National Research Council of Canada
Institute for National Measurement Standards

Uncertainty Evaluation for the Measurement of Gauge Blocks by Optical Interferometry*

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Worth remembering:

> *"Although this Guide provides a framework for assessing uncertainty, it cannot substitute for critical thinking, intellectual honesty, and professional skill. The evaluation of uncertainty is neither a routine task nor a purely mathematical one; it depends on detailed knowledge of the nature of the measurand and of the measurement. The quality and utility of the uncertainty quoted for the result of a measurement therefore ultimately depend on the understanding, critical analysis, and integrity of those who contribute to the assignment of its value."*

"Guide to the Expression of Uncertainty in Measurement", 1993, 1st Edition (International Organization for Standardization, Switzerland), §3.4.8.

Contents

1 Introduction

The *Guide to the expression of uncertainty in measurement* [2], herein referred to simply as the *Guide*, is finding wide application in the measurement science community. The methods laid out in the *Guide* are applicable to many measurement scenarios, however, for successful application of its rules, very specfic measurement cases must be considered one at at time. The details of the measurement are reflected in the uncertainty evaluation, so there will be a unique measurement uncertainty associated with each particular measurement depending on the lab and its practices.

A close correspondance between the model equation and the measurement is paramount to successful evaluation of the measurement uncertainty, and moreover to formulate the approach to the evaluation. In a companion paper [1] we describe in detail gauge block calibration by optical interferometry at NRC. This paper begins with the model equations developed there, and evaluates the uncertainty for the measurement of gauge blocks using the NRC system.

2 The Expression of Uncertainty

The uncertainty evaluation presented in this document follows internationally accepted general rules for the evaluation and expression of uncertainties as laid out in the *Guide*. The *Guide* contains detailed definitions of the statistical concepts and terminology, general equations for standard and expanded uncertainties, as well as recommendations for dealing with special uncertainty cases. Every attempt is made to reference the relevant section of the *Guide* where appropriate.

2.1 Combined Standard Uncertainty

The combined standard uncertainty $u_c(d)$ is an estimate of the standard deviation of the distribution of possible values (or probability distribution) of the deviation from nominal length of the gauge block d, here measured by interferometry. The combined standard uncertainty, as its name implies, is a quadrature sum of the uncertainties $u(x_i)$ of all of the influence factors x_i, each weighted by a sensitivity coefficient [§5.1.2 *Guide*]:

$$u_c^2(d) \quad = \quad \sum_{i=1}^{N} c_i^2 u^2(x_i) + \sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{1}{2}c_{ij}^2 + c_i \cdot c_{ijj}\right] u^2(x_i)u^2(x_j) \tag{1}$$

where $u(x_i)$ are the standard uncertainties attributed to the influence quantities x_i, and where the sensitivity coefficients

$$c_i = \frac{\partial d}{\partial x_i}, \qquad c_{ij} = \frac{\partial^2 d}{\partial x_i \partial x_j}, \qquad c_{ijj} = \frac{\partial^3 d}{\partial x_i \partial x_j^2} \tag{2}$$

are the partial derivatives of the model equation.

It is convenient to think of equation (1) as consisting of two parts: variance terms containing $u^2(x_i)$, and higher order terms containing $u^2(x_i)u^2(x_j)$. In modelling thermal effects in dimensional metrology, the higher order terms always make a significant, if not dominant contribution to the uncertainties. In the subject area of dimensional metrology, the higher order terms should be evaluated with the variance terms as a matter of course, not as a footnote or appendix.

2.1.1 Nominal Length

The nominal length L is used for convenience in the calculation of length dependent coefficients. Referring to equation (1) in Ref. [1], the difference arising in the calculation of these small corrections as a result of substituting L in place of the measured length ℓ is negligible.

2.1.2 Type A and Type B Uncertainty Evaluations

In the *Guide*, there is considerable concern about distinguishing between Type A and Type B uncertainty evaluations. Simply put: Type A evaluations imply statistical evaluation of data collected from repeated measurements. The standard deviation is evaluated from the data and used as the standard uncertainty. Type B uncertainty evaluations are those for which repeated measurements cannot simply isolate the influence, and the uncertainty must be obtained by some other method based on the experience and expertise of the metrologist. The reader is referred to the *Guide* [§4.2, §4.3] for more background.

Here, almost all of the evaluations are Type B. In some cases, repeated measurements are taken in order to isolate an influence parameter. Such repeated measurements can greatly support the arguments made for a Type B evaluation of a parameter; they do not necessarily imply that the evaluation of the uncertainty is Type A.

The rectangular distribution crops up frequently in Type B evaluations, and it is used in several instances in this document. As explained in §4.4.5 of the *Guide* (also see Figure 2, p. 17 *Guide*), if data to estimate the uncertainty distribution of an influence parameter is limited, often an adequate and useful approximation is to assume an upper $+a$ and lower $-a$ bound for a range of equally probable values. The standard uncertainty (§4.3.7 *Guide*) is then given by $a/\sqrt{3}$. Similarly, a reading from a digital read-out of resolution b has a standard uncertainty of $b/\sqrt{12}$ [§F.2.2.1 *Guide*]; the digital reading is considered to be a rectangular distribution bounded by $+a = +b/2$ and $-a = -b/2$.

2.1.3 End Effects and Length Dependence

Gauge block uncertainties can be grouped into those due to length dependent effects and those that are due to end effects. Generally, gauge blocks within a set are all made to the same high quality of end surface finish and geometry. So a set can be assigned a single value to characterize end effects, plus another coefficient to characterize the uncertainty changes with the nominal length. For this reason, the uncertainties are collected into two groups: end effects and length dependent effects. End effects are those arising solely from the quality of the active optical surfaces of the gauge block and their interaction with the measurement system, and by definition are independent of the length of the gauge block. For example, wringing is an end effect. Conversely, length dependent effect arise from the bulk properties of the gauge block and the surrounding medium (e.g. air), and by definition are independent of the end effects. Thermal dilatation is an obvious example of a length dependent influence.

2.2 Expanded Uncertainty

It is desirable to express uncertainties so that for most of the measurements the measured value is within the uncertainty range of the true value. The expanded uncertainty [§6.2 *Guide*]

$$U = ku_c(d), \tag{3}$$

is defined as the combined standard uncertainty multiplied by a coverage factor k. The value of the coverage factor is chosen depending on the approximate level of confidence that would facilitate the interpretation of the uncertainty. Most measurements are expressed with a value of k between 2 and 3. NRC has chosen to express the expanded uncertainty for $k = 2$, corresponding to a confidence level of approximately 95% [§6.2.2 *Guide*]. Multiplying by a coverage factor does not add any new information; it is a convention. The emphasis of this document is on the calculation of the combined standard uncertainty.

It is important that the reader distinguish between the *standard uncertainty* u, which is the $k = 1$ or 1σ value, and the *expanded uncertainty* $U = ku$, where $k > 1$. In the following sections, uncertainties with various k values will be adjusted to the 1σ ($k = 1$) level in order to work with standard uncertainties throughout the evaluation.

3 The Model Equations

For a sophisticated measurement such as interferometry of a gauge block, it is expedient to arrange the model equation such that the influence parameters are as isolated as possible, yet still reflect the relative influences on other parameters for the sake of evaluating correlations. The techniques suggested by the *Guide* can then be applied, in turn, to each of the influences. The following model equations, with the influence parameters, are identified in an order that is convenient for presenting the uncertainty evaluation. The reader may find it helpful to consult Table 1, which summarizes the uncertainty components described in the remainder of the paper.

