 Inflation-risk and real-term premiums

The U.S. inflation-risk and real-term premiums can be derived from the expected excess return on an *n*-period bond:

$$\begin{aligned} E_t(p_{n-1,t+1}) - p_{nt} - y_{1t} = & -\lambda_1 B_{1,n-1}\sigma_1^2 x_{1,t} - \frac{1}{2}B_{1,n-1}^2\sigma_1^2 x_{1t} \\ & -\lambda_2 B_{2,n-1}\sigma_2^2 x_{2,t} - \frac{1}{2}B_{2,n-1}^2\sigma_2^2 x_{2t} \end{aligned} \tag{26}$$

where the terms with x_{1t} represent the inflation-risk premium and the terms with $x_{2,t}$ represent the real-term premium. The two terms not containing λ_1 or λ_2 represent Jensen's inequality, which appears because we are working in logarithms. Note that both the inflation-risk and real-term premiums will depend on maturity and vary over time with the respective factors.

Similarly, the Canadian inflation-risk premium and real-term premium can be derived from the expected excess return on an *n*-period bond:

$$\begin{aligned} E_t(p_{n-1,t+1}{}^*) - p_{nt}{}^* - y_{1t}{}^* = & -\lambda_1{}^* B_{1,n-1}{}^*\sigma_1{}^{*2} x_{1,t}{}^* - \frac{1}{2}B_{1,n-1}{}^{*2}\sigma_1{}^{*2} x_{1t}{}^* \quad . \\ & -\lambda_2{}^* B_{2,n-1}{}^*\sigma_2^2 x_{2,t} - \frac{1}{2}B_{2,n-1}{}^{*2}\sigma_2^2 x_{2t} \end{aligned} \tag{27}$$

3. Data and estimation

3.1 Data

Some recent work on term-structure models, such as Duffie and Singleton (1997) and Gong and Remolona (1997c), have found that a third factor may be needed to fit the entire yield curve and to explain the hump in the volatility curve. Therefore, we focus ourselves to fitting only the 2-year to 10-year range of the yield curve because inflation expectations and inflation risks tend to have larger and more persistent influences on these yields than on the shorter-term yields. At the same time, the assumption of independent real and inflation expectations is more reasonable for these maturities. The sample period runs from 1984:1 to 1998:12.

9. An alternative way of identifying the inflation factor is to use inflation forecast data, which we may consider in future work.

3.1.1 Canadian data

The Canadian monthly data set consists of zero-coupon rates derived from the constant-maturity par-value yields on federal bonds used in Day and Lange (1997).[10]

3.1.2 U.S. data

Monthly data on zero-coupon yields of 2-year to 10-year bonds are from McCulloch and Kwon (1993) and supplemented by the data from the Federal Reserve Bank of New York. In the case of the Federal Reserve data, each zero curve is generated by fitting a cubic spline to prices and maturities of about 160 outstanding coupon-bearing U.S. Treasury securities. The securities are limited to off-the-run Treasuries to eliminate the most-liquid securities and reduce the possible effect of liquidity premiums.

Summary statistics for the annualized CPI inflation and the zero-coupon yields for maturities of 2, 5, and 10 years for the two countries are reported in Table 1. The CPI inflation is constructed from one-month-ahead percentage changes in seasonally adjusted CPI and is annualized by multiplying by 12. Note that average bond yields are lower in the United States but average inflation is higher. Bond yields, however, are more volatile in the United States and inflation is less volatile. The average inflation and yield differentials between the two countries are reported in the last column of Table 1. It is interesting to explain why Canada has a lower inflation rate but higher bond yields throughout the sample.

Figure 1a plots the U.S. and Canadian 2-year yields and Figure 1b plots the 2-year-ahead CPI inflation rates over the sample period. Canadian yields were above U.S. yields for most of the sample period, except in 1984. Canadian inflation was higher than U.S. inflation before 1987, but between 1987 and 1989 inflation in Canada and the United States was very similar. The anti-inflation policy that was introduced in Canada in 1989 and the subsequent introduction of inflation targets in 1991 resulted in a sharp drop in inflation. Canadian inflation has been lower than U.S. inflation since 1989, however bond yields have remained higher in Canada. Figure 1c shows that the inflation differential between Canada and the United States has turned negative since 1988, but the yield differential has remained positive until 1996.

10. The par-value yields are constructed using the Bell method. In the literature, there are two standardized ways to express the term structure; to report a par yield curve consisting of yield to maturity on a par bond or to report a spot rate curve consisting of yields to maturity on zero-coupon bonds. Either way of expressing the term structure requires estimating the term structure from yields to maturity on non-par coupon bonds. However, once constructed, the par yield and the spot rate can be derived from each other using a bootstrap method. For the range of bond yields studied in this paper, only Canadian par yield data is available at the moment. The 10-year par-value yield is from Boothe (1991) up to 1989 and then spliced with the Bank of Canada data base. Both use the Bell model.

3.2 Kalman filtering and maximum-likelihood estimation

Estimation of the model is based on a subset of the available yields that covers the medium-term maturity spectrum. Since the factors are treated as latent variables, they can be backed out using the Kalman filter. Estimation is then by maximum likelihood based on the conditional means and variances of the processes of the factors.[11] To apply the Kalman filter in our estimation, we have to write our models in linear state-space form. The measurement and transition equations are :

$$y_t = A + HX_t + v_t, \tag{28}$$

$$X_{t+1} = C + FX_t + u_{t+1}. \tag{29}$$

In our model, the yields, which are affine functions of the factors, serve as the measurement equations. The factors' stochastic processes, which are AR(1) processes, and the inflation equation, (23), form the transition equations. Thus we have:

$$\begin{bmatrix} \pi_{kt} \\ y_{lt} \\ y_{mt} \\ y_{nt} \\ y_{lt}^* \\ y_{mt}^* \\ y_{nt}^* \end{bmatrix} = \begin{bmatrix} A_\pi \\ a_l \\ a_m \\ a_n \\ a_l^* \\ a_m^* \\ a_n^* \end{bmatrix} + \begin{bmatrix} B_\pi & 0 & 0 \\ b_{1l} & b_{2l} & 0 \\ b_{1m} & b_{2m} & 0 \\ b_{1n} & b_{2n} & 0 \\ 0 & b_{2l}^* & b_{1l}^* \\ 0 & b_{2m}^* & b_{1m}^* \\ 0 & b_{2n}^* & b_{1n}^* \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{1t}^* \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \\ v_{3t} \\ v_{4t} \\ v_{5t} \\ v_{6t} \\ v_{7t} \end{bmatrix} \tag{30}$$

where π_{kt} is the actual inflation rate in equation (23) while y_{lt}, y_{mt}, y_{nt} and y_{lt}^*, y_{mt}^*, y_{nt}^* are zero-coupon yields at time *t* with maturities *l*, *m* and *n* in the United States and Canada respectively. The coefficients in the equation are:

$$a_k = \frac{A_k}{k},\ b_{1k} = \frac{B_{lk}}{k},\ \text{and}\ b_{2k} = \frac{B_{2k}}{k},\ k=l, m, n,$$

which are given by equations (9) through (11), whereas those coefficients with an asterisk are the Canadian counterparts given by equations (17) through (19). The coefficients A_π and B_π are given by equations (24) and (25). The v_{it} expressions are measurement errors distributed with zero-mean and standard-deviation e_i expressions where $i = 1, 2, \ldots, 7$.