From the companion document, the measured deviation from nominal length of the gauge block is:

$$\begin{aligned} d &= l - L \\ d &= l_{\text{fit}} - L + l_t + l_w + l_A + l_\Omega + l_n + l_G + l_\phi \end{aligned} \tag{4}$$

where

- The best-fit solution for gauge block length, based on the method of exact fractions for q wavelengths of light, can be expressed as

  $$l_{\text{fit}} = \frac{1}{q}\sum_{i=1}^{q}(\kappa_i + F_i)\,\frac{\lambda_i}{2}. \tag{5}$$

  Influence parameters impacting the length solution are the measured interference fringe fractions F_i, and the vacuum wavelengths λ_i with uncertainties $u(F_i)$ and $u_c(\lambda_i)$ respectively.

- L is the nominal gauge block length; it is assumed that $u(L) = 0$.

- The gauge temperature correction is

  $$l_t = \theta\alpha L. \tag{6}$$

This correction arises from the gauge block temperature offset $\theta = 20 - t_g$, where t_g is the gauge block temperature in degrees Celcius and 20°C is the ISO standard reference temperature for dimensional measurements [3]. The thermal dilatation coefficient for the gauge material α is provided by the gauge manufacturer in units of ppm/K.[1] Uncertainties associated with this correction include those associated with gauge temperature offset $u_c(\theta)$ and the thermal dilatation coefficient $u(\alpha)$.

- l_w is the length attributed to the thickness of the wringing film. The expectation value of this correction is zero $\langle l_w \rangle = 0$, since the length of the gauge block is defined to include one wringing film [4], however the uncertainty associated with the variation in the wringing film $u(w)$ is non-zero.

- The correction for wavefront errors as a result of imperfect interferometer optics is l_A, with uncertainty $u(l_A)$. The expectation value of this correction is zero $\langle l_A \rangle = 0$, but its uncertainty is non-zero.

- The obliquity correction

$$\begin{aligned} l_\Omega &= \Omega L \\ &= \left(\frac{a^2}{16f^2} + \frac{x^2}{2f^2} \right) L \end{aligned} \quad (7)$$

is a length correction accounting for the shift in phase resulting from the optical design and alignment properties inherent in the NRC interferometer. Its uncertainty is $u_c(l_\Omega)$, which is dependent on collimator lens focal length f, aperture diameter a and lateral offset x, each with associated uncertainties $u(f)$, $u(a)$ and $u(x)$ respectively.

- The refractive index correction is

$$l_n = (n-1)L, \quad (8)$$

where n is the refractive index of air evaluated using a modified version of the Edlén equation. The combined standard uncertainty attributed to the refractive index correction $u_c(l_n)$ includes standard uncertainty components associated with: the empirical fit of the Edlén model $u(E)$; the ambient air density factors of temperature $u_c(t)$ presssure $u_c(p)$, water vapour content or relative humidity $u_c(R)$, CO_2 content $u(CO_2)$; and the vacuum wavelength $u(l_{n,\lambda})$ of the light.

- The gauge block geometry correction l_G accounts for non-flatness and non-parallelism of the gauge block. The uncertainty associated with this correction is $u(l_G)$.

- The phase change correction

$$l_\phi = \frac{1}{n-1} \left(l_p - \sum_{i=1}^{n} l_i \right) \quad (9)$$

is an end effect correction accounting for the difference in the apparent optical length to the mechanical length. Its uncertainty is $u(l_\phi)$.

[1] ppm: parts per million = 1×10^{-6}

TABLE 1: NRC STANDARD UNCERTAINTY COMPONENTS FOR LENGTH MEASUREMENT BY OPTICAL INTERFEROMETRY

STANDARD UNCERTAINTY COMPONENT $u(x_i)$	SOURCE	STANDARD UNCERTAINTY $u(x_i)$	$c_{x_i} \equiv \frac{\partial f}{\partial x_i}$	$u_i(d) \equiv \lvert c_{x_i}\rvert u(x_i)$ [nm] for L [mm]
$u_c(l_{\text{fit}})$	**Best fit length (see text)**			
$u(\lambda_i)$	*Vacuum wavelength			$0.012L$
	He-Ne laser			
	calibration	0.002 ppm	$L/q\lambda_i$	
	1-year drift range	0.01 ppm	$L/q\lambda_i$	
	^{114}Cd lamp vacuum wavelength	0.02 ppm	$L/q\lambda_i$	
$u(F_i)$	*‡Fringe fraction	0.01 fringe	$\lambda_i/2q$	1.2
	Thermal Effects:			
$u_c(\theta)$	Gauge temperature			
$u_c(t)$	calibration, performance	4 mK	αL	$0.046L^\dagger$
$u(\theta_{\text{grad}})$	gauge gradient	5 mK	αL	$0.058L^\dagger$
$u(\alpha)$	Thermal dilatation coefficient	$0.66^\dagger$ ppm/K	$(20 - t_g)L$ $= 50L$ mK	$0.033L^\dagger$
$u(\alpha)u_c(\theta)$	Gauge temperature cross term	(0.66 ppm/K)(6 mK)	L	$0.004L^\dagger$
$u(l_w)$	‡Wringing	6 nm	1	6
	Interferometer Optics:			
$u(l_A)$	‡Wavefront errors	3 nm	1	3
$u_c(l_\Omega)$	Obliquity:			
$u_{c,1}(l_\Omega)$	Source size	5 μm	$0.3 \times 10^{-6} L/\text{mm}$	$0.002L$
$u_{c,2}(l_\Omega)$	Alignment cross term	0.05 mm	$3.3 \times 10^{-6} L/\text{mm}^2$	$0.008L$

‡Uncorrelated terms.

†Based on a steel gauge, where $\alpha = 11.5$ ppm/K.

*Based on $q = 5$ wavelengths applied to the method of exact fractions.

Table 1: NRC Standard Uncertainty Components for Length Measurement by Optical Interferometry Cont'd

Standard Uncertainty Component $u(x_i)$	Source	Standard Uncertainty $u(x_i)$	$c_{x_i} \equiv \frac{\partial f}{\partial x_i}$	$u_i(d) \equiv \lvert c_{x_i} \rvert u(x_i)$ [nm] for L [mm]
$u_c(l_n)$	**Refractive index of air:**			
$u(E)$	Edlén equation	1×10^{-8}	L	$0.010L$
$u_c(t)$	Air temperature:			
	calibration	2.5 mK	$-9.5 \times 10^{-7} L/\mathrm{K}$	$0.002L$
	‡device reading capability	0.1 mK	$-9.5 \times 10^{-7} L/\mathrm{K}$	$0.0001L$
	drift	3 mK	$-9.5 \times 10^{-7} L/\mathrm{K}$	$0.002L$
$u_c(p)$	Air pressure:			
	calibration	50 Pa	$2.7 \times 10^{-9} L/\mathrm{Pa}$	$0.135L$
	‡device reading capability	4 Pa	$2.7 \times 10^{-9} L/\mathrm{Pa}$	$0.011L$
	drift	54 Pa	$2.7 \times 10^{-9} L/\mathrm{Pa}$	$0.146L$
$u_c(R)$	Humidity:			
	calibration	1.2%	$-8.5 \times 10^{-9} L$	$0.010L$
	‡device reading capability	0.1%	$-8.5 \times 10^{-9} L$	$0.001L$
	drift	1%	$-8.5 \times 10^{-9} L$	$0.009L$
$u_c(\lambda_v)$	Vacuum wavelength	0.01 ppm	$-1.2 \times 10^{-5} L$	*negl*
$u(l_G)$	‡Gauge block departure from flatness/parallel	2 nm	1	2
$u_c(l_\phi)$	Phase-change correction	6 nm	1	6
			Combined Standard Uncertainty:	$u^2(d) = 9.3^2 + 0.216^2 L^2$ nm^2
			Linearized Expanded Uncertainty for Coverage Factor $k = 2$:	$U = 19 + 0.28L$ nm

Source	Colour	Vacuum Wavelength λ_v [nm]	1σ Uncertainty $u_c(\lambda_v)$ [ppm·λ]
He-Ne Laser	red	632.991 162	0.01
^{114}Cd Lamp	red	644.024 80	0.03
	green	508.723 79	0.03
	blue	480.125 21	0.03
	violet	467.945 81	0.03

Table 2: Optical sources and their uncertainties.

4 Uncertainty Evaluations

Applying (1) to (4) yields the combined standard uncertainty in d in terms of the measured quantities and corrections:

$$\begin{aligned} u_c^2(d) &= c_{l_{\text{fit}}}^2 u_c^2(l_{\text{fit}}) + c_{l_t}^2 u^2(l_t) + c_{l_w}^2 u^2(l_w) + c_{l_A}^2 u_c^2(l_A) \\ &\quad + c_{l_\Omega}^2 u_c^2(l_\Omega) + c_{l_n}^2 u_c^2(l_n) + c_{l_G}^2 u^2(l_G) + c_{l_\phi}^2 u^2(l_\phi) \end{aligned} \tag{10}$$

where each contribution is described in detail below.

Definition of the variables used in the evaluation of the standard uncertainties and the values of the uncertainty components are catalogued in Table 1. This section provides a discussion of each case individually. It is noted that the tabulated uncertainties relate to sensors sometimes by themselves, but more often inherent in systems specific to the NRC gauge block interferometer. The calibration of these systems and associated uncertainties cannot necessarily be applied to other measurement systems other than the NRC gauge block interferometer.

4.1 Length Evaluation Based on the Method of Exact Fractions

4.1.1 Source Vacuum Wavelengths

The light sources used in the gauge block length measurements, and their combined standard uncertainties are listed in Table 2. Calibration of the laser vacuum wavelength is done in-house against the NRC primary standard He-Ne laser stabilized by saturated absorption in $^{127}I_2$ [5]. The calibration uncertainty of the laser is 1.3 fm (1 MHz) or 0.002 ppm·λ; the one year drift of the laser frequency is on average 13 fm (5 MHz) or 0.01 ppm·λ. The combined standard uncertainty in the laser vacuum wavelength is therefore

$$\begin{aligned} u_c(\lambda) &= \sqrt{(0.002 \text{ ppm}\cdot\lambda)^2 + (0.01 \text{ ppm}\cdot\lambda)^2} \\ &= 0.01 \text{ ppm}\cdot\lambda. \end{aligned} \tag{11}$$

The electrodeless cadmium lamp operates in accordance with the 1963 CIPM *La Définition du Mètre* [6]. The uncertainty of 0.07 ppm·λ assigned in the CIPM documentation is assumed to be at the 99% confidence level, corresponding to a coverage factor of $k = 2.58$ [§4.3.4 *Guide*], so

$$\begin{aligned} u_c(\lambda) &= \frac{0.07 \text{ ppm}\cdot\lambda}{2.58} \\ &= 0.03 \text{ ppm}\cdot\lambda. \end{aligned} \tag{12}$$

4.1.2 Fringe Fraction Measurement

NRC's length evaluation based on equation (72) is performed by a regression-style computer program (see Appendix A). The gauge length evaluation begins with the measurement of the fractional fringe order at each of $q = 5$ wavelengths. The computer program determines the best fit of the integral interference orders κ_i corresponding to their measured fractions F_i and known λ_i. The solution for the gauge length averaged over the q wavelengths used to make the measurement is:

$$l_{\text{fit}} = \sum_{i=1}^{q} \left(\frac{\kappa_i + F_i}{q} \right) \frac{\lambda_i}{2}. \tag{13}$$

Following equation (1) for evaluating the uncertainty,

$$u_c^2(l_{\text{fit}}) = c_{\kappa_i}^2 u^2(\kappa_i) + c_{F_i}^2 u^2(F_i) + c_{\lambda_i}^2 u^2(\lambda_i), \tag{14}$$

where

$$c_{\kappa_i} = \frac{\partial l_{\text{fit}}}{\partial \kappa_i}, \qquad c_{F_i} = \frac{\partial l_{\text{fit}}}{\partial F_i}, \qquad c_{\lambda_i} = \frac{\partial l_{\text{fit}}}{\partial \lambda_i}. \tag{15}$$

Table 3 displays the sensitivity coefficients for the variance and cross terms, following the procedure outlined above. Matching the sensitivity coefficients with the uncertainties and including the summation notation

	Variance Terms	Cross Terms						
	$c_i = \frac{\partial l_{\text{fit}}}{\partial x_i}$	$c_{ij} = \frac{\partial^2 l_{\text{fit}}}{\partial x_i \partial x_j}$				$c_{ijj} = \frac{\partial^3 l_{\text{fit}}}{\partial x_i \partial^2 x_j}$		
x_i		$x_j =$	κ_i	F_i	λ_i	κ_i	F_i	λ_i
κ_i	$\frac{\lambda_i}{2q}$		0	0	$\frac{1}{2q}$	0	0	0
F_i	$\frac{\lambda_i}{2q}$		0	0	$\frac{1}{2q}$	0	0	0
λ_i	$\frac{\kappa_i + F_i}{2q}$		$\frac{1}{2q}$	$\frac{1}{2q}$	0	0	0	0

Table 3: Sensitivity coefficients for evaluated length.

yields the following contributions to the combined uncertainty:

$$\begin{aligned} u_c^2(l_{\text{fit}}) = & \sum_{i=1}^{q} \left(\frac{\lambda_i}{2q}\right)^2 u^2(\kappa_i) \\ + & \sum_{i=1}^{q} \left(\frac{\lambda_i}{2q}\right)^2 u^2(F_i) \\ + & \sum_{i=1}^{q} \left(\frac{\kappa_i + F_i}{2q}\right)^2 u^2(\lambda_i) \\ + & \sum_{i=1}^{q} \left(\frac{1}{2q}\right)^2 u^2(\kappa_i)\, u^2(\lambda_i) \\ + & \sum_{i=1}^{q} \left(\frac{1}{2q}\right)^2 u^2(F_i)\, u^2(\lambda_i). \end{aligned} \tag{16}$$

Considering that κ_i represents an integral count of interference orders for the gauge length, and that it is assumed that the order sorting algorithm has chosen the correct length solution, then $u(\kappa_i) \equiv 0$. An incorrect value of κ_i would constitute a blunder [§3.4.7 *Guide*]. Contributions from cross terms are found to be negligibly small; terms making significant contributions are

$$u_c^2(l_{\text{fit}}) = \sum_{i=1}^{q} \left(\frac{\lambda_i}{2q}\right)^2 u^2(F_i) + \sum_{i=1}^{q} \left(\frac{\kappa_i + F_i}{2q}\right)^2 u^2(\lambda_i). \tag{17}$$

The standard uncertainty in reading a fringe fraction has been determined experimentally (Type A) to be 0.01 fringe, taken from 5 readings of a fringe fraction. This value is representative of all fraction measurements, and is independent of wavelength. Uncertainty contributions attributed to the vacuum wavelengths of light are listed in Table 2. For convenience, the second term in (17) can be re-written using (72) so that

$$\frac{\kappa_i + F_i}{2q} = \frac{L}{q\lambda_i}.$$

Thus for the $q = 5$ wavelengths used in the NRC gauge block calibration and referring to Table 2,

$$\begin{aligned} u_c^2(l_{\text{fit}}) &= \sum_{i=1}^{5} \left(\frac{\lambda_i}{2 \cdot 5}\right)^2 (0.01 \text{ fringe})^2 + \sum_{i=1}^{5} \left(\frac{L}{5\lambda_i}\right)^2 u^2(\lambda_i) \\ &= (0.633 \text{ nm})^2 + (0.644 \text{ nm})^2 + (0.509 \text{ nm})^2 + (0.480 \text{ nm})^2 + (0.468 \text{ nm})^2 \\ &\quad + (0.002L)^2 + (0.006L)^2 + (0.006L)^2 + (0.006L)^2 + (0.006L)^2 \\ &= (1.2 \text{ nm})^2 + (0.012L \text{ nm})^2, \end{aligned} \tag{18}$$

for L in millimetres.