11. de Jong (1997) discusses some empirical problems related to the estimation of the parameters by maximum likelihood and/or quasi-maximum-likelihood methods. However, he finds that for parameters typically found in estimates of term-structure model, the simulation results in Lund (1997) suggest that the bias in the QML estimator is not particularly large.

The transition equations correspond to equations (5), (6), and (13):

$$\begin{bmatrix} x_{1t} \\ x_{2t} \\ x_{1t}^* \end{bmatrix} = \begin{bmatrix} (1-\phi_1)\mu_1 \\ (1-\phi_2)\mu_2 \\ (1-\phi_1^*)\mu_1^* \end{bmatrix} + \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_1^* \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \\ x_{1,t-1}^* \end{bmatrix} + \begin{bmatrix} x_{1,t-1}^{1/2} u_{1t} \\ x_{2,t-1}^{1/2} u_{2t} \\ (x_{1,t-1}^*)^{1/2} u_{1t}^* \end{bmatrix}, \quad (31)$$

where the shocks u_{1t}, u_{2t}, and u_{1t}^* are distributed normally with mean-zero and standard errors σ_1, σ_2 and σ_1^*. Note that in standard linear state-space models, no restrictions link the measurement equations and the transition equations. In our model, however, the arbitrage conditions serve as over-identifying restrictions that link the coefficients of these two equations. The arbitrage conditions are given by (9) through (11) and (17) through (19); the initial values are set by (8) and (16).

4. Results

4.1 Parameter estimates

Table 2 reports the parameter estimates for the model. The $(1-\phi)$ expressions measure the rates of mean reversion; our parameter estimates of ϕ are very close to 1, suggesting very slow mean reversion. The σ expressions measure factors' volatilities, and the μ expressions are the long-run means of the factors. λ_1 and λ_1^* are the prices of inflation risk and λ_2 and λ_2^* measure the prices of real risks. In estimating the model, we allow the prices of risks to be different in the two countries. However, the prices of both real and inflation risks turn out to be almost identical.

In evaluating the model, we rely on the implications of the parameters for inflation expectations and risk premiums rather than on individual estimates. To do this, we back out from the model conditional forecasts of the inflation and real factors, derive the implied expectations and risk premiums, and then examine how these implied variables behave over time. In particular, we can examine how they vary over time in light of the events over the sample period. Finally, we also examine how well the implied average yield curves fit the actual curves for the two countries.

4.2 Implied yields and inflation expectations

Figures 2a through 2c plot the implied Canadian and U.S. 2-year yields and inflation expectations. Comparing Figures 1a and 2a, we find that the implied yields for Canada and the United States follow a similar relationship to that of the actual yields. The Canadian implied yields are higher than the U.S. implied yields for most of the sample period. Figure 2b shows that until mid-1996, Canadian inflation expectations were higher than U.S. inflation expectations, which is in contrast to the actual inflation series depicted in Figure 1b. As a result, Figure 2c shows that both the yield and inflation differentials are positive for most of the period after 1989; the differentials reached their peaks in the early 1990s.

Figures 3a and 3b plot the actual and implied 2-year yields for Canada and the United States. The model does a good job of producing time series of implied bond yields that mimic actual bond yields.

Figure 4a plots the 2-year-ahead actual inflation and inflation expectations in the United States and Figure 4b plots those in Canada. Note that actual 2-year-ahead inflation is only available up to December 1996. One-period-ahead inflation expectations are backed out from the model's conditional forecasts of $x_{1,t}$ and $x_{1,t}^*$, and from equation (21). We can then calculate the 24-month-ahead inflation expectations by accumulating them over the same horizon. Figure 4a shows that the derived U.S. inflation expectations closely follow actual inflation, especially after 1991. This is in sharp contrast to the results found in Fung and Remolona (1998). In that paper, inflation expectations are substantially below actual U.S. inflation. The results in this paper are a significant improvement because using actual U.S. inflation in the estimation allows us to better identify the U.S. inflation factor.

Figure 4b plots Canadian inflation expectations, actual inflation, and the survey data on inflation.[12] From 1984 to 1989, the three lines were fairly close, but in 1989 both the survey data and derived inflation expectations missed the sharp decline in inflation. This suggests that the public was slow to react to the Bank of Canada's low-inflation policy but that the bond market was even slower to respond. However, the Bank was slowly gaining credibility. Since 1993, the survey data has moved closely with actual inflation, however the derived inflation expectations have still been above actual inflation by more than 1 percentage point. One reason that the derived Canadian inflation expectations are higher than actual inflation is that actual Canadian yields are higher than U.S. yields, while actual Canadian inflation is lower. In the model, actual U.S. inflation helps to extract U.S. inflation expectations that fit actual U.S. inflation well. With the assumption of a common real factor, higher Canadian bond yields imply higher Canadian inflation expectations and/or inflation risk than those of the United States. Thus, while Canadian inflation became lower than U.S. inflation in 1989, we find that inflation expectations have been substantially higher than actual inflation since then. Since 1997, however, the derived inflation expectations have moved

12. Canadian 2-year-ahead inflation expectations from Consensus Forecasts only began in 1990. Thus we use 1-year-ahead inflation expectations from the Conference Board of Canada to supplement the series. Note that the data are used only for comparison purposes, but not for estimation of the model.

closely with the survey data at around an inflation rate of 2 per cent. This suggests that the market expects inflation to remain stable at the mid-point of the Bank of Canada's inflation-target range.

4.3 Inflation and real risks

Revisions in inflation expectations are a source of risk that appears to have been priced by the bond market in the 1980s and 1990s. Since the magnitudes of the revisions are related to the level of the expectations, risk premiums vary over time. The estimates of the prices of risk, λ_1 and λ_2, allow us to calculate inflation and risk premiums by applying the model's conditional forecasts of $x_{1,t}$ and $x_{1,t}^*$ as well as $x_{2,t}$ to the relevant terms in equations (26) and (27). In Figures 5a and 5b, we graph the estimated inflation and real-risk premiums for the 5-year yield in Canada and the United States.[13] These risk premiums show substantial time variation through the entire sample period. Figure 5a shows that the inflation-risk premium is higher in Canada than the United States over the sample period. The Canadian inflation-risk premium peaked in 1991, and since then has declined slowly to a similar level as the U.S. inflation-risk premium. Figure 5b shows that the real-risk premium is exactly the same for both countries, although we allow the price of real risk to be different. Note that the real-risk premium has been slowly declining since mid-1984 and has remained rather stable at a very low level since 1992.

Campbell and Shiller (1996) estimate the size of the inflation-risk premium in the United States, defined as the average excess return on an inflation-sensitive asset that is attributable to its inflation sensitivity, using two different methods. In the first method, they assume that the average excess return on a nominal 5-year bond over that of a comparatively riskless asset such as a nominal 3-month Treasury bill is entirely accounted for by its inflation-risk premium. Over the sample period 1953–94, they estimate a risk premium of 70 to 100 basis points on a 5-year nominal bond.[14] In the second method, they use asset-pricing theory to try to judge what risk premium is implied by the covariance of bond returns with relevant state variables. They use the return on a proxy for the market portfolio, such as a value-weighted stock index, and the growth rate of aggregate consumption. They obtain an implied risk premium of about 90 to about 150 basis points. Thus they suggest that a best guess might be 50 to 100 basis points for a 5-year zero-coupon bond. Gong and Remolona (1997b) estimate the inflation-risk premium in the United States to be time-varying, ranging from around 50 to around 150 basis points.

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13. We report the 5-year risk premium because it allows us to compare our results with estimates from other studies.

14. This estimate could be interpreted as the upper bound for the inflation-risk premium because of the possible presence of a real-risk premium.

In our model, over the sample period 1984–98 the inflation-risk premiums in Canada and the United States vary between approximately 10 basis points and approximately 90 basis points. The average inflation-risk premiums are 57 basis points in Canada and 21 basis points in the United States, with a differential of about 36 basis points. The inflation-risk premiums derived in the model are in line with those found in the literature. Figure 5b shows that the real-risk premium varies over a range of 0 to 57 basis points. The average total risk premiums for the 5-year rates are 72 basis points for Canada and 36 basis points for the United States; these are also in line with previous findings.

4.4 Actual and implied yield differentials

One question often asked when working with term-structure models is how well the implied yield curve from the model fits the actual average yield curve over the sample period. Figures 6a and 6b plot the actual and implied yield curves in the United States and Canada, respectively. The implied U.S. yield curve gives a good fit of the actual yield curve. The implied Canadian yield curve fits the actual curve well between the 1- and 10-year maturities. This is probably because we estimate the model using only medium-term bond yields. The actual Canadian yield curve is rather flat, with a steep slope at the short end of the maturity spectrum — less than 12-months. We may be able to get a better fit of the curve by including short-term bond yields in our estimation. However, including short-term yields would make it harder to justify our assumption of independent inflation and real factors.

In a two-country model, it may also be interesting to look at how well the three factors reproduce the shape of the average yield differential curve, because if the model is misspecified it will affect the implied yield curves in the two countries in more or less the same way. Figure 6c plots the actual and implied Canada–U.S. yield differentials across maturities up to 10 years. The actual yield differential curve is mainly downward-sloping, except the slight upward slope at the short end. The curve is almost flat for maturities of 5 years and above. The implied yield differential curve is also downward-sloping starting at the 3-year maturity and does not have a very close fit to the actual curve.

5. Conclusions

In this paper, we construct a two-country, multi-factor affine term-structure model to estimate inflation expectations and risk premiums in Canada and the United States using bond yields of 2-, 5- and 10-year maturities as well as actual U.S. inflation. The results suggest that there is useful and substantial information that can be extracted from the yield curve, especially when countries that have integrated financial markets are estimated jointly.

A few other issues, however, deserve further investigation. First, in future work, we could also include actual Canadian inflation in our estimation in order to get better estimates for the inflation expectations in Canada. Thus, we could compare the results with two separate two-factor models to examine whether estimating bond yields of the two countries jointly would provide more information than estimating two separate closed-economy models. Second, we could allow for an extra real idiosyncratic shock that affects only Canadian yields but not U.S. yields, or allow the same real shock to affect the two countries differently.

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Appendix 1: Recursive Restrictions

We start with the general pricing equation:

$$p_{nt} = E_t[m_{t+1} + p_{n-1,t+1}] + \frac{1}{2}Var_t[m_{t+1} + p_{n-1,t+1}].$$

The short rate is derived by setting $p_{0,t} = 1$:

$$y_{1t} = -p_{1,t} = -E_t(m_{t+1}) - \frac{1}{2}Var_t(m_{t+1}),$$
$$= \left(1 - \frac{1}{2}\lambda_1^2\sigma_1^2\right)x_{1t} + \left(1 - \frac{1}{2}\lambda_2^2\sigma_2^2\right)x_{2t},$$

showing the short rate to be linear in the factors.

Now, we guess that the price of an n-period bond is affine:

$$-p_{nt} = A_n + B_{1n}x_{1t} + B_{2n}x_{2t}.$$

We verify that there exist A_n, B_{1n} and B_{2n} that satisfy the general pricing equation:

$$-p_{nt} = -E_t[m_{t+1} + p_{n-1,t+1}] - \frac{1}{2}Var_t[m_{t+1} + p_{n-1,t+1}].$$
$$= (A_{n-1} + (1-\phi_1)\mu_1 B_{1,n-1} + (1-\phi_2)\mu_2 B_{2,n-1})$$
$$+ \left(1 + \phi_1 B_{1,n-1} - \frac{1}{2}(\lambda_1 + B_{1,n-1})^2{\sigma_1}^2\right)x_{1,t}$$
$$+ \left(1 + \phi_2 B_{2,n-1} - \frac{1}{2}(\lambda_2 + B_{2,n-1})^2{\sigma_2}^2\right)x_{2,t}.$$

Now, by matching coefficients we have

$$A_n = A_{n-1} + (1-\phi_1)\mu_1 B_{1,n-1} + (1-\phi_2)\mu_2 B_{2,n-1}$$
$$B_{1,n} = 1 + \phi_1 B_{1,n-1} - \frac{1}{2}(\lambda_1 + B_{1,n-1})^2{\sigma_1}^2$$
$$B_{2,n} = 1 + \phi_2 B_{2,n-1} - \frac{1}{2}(\lambda_2 + B_{2,n-1})^2{\sigma_2}^2.$$

Appendix 2. Kalman Filtering Procedure[15]

For the state-space models in Section 4, the measurement and transition equations can be written in the following matrix form:

Measurement equation:

$$y_t = A + HX_t + v_t,$$

where $v_t \sim N(0, R)$.

Transition equation:

$$X_{t+1} = C + FX_t + u_{t+1},$$

where $u_{t+1|t} \sim N(0, Q_t)$.