4.2 Uncertainty Attributed to Thermal Effects

Recall from (6) that the temperature correction is

$$l_t = \theta\alpha L. \tag{19}$$

Table 4 lists the sensitivity coefficients for the variance and cross terms. Matching the sensitivity coefficients

	Variance Terms	Cross Terms						
	$c_i = \frac{\partial l_t}{\partial x_i}$	$c_{ij} = \frac{\partial^2 l_t}{\partial x_i \partial x_j}$			$c_{ijj} = \frac{\partial^3 l_t}{\partial x_i \partial^2 x_j}$			
x_i		$x_j =$ L	α	θ	L	α	θ
L	$\alpha\theta$	0	θ	α	0	0	0
α	$L\theta$	θ	0	L	0	0	0
θ	αL	α	L	0	0	0	0

Table 4: Sensitivity coefficients for thermal effects.

with the uncertainties yields the following contributions to the combined uncertainty:

$$\begin{aligned} u_c^2(l_t) = & \quad (\alpha\theta)^2\, u^2(L) \\ & + \quad (L\theta)^2\, u^2(\alpha) \\ & + \quad (\alpha L)^2\, u^2(\theta) \\ & + \quad \theta^2\, u^2(L)\, u^2(\alpha) \\ & + \quad \alpha^2\, u^2(L)\, u^2(\theta) \\ & + \quad L^2\, u^2(\alpha)\, u^2(\theta). \end{aligned} \tag{20}$$

Terms making significant contributions are considered in detail below.

4.2.1 Uncertainty in the Gauge Block Temperature Measurement

To evaluate the uncertainty component in (20) attributed to the measurement of gauge block temperature $(\alpha L)^2 u^2(\theta)$, the combined uncertainty in the gauge temperature measurement is used, where $u_c(\theta) = u_c(t_g)$ includes two components: the combined standard uncertainty attributed to temperature measurement using the thermistors at NRC $u_c(t)$ (this includes components for traceable calibration, reading capability, and drift between calibrations, as described below) and the standard uncertainty attributed to possible temperature gradients within the gauge block $u(\theta_{\text{grad}})$. Expressing $u^2(\theta)$ as a combined uncertainty consisting of a quadrature sum:

$$\begin{aligned} (\alpha L)^2 u^2(\theta) &= (\alpha L)^2 u_c^2(\theta) \\ &= (\alpha L)^2 \left(u_c^2(t) + u^2(\theta_{\text{grad}}) \right). \end{aligned} \tag{21}$$

The evaluation is for the case of a steel gauge block, where $\alpha = 11.5$ ppm/K and L is in millimetres.

The combined standard uncertainty attributed to temperature measurement based on thermistor calibration and performance is (see §4.5.4 below)

$$\begin{aligned} u_c(t) &= \sqrt{(2.5 \text{ mK})^2 + (0.1 \text{ mK})^2 + (3 \text{ mK})^2} \\ &= 4 \text{ mK}, \end{aligned} \tag{22}$$

therefore the contribution to the combined uncertainty in the deviation from nominal length is

$$\begin{aligned} u_c(t)\alpha L &= (0.004 \text{ K})(11.5 \times 10^{-6}/\text{K})L \\ &= 0.046L \text{ nm}. \end{aligned} \tag{23}$$

The thermal stability of the NRC laboratory and our experience characterizing the temperature drift properties within the interferometer enclosure are such that the contribution to the uncertainty reflecting potential temperature gradients in the gauge is 5 mK or

$$\begin{aligned} u(\theta_{\mathrm{grad}})\alpha L &= (0.005\ \mathrm{K})(11.5\times 10^{-6}/\mathrm{K})L \\ &= 0.058L\ \mathrm{nm}. \end{aligned} \tag{24}$$

4.2.2 Uncertainty in the Thermal Dilatation Coefficient

The standard uncertainty in the thermal dilatation coefficient $u(\alpha)$ is taken to be 10% of the manufacturers stated value, with a rectangular distribution, as a conservative bound. The gauge temperature value is estimated and recorded for each gauge block; θ is usually not more than 50 mK. For the sake of this example, assume the gauge temperature was measured to be 19.950°C, therefore the component of equation (20) representing the length dependent variance contribution attributed to the thermal dilatation coefficient for a steel gauge block is

$$\begin{aligned} u(\alpha)\theta L &= \frac{10\%\cdot 11.5\times 10^{-6}/\mathrm{K}}{\sqrt{3}}(0.050\ \mathrm{K})L \\ &= 0.033L\ \mathrm{nm}, \end{aligned} \tag{25}$$

for L in millimetres. In the evaluation of the uncertainty for an entire set of gauge blocks, NRC chooses to select the largest value of θ observed during the measurements of the set since the deviation from 20°C is small and quite consistent throughout the set. A more rigorous treatment would evaluate the uncertainty for each gauge block of the set individually, in which case the value of θ measured for each individual gauge block would be used here.

4.2.3 Higher Order Uncertainty Terms in the Thermal Dilatation and Temperature Measurement

The term in equation (20) making a significant contribution to the combined uncertainty in the length measurement is $L^2u^2(\alpha)u^2(\theta)$, where the uncertainty in the gauge temperature measurement $u^2(\theta)$ is a combined uncertainty as described above. Substituting values:

$$\begin{aligned} u(\alpha)u_c(\theta)L &= (0.66\times 10^{-6}/\mathrm{K})\sqrt{(4\ \mathrm{mK})^2+(5\ \mathrm{mK})^2}L \\ &= 0.004L\ \mathrm{nm}. \end{aligned} \tag{26}$$

4.3 Wringing Film

The variation in measurement results due to the wringing film has been evaluated experimentally by repeated measurement of an $L = 2.5$ mm tungsten carbide gauge block both with single-wring and re-wringing measurements. The 1σ standard deviation attributed to wringing effects is $u(w) = 6$ nm. This is an end effect uncertainty. The sensitivity coefficient is

$$c_{l_w} = \frac{\partial d}{\partial l_w} = 1. \tag{27}$$

4.4 Interferometer Optics

4.4.1 Wavefront Errors

The optical components comprising the interferometer were custom made by the optical shop at NRC to meet superior quality standards specifically for this interferometric application. Also, the optical flats used as platens in the NRC gauge block interferometer are of superior flatness (better than $\lambda/20$). The optical quality of the interferometer is such that correction to gauge block length due to wavefront distortion effects is not warranted, however the uncertainty attributed to non-ideal optics and wavefront abberations is determined by performing a length measurement on an optical flat without a gauge block. Instead, a wire that has been bent to represent the outline of a gauge block is laid on the optical flat, so that fringe measurements taken on this gauge-block outline are really just on the optical flat. Worst-case observations in repeated testing excercises for a value which is typically zero are about 5 nm. This value represents a bounded region, therefore the standard uncertainty associated with this end effect is $u(l_A) = 5/\sqrt{3} = 3$ nm. The sensitivity coefficient is

$$c_{l_A} = \frac{\partial d}{\partial l_A} = 1. \tag{28}$$

4.4.2 Obliquity

The obliquity correction, described in more detail in [1] is:

$$l_\Omega = \left(\frac{a^2}{16f^2} + \frac{x^2}{2f^2} \right) L \tag{29}$$

where a is the fibre diameter, f is the focal length of the collimator lens, and x is the lateral offset distance of the collimator aperture from the optic axis of the instrument.

Following the $x \equiv 0$ design characteristic of the Twyman-Green interferometer, the expectation value $\langle x \rangle = 0$, so the obliquity correction to the length measurement includes only the first term in (29). Even though the second term in (29) is zero, the non-zero uncertainty attributed to this alignment shows up in the higher order uncertainty components.