The Kalman filter procedure of this state-space model is the following:

1. Initialize the state-vector S_t.

The recursion begins with a guess, $S_{1|0}$, usually given by

$$\hat{S}_{1|0} = E(S_1).$$

The associated mean square error (MSE) is

$$P_{1|0} \equiv E[(S_1 - \hat{S}_{1|0})(S_1 - \hat{S}_{1|0})'] = Var(S_1).$$

The initial state S_1 is assumed to be $N(\hat{S}_{1|0}, P_{1|0})$.

2. Forecast y_t.

Let I_t denote the information set at time *t*. Then

$$\hat{y}_{t|t-1} = A + HE[S_t|I_{t-1}] = A + H\hat{S}_{t|t-1}.$$

The forecasting MSE is

$$E[(y_t - \hat{y}_{t|t-1})(y_t - \hat{y}_{t|t-1})'] = HP_{t|t-1}H' + R.$$

15. See also Hamilton (1994) for a more complete description of the procedure.

3. Update the inference about S_t given I_t.

Knowing y_t helps to update $S_{t|t-1}$ by the following: Write

$$S_t = \hat{S}_{t|t-1} + (S_t - \hat{S}_{t|t-1})$$

$$y_t = A + H\hat{S}_{t|t-1} + H(S_t - \hat{S}_{t|t-1}) + v_t.$$

We have the following joint distribution:

$$\begin{bmatrix} S_t \middle| I_{t-1} \\ y_t \middle| I_{t-1} \end{bmatrix} \sim N\left(\begin{bmatrix} \hat{S}_{t|t-1} \\ A + H\hat{S}_{t|t-1} \end{bmatrix}, \begin{bmatrix} P_{t|t-1} & P_{t|t-1}H' \\ HP_{t|t-1} & HP_{t|t-1}H' + R \end{bmatrix}\right).$$

Thus,

$$\hat{S}_{t|t} \equiv E[S_t | y_t, I_{t-1}] = \hat{S}_{t|t-1} + P_{t|t-1}H'(HP_{t|t-1}H' + R)^{-1}(y_t - HS_{t|t-1} - A)$$

$$P_{t|t} \equiv E[(S_t - \hat{S}_{t|t})(S_t - \hat{S}_{t|t})'] = P_{t|t-1} - P_{t|t-1}H'(HP_{t|t-1}H' + R)^{-1}HP_{t|t-1}.$$

4. Forecast t S_{t+1} given I_t.

$$\hat{S}_{t+1|t} = E[S_{t+1} | I_t] = F\hat{S}_{t|t}$$

$$P_{t+1|t} = E[(S_{t+1} - \hat{S}_{t+1|t})(S_{t+1} - \hat{S}_{t+1|t})'] = FP_{t|t}F' + Q_t.$$

5. To calculate the maximum likelihood estimation of parameters, the likelihood function can be constructed recursively as:

$$\log L(Y_T) = \sum_{t=1}^{T} \log f(y_t | I_{t-1}),$$

where $f(y_t | I_{t-1}) = (2\pi)^{-0.5} |H'P_{t|t-1}H + R|^{-0.5} \times$

$$\exp\left\{-\frac{1}{2}(y_t - A - H\hat{S}_{t|t-1})'(H'P_{t|t-1}H + R)^{-1}(y_t - A - H\hat{S}_{t|t-1})\right\}$$

for $t = 1, 2, \ldots, T$.

Parameter estimates can then be estimated based on the numerical maximization of the likelihood function.

Table 1: Summary Statistics

Sample: January 1984 to August 1995							
	United States			Canada			Canada–US differentials
Variable	Mean	Standard deviations	First order autocorrelation	Mean	Standard deviations	First order autocorrelation	
CPI inflation	3.18	2.09	0.46	2.83	2.68	0.23	–0.35
2-year bond yield	6.86	1.93	0.98	7.85	2.19	0.98	0.99
5-year bond yield	7.45	1.88	0.98	8.21	1.94	0.98	0.76
10-year bond yield	7.86	1.81	0.98	8.70	1.82	0.98	0.84

Table 2: Parameter Estimates[a]

	Sample: January 1984 to Decmber 1998	
Inflation parameters		
ϕ_1	0.94	(0.5015)
ϕ_1^*	0.97	(0.4381)
μ_1	4.62	(3.1936)
μ_1^*	5.67**	(2.3008)
λ_1	–7.29**	(3.0013)
λ_1^*	–13.62**	(3.7844)
σ_1	0.1033	(0.0990)
σ_1^*	0.0568	(0.0531)
Real-return parameters		
ϕ_2	0.97	(0.6725)
μ_2	9.83	(8.5447)
λ_2	–7.07**	(2.9168)
λ_2^*	–7.06**	(2.9099)
σ_2	0.1667**	(0.07)
Standard deviation of measurement errors		
e_1	1.4916	
e_2	0.3247	
e_3	1.0714	
e_4	1.4507	
e_5	0.6296	
e_6	0.8577	
e_7	1.1829	
Mean log likelihood	–6.18	

a. ** indicates statistical significance at the 5-per-cent level. For the ϕ statements, we report significant difference from 1 instead of 0.