Uncertainty Attributed to Source Size The combined standard uncertainty is calculated by applying (1) to (29):

$$u_{c,1}^2(l_\Omega) = \left(\frac{aL}{8f^2} \right)^2 u^2(a) + \left(\frac{-a^2 L}{8f^3} \right)^2 u^2(f). \tag{30}$$

Substituting values for the fibre diameter $a = 600$ μm with $u(a) = 5$ μm, and for the lens focal length $f = 463$ mm, $u(f) = 0.15$ mm, the contribution of the variance terms to the combined standard uncertainty in length is

$$u_{c,1}(l_\Omega) = 0.002L. \tag{31}$$

The second term in (30) involving $u^2(f)$ is negligible.

	Variance Terms	Higher Order Terms							
	$c_i = \frac{\partial l_\Omega}{\partial x_i}$	$c_{ij} = \frac{\partial^2 l_\Omega}{\partial x_i \partial x_j}$				$c_{ijj} = \frac{\partial^3 l_\Omega}{\partial x_i \partial^2 x_j}$			
x_i		$x_j =$	x	a	f	$x_j =$	x	a	f
x	$\frac{xL}{f^2}$		$\frac{L}{f^2}$	0	$\frac{-2xL}{f^3}$		0	0	$\frac{6xL}{f^4}$
a	$\frac{aL}{8f^2}$		0	$\frac{L}{8f^2}$	$\frac{-aL}{4f^3}$		0	0	$\frac{3aL}{4f^4}$
f	$\frac{-L}{f^3}\left(x^2+\frac{a^2}{8}\right)$		$\frac{-2xL}{f^3}$	$\frac{-aL}{4f^3}$	$\frac{3L}{f^4}\left(x^2+\frac{a^2}{8}\right)$		$\frac{-2L}{f^3}$	$\frac{-L}{4f^3}$	$\frac{-12L}{f^5}\left(x^2+\frac{a^2}{8}\right)$

Table 5: Sensitivity coefficients in the obliquity correction uncertainty.

Higher Order Uncertainty Terms Attributed to Alignment To evaluate the sensitivity coefficients for the higher order terms, it is more convenient to re-write the obliquity correction in the form of

$$l_\Omega = \frac{L}{2f^2}\left(x^2 + \frac{a^2}{8}\right), \tag{32}$$

and to use a table format in the book-keeping of the terms.

Using the partial derivatives given in Table 5, the following coefficients are evaluated:

$$
\begin{aligned}
c_{xx}^2 &= \left(\frac{L}{f^2}\right)^2, && (33)\\
c_{aa}^2 &= \left(\frac{L}{8f^2}\right)^2, && (34)\\
\frac{1}{2}c_{ff}^2 + c_f \cdot c_{fff} &= -\frac{15}{64}\cdot\frac{a^4L^2}{f^8}, && (35)\\
\frac{1}{2}c_{xf}^2 + c_x \cdot c_{xff} &= \frac{8x^2L^2}{f^6}, && (36)\\
\frac{1}{2}c_{fx}^2 + c_f \cdot c_{fxx} &= \frac{2a^2L^2}{8f^6}, && (37)\\
\frac{1}{2}c_{af}^2 + c_a \cdot c_{aff} &= \frac{a^2L^2}{8f^6}, && (38)\\
\frac{1}{2}c_{fa}^2 + c_f \cdot c_{faa} &= \frac{a^2L^2}{16f^6}. && (39)
\end{aligned}
$$

The uncertainty in the lateral alignment of the optical fibre end and exit aperture to the optic axis of the interferometer is $u(x) = 0.05$ mm. Values for the other quantities are as mentioned above. Performing the multiplications, all of the cross terms are either very small or zero, with the exception of the term $i = j = x$. The length dependent uncertainty is therefore

$$
\begin{aligned}
u_{c,2}^2(l_\Omega) &= \frac{1}{2}c_{xx}^2 u^4(x)\\
u_{c,2}(l_\Omega) &= \sqrt{\frac{1}{2}}\left(\frac{L}{f^2}\right)u^2(x)\\
&= 0.008L. && (40)
\end{aligned}
$$

4.5 Refractive Index of Air

Gauge blocks are measured in a laboratory under ambient conditions; however, the metre is defined in terms of the distance that light travels in a vacuum. The refractive index of air n alters the wavelength according to $\lambda_v = n\lambda_{air}$. In most laboratories, as well as at NRC, the refractive index is determined by measuring the properties affecting the density of the air, and then calculating the index using a modified version of the Edlén equation [7, 8].

4.5.1 The Modified Edlén Equation

The modified Edlén equation used in NRC gauge block interferometer measurements is [1]:

$$\begin{aligned}(n-1)\times 10^8 &= \left(8342.54+\frac{2406147}{130-\gamma^2}+\frac{15998}{38.9-\gamma^2}\right)\left(\frac{p}{96095.43}\right)\\ &\quad\times\left(\frac{1+10^{-8}(0.601-0.00972t)p}{1+0.0036610t}\right)\\ &\quad -R(8.753+0.036588t^2)(0.037345-0.000401\gamma^2),\end{aligned} \tag{41}$$

where p represents air pressure in Pascal units, t represents temperature in degrees Celcius, R represents relative humidity in percent, and $\gamma = 1/\lambda$ is the vacuum wavenumber in μm^{-1} units. The uncertainty attributed to the empirical determination of the numerical coefficients in this equation is 1×10^{-8} at the 1σ level of confidence [8].

4.5.2 Combined Standard Uncertainty and Sensitivity Coefficients

The combined standard uncertainty $u_c(l_n)$ in the length correction $l_n = (n-1)L$ is determined by applying (1) to l_n, considering $(n-1)$ as expressed in (41), so that

$$\begin{aligned}u_c^2(l_n) &= \left(\frac{\partial l_n}{\partial p}\right)^2 u_c^2(p)+\left(\frac{\partial l_n}{\partial t}\right)^2 u_c^2(t)+\\ &\quad\left(\frac{\partial l_n}{\partial R}\right)^2 u_c^2(R)+\left(\frac{\partial l_n}{\partial \lambda}\right)^2 u_c^2(\lambda).\end{aligned} \tag{42}$$

The sensitivity coefficients are evaluated by first performing the partial derivatives of (42) using (41):

$$\frac{\partial l_n}{\partial R} = -\left(8.753+0.036588t^2\right)\left(0.037345-0.000401\gamma^2\right)\times 10^{-8}L \tag{43}$$

$$\begin{aligned}\frac{\partial l_n}{\partial p} &= \frac{10^{-8}}{96095.43(1+0.0036610t)}\left(8342.54+\frac{2406147}{130-\gamma^2}+\frac{15998}{38.9-\gamma^2}\right)\\ &\quad\times\left(1+2\cdot 10^{-8}p(0.601-0.00972t)\right)L\end{aligned} \tag{44}$$

$$\begin{aligned}\frac{\partial l_n}{\partial t} &= \frac{-10^{-8}}{96095.43(1+0.0036610t)^2}\left(8342.54+\frac{2406147}{130-\gamma^2}+\frac{15998}{38.9-\gamma^2}\right)\\ &\quad\times\left[0.00972(1+0.003661t)p^2\cdot 10^{-8}L\right.\\ &\quad\left.+0.003661p\left(1+10^{-8}p(0.601-0.00972t)\right)\right]L\end{aligned} \tag{45}$$

$$\begin{aligned}\frac{\partial l_n}{\partial \lambda} &= -10^{-8}\gamma^2\left[\left(\frac{2406147\gamma}{(130-\gamma^2)^2}+\frac{15998\gamma}{(38.9-\gamma^2)^2}\right)\right.\\ &\quad\times\frac{2p}{96095.43(1+0.003661t)}(1+10^{-8}p(0.601-0.00972t))L\\ &\quad\left.+2R\gamma\cdot 0.000401\left(8.753+0.036588t^2\right)\right]L.\end{aligned} \tag{46}$$

Routine environmental conditions in the NRC laboratory are used in the numerical evaluation of the partial derivatives; they are: $t = 20°\text{C}$, $\lambda = 0.633$ μm, $p = 101325$ Pa, and $R = 44\%$. Substitution yields the following values for the sensitivity coefficients:

$$\left.\frac{\partial l_n}{\partial R}\right|_{t,\lambda} = -8.5\times 10^{-9}L/\%, \tag{47}$$

$$\left.\frac{\partial l_n}{\partial p}\right|_{t,\lambda} = 2.7 \times 10^{-9} L/\mathrm{Pa}, \tag{48}$$

$$\left.\frac{\partial l_n}{\partial t}\right|_{t,\lambda,p,R} = -9.5 \times 10^{-7} L/\mathrm{K}, \tag{49}$$

$$\left.\frac{\partial l_n}{\partial \lambda}\right|_{t,\lambda,p,R} = -1.2 \times 10^{-5} L/\mu\mathrm{m}. \tag{50}$$

4.5.3 Evaluation of Water Vapour Partial Pressure

The Edlén equation originally considered the vapour partial pressure of water ρ explicitly, rather than in terms of the relative humidity R. The vapour partial pressure is related by definition to the saturation vapour pressure ρ_s and the relative humidity

$$R \equiv \frac{\rho}{\rho_s} \times 100. \tag{51}$$

NRC calculates ρ_s from a quadratic function of temperature fitted through saturation pressure values spanning room temperature, using the 1984 NBS/NRC Steam Tables [9] so that:

$$\rho = R(8.753 + 0.036588t^2), \tag{52}$$

where ρ is in Pascal units, R is in percent and t is in degrees Celcius. The relative uncertainty attributed to equation (52) itself, based on the vapour partial pressure, is approximately one part in 10^3. It is known that a change in relative humidity of -1% corresponds to a change in refractive index of 1×10^8 [10], and the uncertainty attributed to the empirical fit of the Edlén equation itself is also 1×10^8 (see above), therefore the impact on the uncertainty evaluation resulting from the application of equation (52) to the refractive index of air is deemed negligible.

4.5.4 Measured Parameters Influencing the Refractive Index of Air

The combined uncertainty attributed to the influence of measuring air temperature, pressure and relative humidity all similarly include the following three uncertainty components:

- accuracy and calibration of the sensor,
- sensor reading capability, and
- drift of the sensor between calibrations.

Each sensor type is discussed below according to these criteria. The combined standard uncertainty of the sensor then has the form:

$$u_c^2(x) = u^2(x_{\mathrm{cal}}) + u^2(x_{\mathrm{read}}) + u^2(x_{\mathrm{drift}}). \tag{53}$$

Length dependent uncertainties assume L in millimetres.

Air Temperature The combined standard uncertainty in the NRC thermometry calibration of the bead-in-glass thermistors used in the gauge block interferometer is 2.5 mK. Multiplying by the sensitivity coefficient of (49):

$$\begin{aligned} u(t_{\text{cal}}) &= (0.002\,5 \text{ K})(-9.5 \times 10^{-7} L/\text{K}) \\ &= -0.002L \text{ nm}. \end{aligned} \tag{54}$$

The digital resolution of the thermistor reading is 0.4 mK, therefore the uncertainty associated with the device reading capability is

$$\begin{aligned} u(t_{\text{read}}) &= \frac{0.000\,4 \text{ K}}{\sqrt{12}}(-9.5 \times 10^{-7} L/\text{K}) \\ &= -0.000\,1L \text{ nm}. \end{aligned} \tag{55}$$

Drift within the time interval between calibrations is about 3 mK, assumed rectangularly distributed, therefore

$$\begin{aligned} u(t_{\text{drift}}) &= \frac{0.003 \text{ K}}{\sqrt{3}}(-9.5 \times 10^{-7} L/\text{K}) \\ &= -0.002L \text{ nm}. \end{aligned} \tag{56}$$

Air Pressure The standard uncertainty in the NRC pressure calibration is 0.05% of the pressure reading. Pressure readings are in the neighborhood of 100 kPa, therefore the 1σ uncertainty in the device calibration is 50 Pa. Multiplying by the sensitivity coefficient $2.7 \times 10^{-9} L/\text{Pa}$ from (48), the length dependent uncertainty becomes

$$\begin{aligned} u(p_{\text{cal}}) &= (50 \text{ Pa})(2.7 \times 10^{-9} L/\text{Pa}) \\ &= 0.135L \text{ nm}. \end{aligned} \tag{57}$$

The device reading capability is 13 Pa on a digital meter, therefore

$$\begin{aligned} u(p_{\text{read}}) &= \frac{13 \text{ Pa}}{\sqrt{12}}(2.7 \times 10^{-9} L/\text{Pa}) \\ &= 0.011L \text{ nm}. \end{aligned} \tag{58}$$

The standard uncertainty attributed to the average 1-year drift between calibrations is 54 Pa, therefore

$$\begin{aligned} u(p_{\text{drift}}) &= (54 \text{ Pa})(2.7 \times 10^{-9} L/\text{Pa}) \\ &= 0.146L \text{ nm}. \end{aligned} \tag{59}$$

Humidity The accuracy quoted by the manufacturer of the humidity sensor is 2% in the range for which it is operated, which is assumed to be rectangularly distributed. Using the sensitivity coefficient $-8.5 \times 10^{-9}/\%$ from (47), the contribution to the combined uncertainty in the length measurement is then

$$\begin{aligned} u(R_{\text{cal}}) &= \frac{2\%}{\sqrt{3}}(-8.5 \times 10^{-9}) L/\% \\ &= -0.010L \text{ nm}. \end{aligned} \tag{60}$$

The device reading capability is 0.1% on a digital meter, therefore

$$\begin{aligned} u(R_{\text{read}}) &= \frac{0.1\%}{\sqrt{12}}(-8.5 \times 10^{-9}) L/\% \\ &= -0.001L \text{ nm}. \end{aligned} \tag{61}$$

The uncertainty attributed to the 1-year average drift between calibrations is 1% or

$$\begin{aligned} u(R_{\text{drift}}) &= (1\%)(-8.5 \times 10^{-9})L/\% \\ &= -0.009L \text{ nm}. \end{aligned} \quad (62)$$

4.5.5 Vacuum Wavelength

The combined uncertainty attributed to the vacuum wavelength through its contribution to the refractive index of air is very small. Using values from Table 2 and equation (50),

$$u_c(l_{n,\lambda}) = (0.01 \times 10^{-6} \cdot 0.633\ \mu\text{m})(-1.2 \times 10^{-5} L/\mu\text{m}) \quad (63)$$

which is negligible.

4.5.6 CO_2 Content

In the revised version of the Edlén equation [8], Birch and Downs assumed an average value of 450 ppm for the CO_2 content of air and assumed a 1σ uncertainty of 57 ppm. No further corrections are made as to the actual content of CO_2 present in the labaoratory at NRC in our application of the modified Edlén equation. It is assumed that the uncertainty attributed to the CO_2 content is incorporated in the uncertainty in the Edlén equation.

4.6 Gauge Block Departure from Flatness and Parallelism

At NRC, l_G is assumed to be zero since the technique of optical interferometry demands superior quality gauge blocks. To evaluate the uncertainty resulting from slight deviations in gauge geometry by either imperfect flatness or parallelism of the measuring faces, the geometry causing the most pronounced variation in central length is considered. Referring to Figure 1, non-parallelism has a larger impact on the central length measurement than a deviation in flatness of similar magnitude. Furthermore, for the same magnitude departure from parallelism the effect will be more pronounced in the crosswise direction than in the lengthwise direction. The worst case scenario of deviation from crosswise parallelism is selected for estimating the uncertainty due to poor gauge block geometry.