Figure 1a: U.S. and Canadian 2−Year Yields, 1984:1 to 1996:12

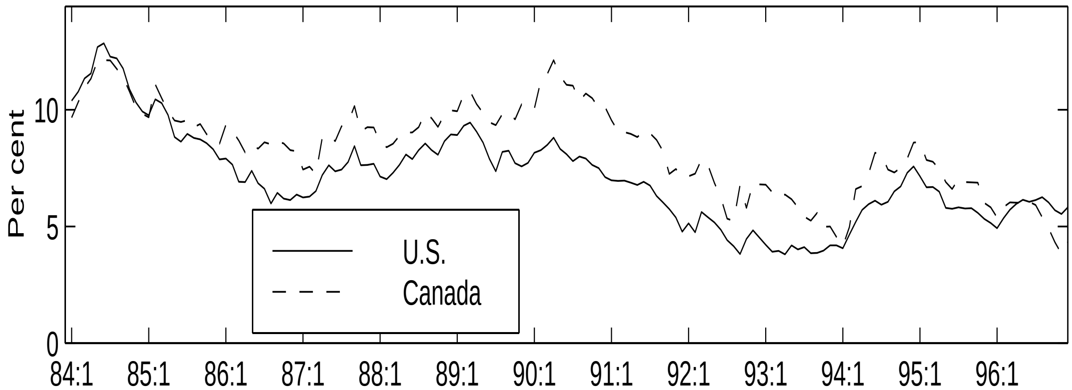


Figure 1b: U.S. and Canadian 2−Year Inflation, 1984:1 to 1996:12

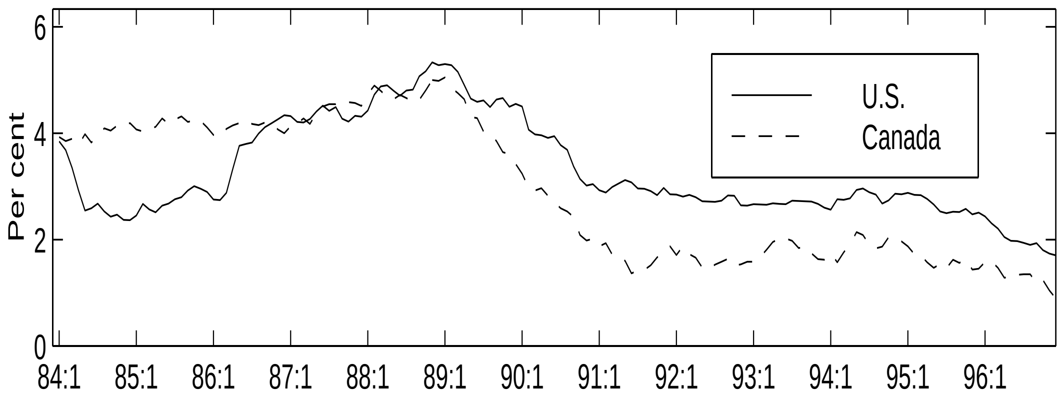


Figure 1c: Actual 2−Year Inflation and Yield differentials, 1984:1 to 1996:12

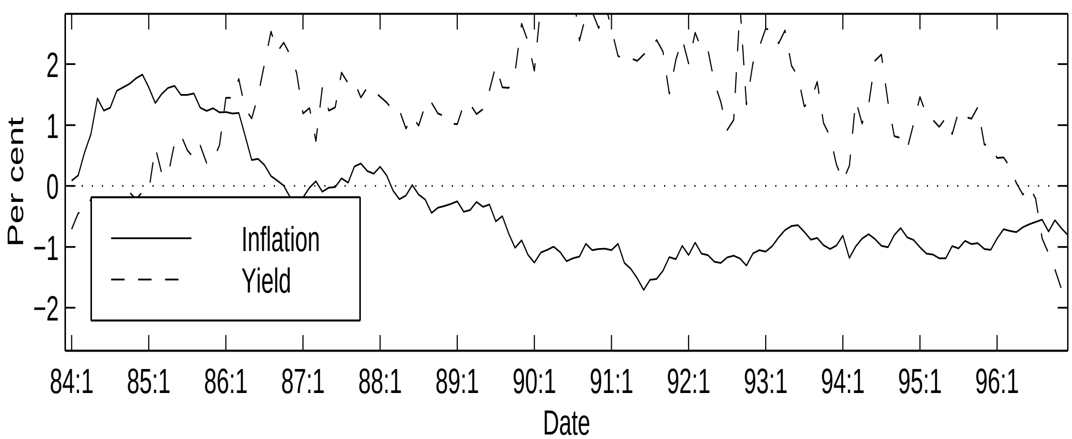


Figure 2a: Implied U.S. and Canadian 2−Year Yields, 1984:1 to 1998:12

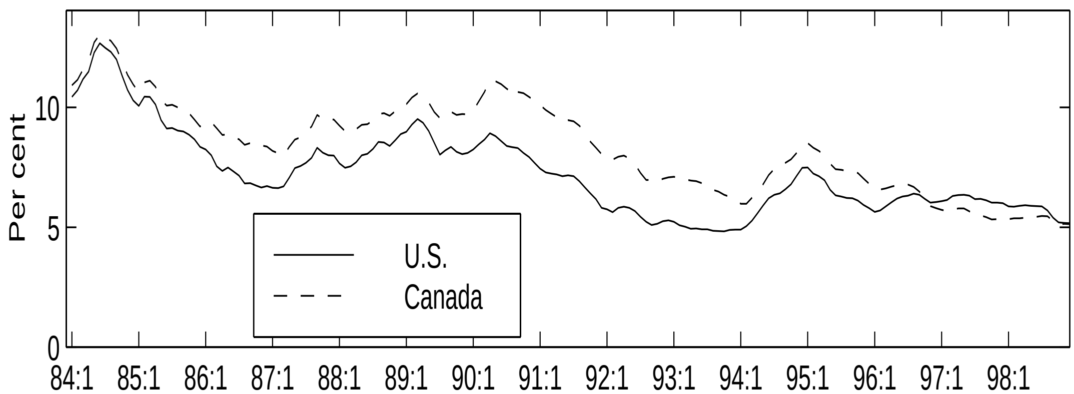


Figure 2b: U.S. and Canadian 2−Year−Ahead Inflation Expectations, 1984:1 to 1998:12