Our ability to point to the centre of a rectangular gauge block is typically $1/20^{th}$ of the 9 mm width. The 2σ uncertainty in pointing to the centre of the gauging surface is then 0.5 mm. The uncertainty in the length is taken as the wedge height difference between the measured length at the reference point and the length at the maximum offset point due to poor pointing, as demonstrated in the inset of Figure 1. Using similar triangles, the resultant central length difference for a 50 nm deviation in parallelism[2] along the crosswise direction of the gauge is

$$\tan\epsilon = \frac{50 \text{ nm}}{9 \text{ mm}} = \frac{y}{0.5 \text{ mm}} \quad (64)$$

and therefore

$$u(l_G) \simeq 2 \text{ nm} \quad (65)$$

[2]International Standard ISO 3650-1978(E) [4] gives the maximum permitted deviation from flatness for the 00 grade gauge block as 50 nm for blocks of nominal length up to and including 150 mm. Only high grade gauge blocks can be calibrated by interferometry — most of these gauges have flatness and parallelism deviations less than 50 nm. In cases of larger deviations in geometry, the uncertainty is evaluated separately for the individual gauge.

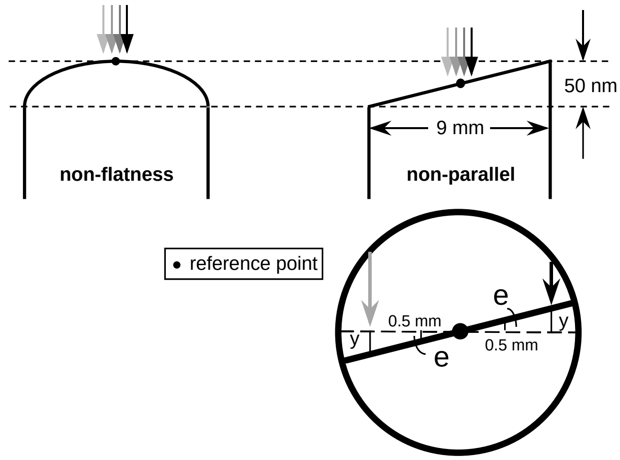

Figure 1: Effect of non-flatness and non-parallelism on the measurement of the central length of a gauge block. In the ideal case, a larger variation is observed in trying to target the reference point on a non-parallel rather than non-flat gauge block. The variable y in the inset represents the uncertainty in the central length measurement attributed to non-parallelism.

at 1σ. The sensitivity coefficient

$$c_{l_G} = \frac{\partial d}{\partial l_G} = 1. \tag{66}$$

4.7 Phase-Change Correction

The phase-change correction is an experimentally measured end effect that is evaluated for each set of gauge blocks of like material and surface finish by the method of pack experiments. A pack (or stack) experiment, consists of measuring four short gauge blocks individually, then wringing them into a pack and measuring the pack the same way the individual gauge blocks were measured. The difference in the measured length of the pack and the sum of the measured lengths of the four gauge blocks gives the length difference attributed to reflection effects for the gauges nested in the pack. Expressed mathematically, this phase change correction is

$$l_\phi = \frac{1}{m-1}\left(l_p - \sum_{i=1}^{m} l_i\right) \tag{67}$$

where l_p represents the measured length of the pack, and l_i the measured lengths of the m individual gauges making up the pack (see [1] for detail).

The evaluation of the phase-change correction by the method of a pack experiment is a measure of the differences in like-measurements. Because it is a relative measurement, the perfectly correlated uncertainty components will sum to zero in the addition of the combined standard uncertainty components for the pack experiment (see Appendix B for detail). The uncertainty components remaining will be those that are characterized by stochastic processes. These include most end-effects and individual instrument reading uncertainties.

The combined standard uncertainty attributed to the phase-change correction is therefore the sum of the uncorrelated components of the length measurement. Applying equation (1) for evaluating the uncertainty to equation (78), the uncertainty in the phase change correction is then:

$$u_c^2(l_\phi) \quad = \quad \frac{1}{(m-1)^2}u^2(l_p) + \frac{1}{(m-1)^2}\sum_{i=1}^{m} u^2(l_i)$$

$$= \frac{1}{(m-1)^2} \sum_{i=1}^{m+1} u_{\mathrm{uc}}^2(l_i), \tag{68}$$

where the sum over $m+1$ gauge blocks includes each of the m constituent gauges and the pack, and $u_{\mathrm{uc}}^2(l)$ are the uncorrelated uncertainty components (marked with a double dagger in Table 1). Gauge blocks used in the pack experiments are short, so the length dependent uncorrelated uncertainties are small. The components making significant contributions are the end effect uncertainties:

$$u_c^2(l_\phi) \simeq \frac{(m+1)}{(m-1)^2} \left[u_c^2(F_i) + u^2(l_w) + u^2(l_A) + u^2(l_G)\right]. \tag{69}$$

For four gauge blocks used in the pack experiment, and substituting values for the uncertainties from Table 1,

$$\begin{aligned} u_c^2(l_\phi) &= \frac{5}{9}\left[(1.2\text{ nm})^2 + (6\text{ nm})^2 + (3\text{ nm})^2 + (2\text{ nm})^2\right] \\ u_c(l_\phi) &= 6\text{ nm}. \end{aligned} \tag{70}$$

The uncertainty attributed to differences in surface roughness between the platen and gauge block surfaces is not separately accounted for. Effects stemming from differences in the reflection properties of materials, such as the surface roughness, are considered to be integrated in the result of the pack experiments.

5 Conclusion

The expanded uncertainty in the deviation from nominal length for a single measurement of a gauge block has been evaluated. Standard uncertainty components are summarized in Table 1. Performing the quadrature sum of length and end effects separately, the expanded uncertainty is

$$U = 2\sqrt{9.3^2 + 0.22^2 L^2}\text{ nm} \tag{71}$$

for L in millimetres. One must bear in mind the uncertainty in the uncertainties, and not attach too many significant digits to the final expression. The expanded uncertainty U represents a confidence level of approximately 95% that the measured value is within $\pm U$ of the value of the measurand. The expanded uncertainty is obtained by multiplying the standard uncertainty u_c of the length measurements (assumed to be normally distributed) by a coverage factor $k = 2$.

Equation (71) above gives the value for the estimated expanded uncertainty. Some clients prefer the simpler linear approximation $U = 19 + 0.28L$ nm, determined by the equation of a straight line $y = a + bL$ joining the points represented by $L = 0$ mm and $L = 100$ mm. The linear approximation must be specified for a restricted range of nominal length for which it is valid. For short gauge blocks this range is 0–100 mm. The linear approximation will have a negligible difference from the quadrature sum in the case that bL is small compared to a, or conversly if a is small compared to bL. The latter is more likely the case for shop-floor calibrations where temperature effects dominate the uncertainty evaluation. The largest difference between the linear approximation and the quadrature sum will be for the case where a and bL are of comparable magnitudes. This is the case of lowest-uncertainty calibrations performed at national labs, where the systems have been optimized specifically to reduce the largest components of uncertianty. In our example, the maximum difference between the linear approximation and the quadrature sum occurs for $L = 37$ mm, where $U_{\mathrm{linear}} = 29$ nm and $U_{\mathrm{quad}} = 25$ nm. Note that the linear approximation always gives a more conservative (*i.e.,* larger) value than the quadrature sum.

6 Acknowledgements

The authors thank John Stoup (NIST), Andrew Lewis (NPL) and Nick Brown (CSIRO) for interesting discussions, as well as NRC colleagues Kostadin Doytchinov and George Chapman for scrutinizing the manuscript.

References

[1] Decker, J.E. and Pekelsky, J.R., 1996, "Gauge Block Calibration by Optical Interferometry at the National Research Council of Canada," Measurement Science Conference, 23, 24 January 1997, Pasadena, California, 17 pages. [NRC 40002]

[2] "Guide to the Expression of Uncertainty in Measurement", 1993, 1^{st} Edition (International Organization for Standardization, Switzerland), 101 pages.