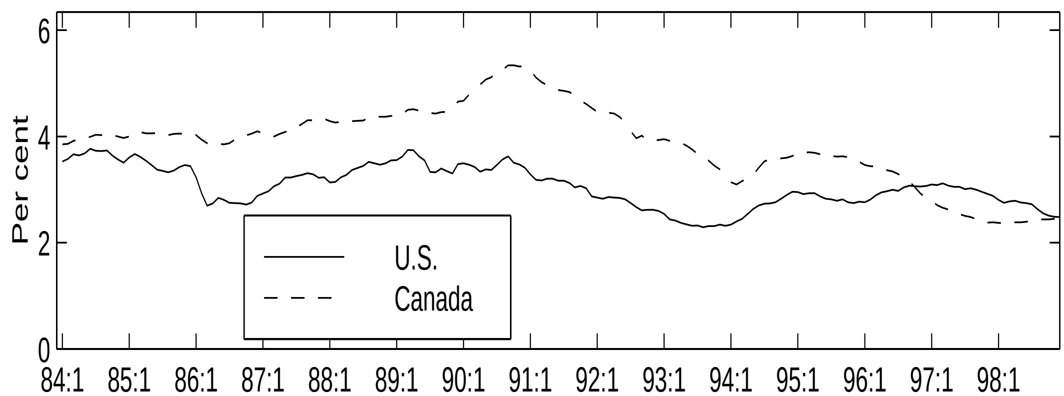


Figure 2c: Implied 2−Year Inflation and Yield differentials, 1984:1 to 1998:12

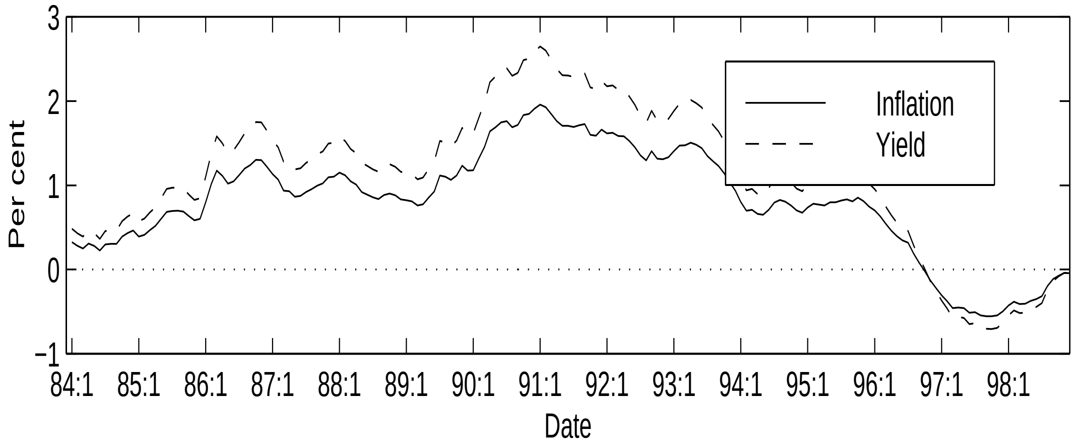


Figure 3a: U.S. Actual and Implied 2−Year Yields, 1984:1 to 1998:12

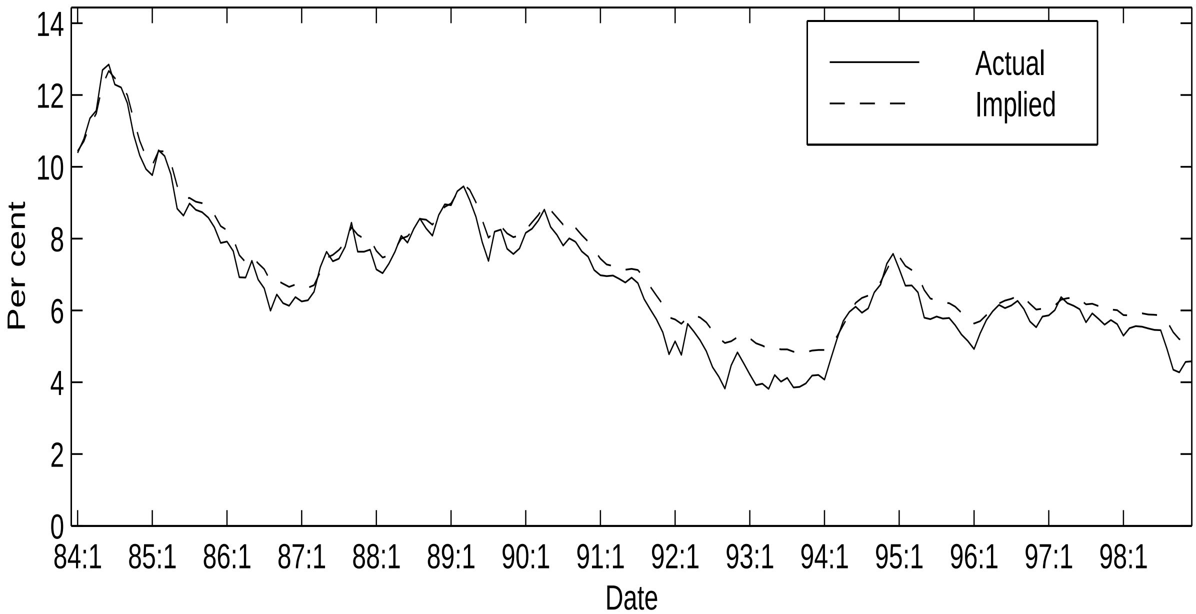


Figure 3b: Canadian Actual and Implied 2−Year Yields, 1984:1 to 1998:12

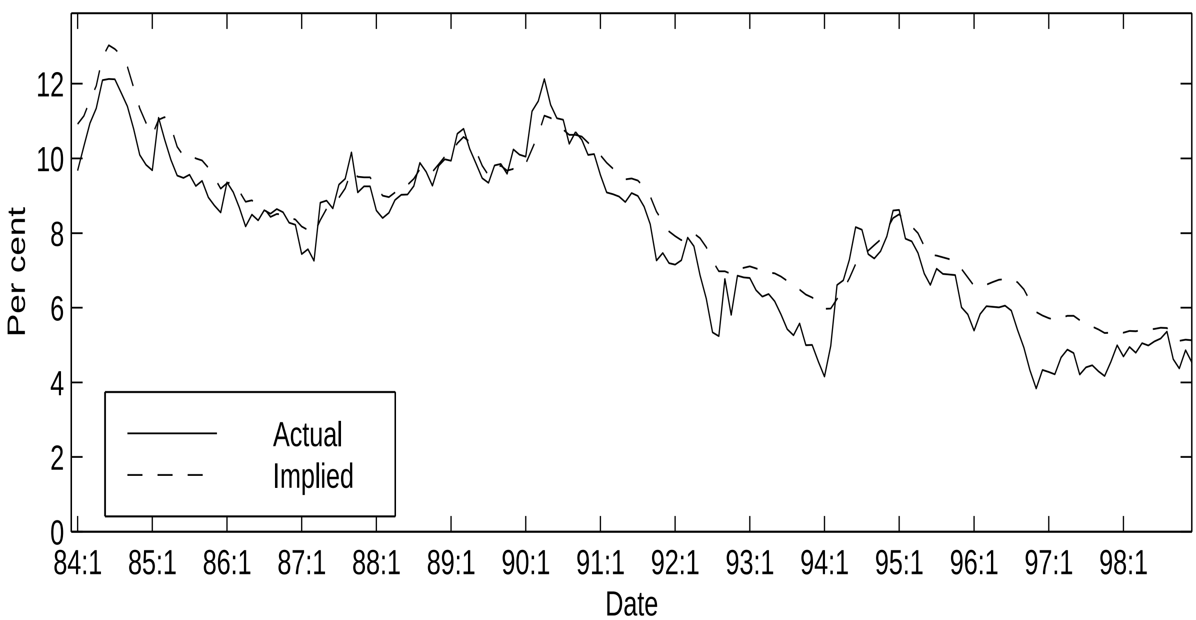


Figure 4a: U.S. 2−Year Ahead Inflation Expectations and Actual Inflation

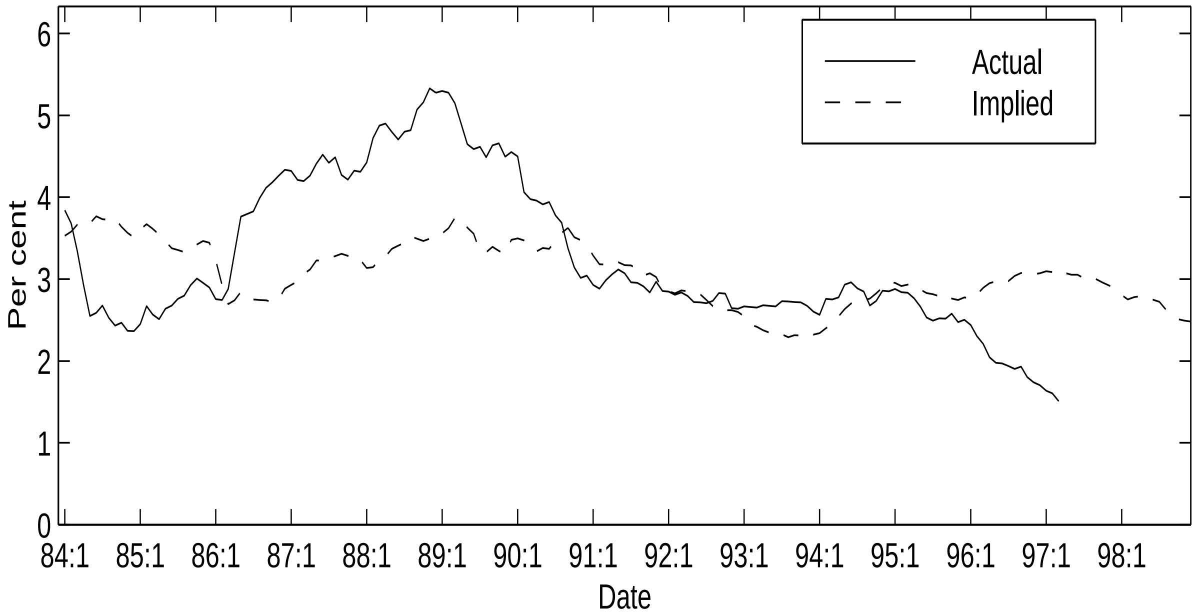


Figure 4b: Canadian 2−Year Ahead Inflation Expectations and Actual Inflation

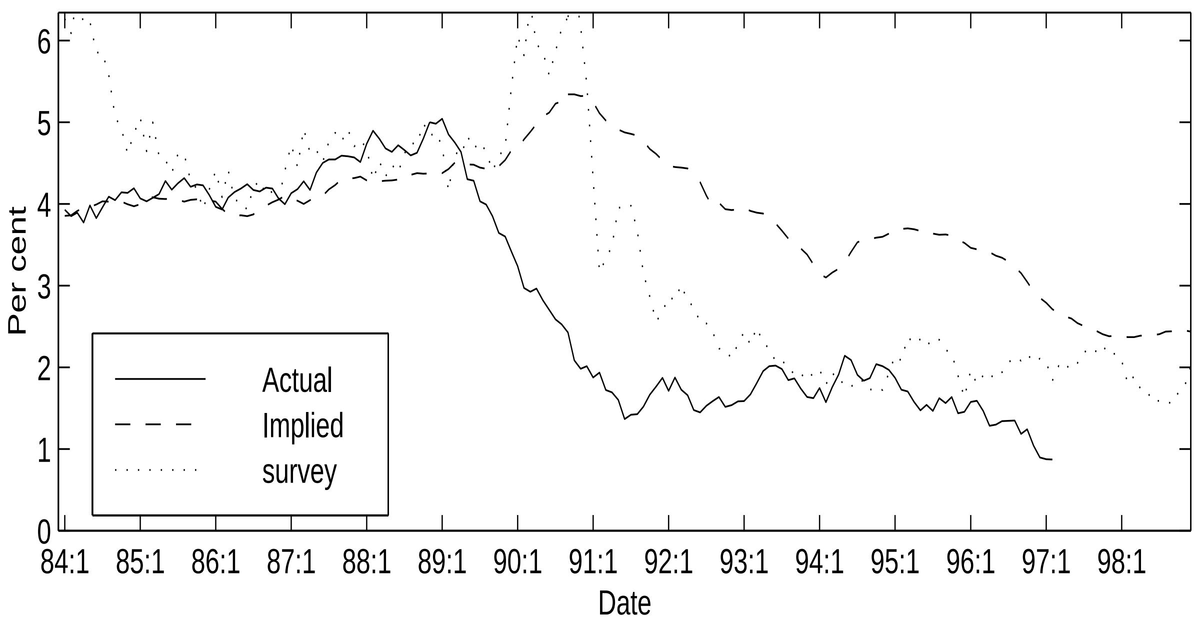


Figure 5a: 5−Year Inflation−Risk Premiums, 1984:1 to 1998:12

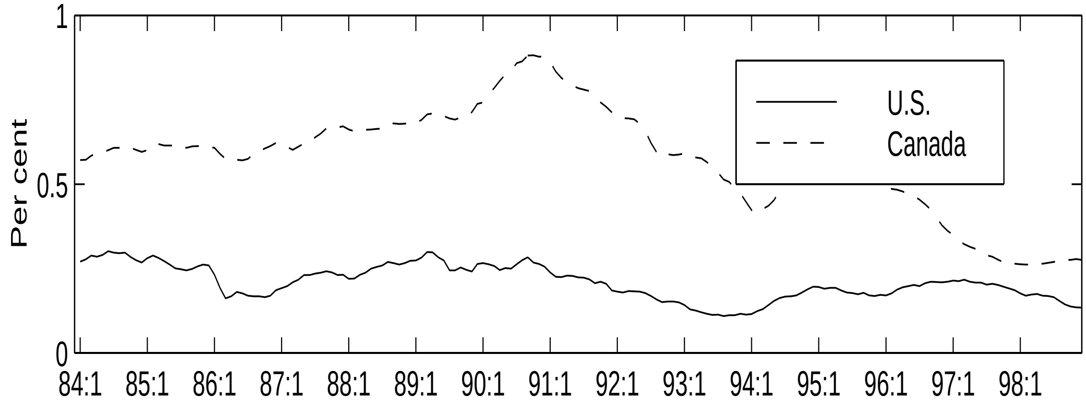


Figure 5b: 5−Year Real−Risk Premiums, 1984:1 to 1998:12

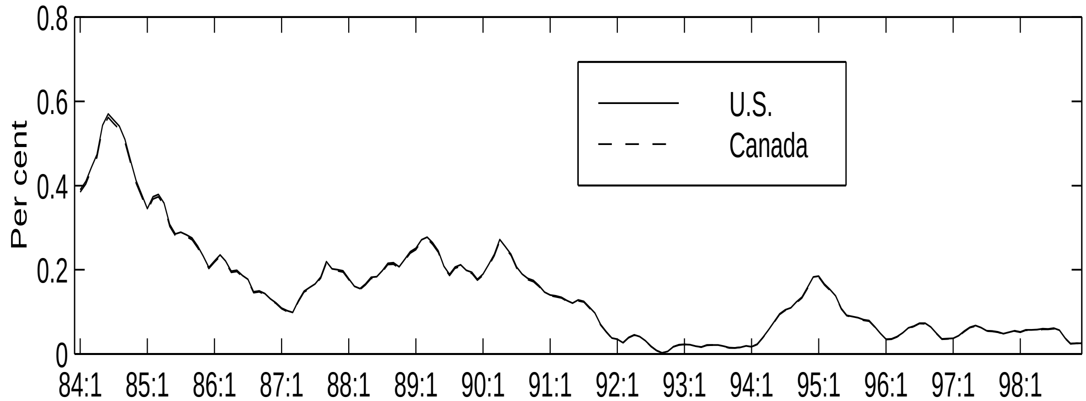


Figure 5c: Implied U.S. and Canadian Real Rate Expectation, 1984:1 to 1998:12

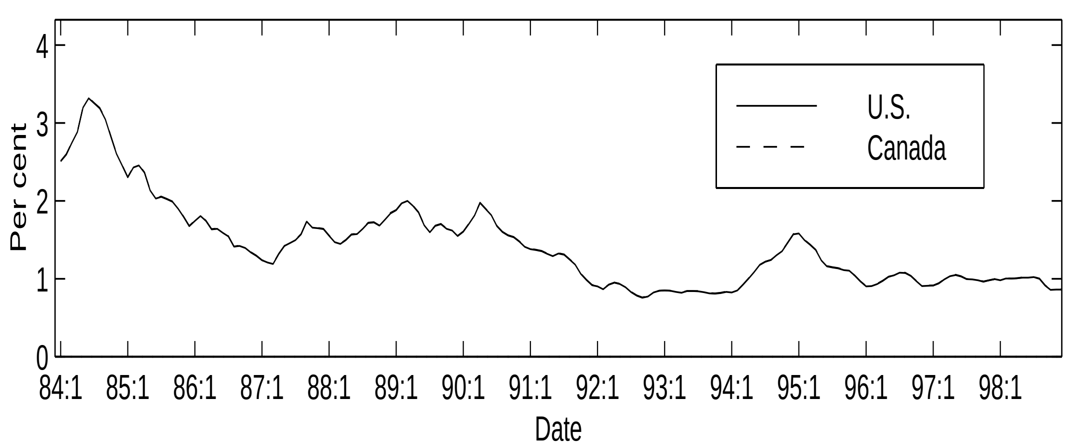


Figure 6a: Actual and Implied U.S. yields

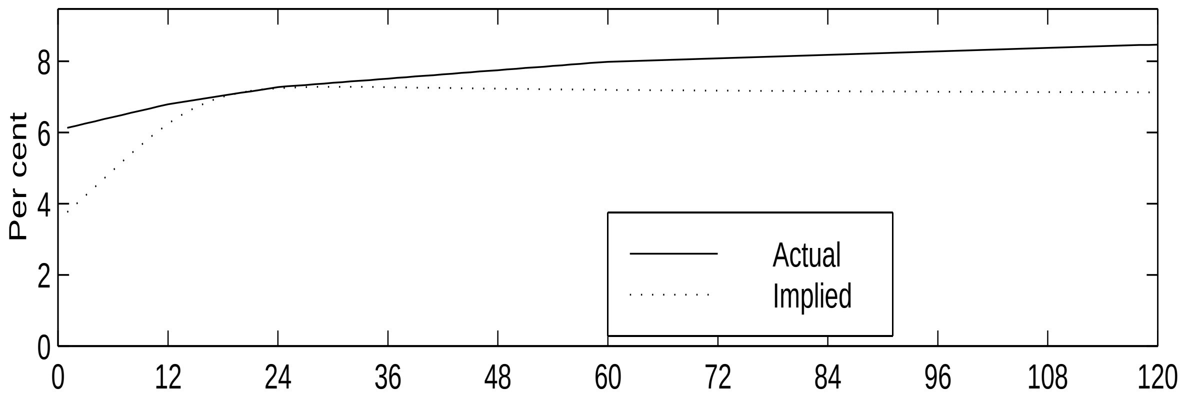


Figure 6b: Actual and Implied Canadian Yields

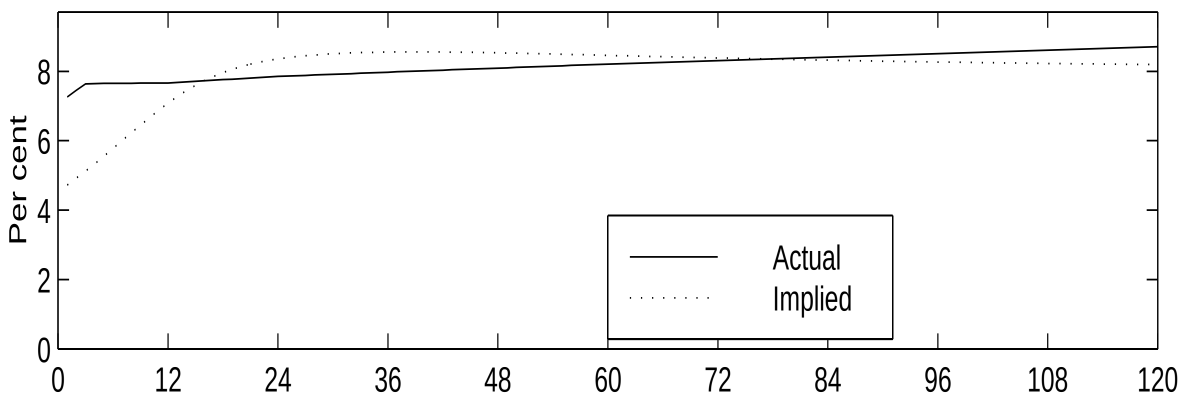
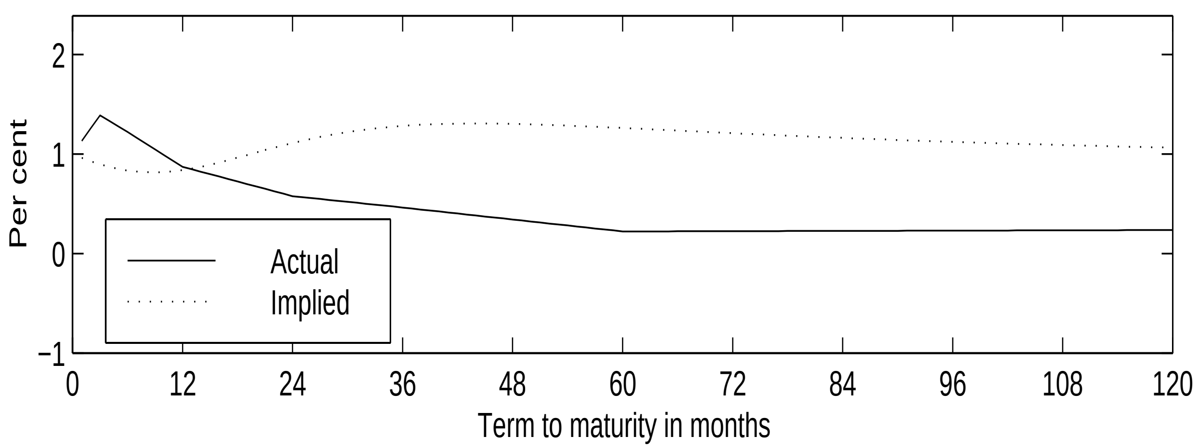


Figure 6c: Actual and Implied Yield Differentials



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