[3] "Standard Reference Temperature for Industrial Length Measurements," ISO 1–1975(E), 1988, *ISO Standards Handbook 33: Applied Metrology — Limits, fits and surface properties*, 1^{st} Edition, (ISO Central Secretariat, Switzerland), p. 223.

[4] "Gauge Blocks," ISO 3650–1978(E), 1988, *ISO Standards Handbook 33: Applied Metrology — Limits, fits and surface properties*, 1^{st} Edition, (ISO Central Secretariat, Switzerland), pp. 778–786.

[5] Quinn T.J., "*Mise en Pratique* of the definition of the metre (1992)," *Metrologia*, 1994, **30**, 523–541.

[6] *Comité Consultatif pour la Définition du Mètre,* 3rd session, 1962, pp. 18–19 and *Procès-Verbaux CIPM,* 52nd session, 1963, pp. 26–27.

[7] Edlén B., 1966, "The refractive index of air," *Metrologia*, **2**(2), pp. 71–80.

[8] Birch, K.P. and Downs, M.J., 1994, "Correction to the updated Edlén equation for the refractive index of air", *Metrologia*, **31**, pp. 315–316.

[9] Haar L., Gallagher J.S., Kell G.S., 1984, "NBS/NRC Steam Tables: Thermodynamic and transport properties and computer programs for vapor and liquid states of water in SI units," (Hemisphere Publishing Corp., Washington), p. 9.

[10] Estler, W.T., 1985, "High-accuracy Displacement Interferometry in Air," *Appl. Opt.,* **24**, pp. 808–815.

[11] Born, M. and Wolf, E., 1980, Principles of Optics, 6^{th} ed., (Pergamon Press, New York), pp. 286–306.

[12] Michelson, A.A. and Benoit, J.R., 1895, "Valeur du mètre en longueurs d'ondes," *Trav. et Mem. Int. Bur. Poids et Mes.,* **11**, p. 1, and Benoit J.R., 1898, "Application des phénomènes d'interférence à des déterminations métrologiques," *J. de Phys.,* **3**(VII), pp. 57–68.

A Evaluation of Length From Fringe Fraction Measurements

As discussed in [1], gauge block length is determined by measuring the fractional difference in the interference pattern formed on the top of the gauge block and the one formed on the top surface of the optical flat. The calculation deducing the gauge length from these fraction measurements is based on the traditional method of exact fractions [11, 12] which exploits the fact that a given length l can be represented by a unique set of interference orders m_i for wavelengths λ_i:

$$l = m_1 \frac{\lambda_1}{2} = m_2 \frac{\lambda_2}{2} = \ldots = m_i \frac{\lambda_i}{2}, \tag{72}$$

where m_i are real numbers consisting of an integer portion κ_i and a fractional portion F_i.

NRC's length evaluation based on equation (72) is performed by a regression-style computer program. The measurement begins by measuring fringe fractions F_i for each of the five wavelengths λ_i listed in Table 2. At NRC, the measured fractions are corrected for the refractive index of air n so that they represent fractions that would have been measured in vacuum. Vacuum wavelengths are used in the fitting routine. The computer program steps through test lengths for the deviation from nominal length, probing the value of the residual $\delta = \sum_i (F_i - \hat{F}_i)$ summed over all the wavelengths used in the measurement, where F_i is the measured fraction, and $\hat{F}_i$ the calculated fraction based on the test length. The criterion for choosing a solution is the minimum residual sum δ.

The solution for the *deviation from nominal length* corresponding to the best fit has the form of:

$$d_{\text{fit}} = \frac{1}{q} \sum_{i=1}^{q} \left[(\kappa_i + F_i) \frac{\lambda_i}{2} - L + (n-1)L \right]. \tag{73}$$

The algorithm returns a value for $d_{\text{fit},i}$ having unique values of κ_i and F_i for each of the λ_i. Because of the noise in the experimental measurement, the values $d_{\text{fit},i}$ differ slightly. Equation (73) expresses the average of the $d_{\text{fit},i}$ over the q wavelengths applied in the measurement. Written another way:

$$d_{\text{fit}} = \sum_{i=1}^{q} \left(\frac{\kappa_i + F_i}{q} \right) \frac{\lambda_i}{2} - L + (n-1)L. \tag{74}$$

The subtraction of the nominal length L and the refractive index correction $(n-1)L$ are taken out of the summation in order to simplify the calculations. In this document, the deviation from nominal length is

$$d_{\text{fit}} = l_{\text{fit}} - L, \tag{75}$$

which leads to equation (5). For purposes of simplifying the uncertainty evaluation, the deviation from nominal length will be written in this form. The refractive index correction is treated separately. Particular details concerning the evaluation of l_{fit} based on NRC's application of the method of exact fractions will be ignored. Differences arising in the uncertainty as a result of this simplification are deemed negligible.

B Correlated Components in the Pack Experiment

To demonstrate the zero sum of the correlated uncertainty components in the pack experiment, let us consider that the combined standard uncertainty in the length measurement $u_c(l)$ is made up of correlated and independent components of uncertainty. These two aspects of the uncertainty are represented by their respective quadrature sums:

$$u_c^2(l) = u^2(l^0) + u^2(l'), \tag{76}$$

where $u(l^0)$ represents the quadrature sum of the correlated uncertainties and $u(l')$ that of the uncorrelated components. For simplicity, the superscript notation will be dropped, and we turn our interest to the quadrature sum of the correlated uncertainty components $u(l_i)$ for gauge i having measured length l_i. Typically the correlated uncertainties are length dependent, and their sum takes on the simple form of

$$u(l_i) = \beta L_i, \tag{77}$$

where β is a constant.

It is useful here to write the phase change correction in the following form:

$$l_\phi = \frac{1}{m-1}\sum_{k=1}^{m+1} s_k l_k. \quad s_k = \begin{cases} +1 & \text{if } k=1 \\ -1 & \text{otherwise} \end{cases} \tag{78}$$

The general expression for the combined standard uncertainty in the pack experiment including correlated uncertainty components is [§5.2.2 *Guide*]

$$u_c^2(l_\phi) = \sum_{i=1}^{N}\left(\frac{\partial l_\phi}{\partial l_i}\right)^2 u^2(l_i) + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(\frac{\partial l_\phi}{\partial l_i}\right)\left(\frac{\partial l_\phi}{\partial l_j}\right) u(l_i)u(l_j)r(l_i,l_j). \tag{79}$$

The combined uncertainty is a sum over the N influence parameters, where the correlation coefficient $r(l_i, l_j)$ is unity for correlated components and zero for the uncorrelated components. As we are considering only the correlated uncertainties in the repeated length measurement for the phase-change correction, the correlation coefficient is unity. The number of parameters in the sum of (79) for the pack experiment is $N = m + 1$. Substituting (77) and (78) into (79):

$$u_c^2(l_\phi) = \sum_{i=1}^{m+1}\left(\frac{s_i}{m-1}\right)^2(\beta L_i)^2 + 2\sum_{i=1}^{m}\sum_{j=i+1}^{m+1}\frac{s_i s_j}{(m-1)^2}(\beta L_i)(\beta L_j). \tag{80}$$

Separating out terms involving $i = 1$, which as expressed in (78) is the index representing the pack,

$$u_c^2(l_\phi)\frac{(m-1)^2}{\beta^2} = L_p^2 + \sum_{i=2}^{m+1} L_i^2 + 2\sum_{i=1}^{1}\sum_{j=i+1}^{m+1} s_1 s_j L_p L_j + 2\sum_{i=2}^{m}\sum_{j=i+1}^{m+1} s_i s_j L_i L_j. \tag{81}$$

Further combining terms, use is made of equation (78) and the conditions of s_k, as well as the expression of the pack *nominal* length as the sum of the nominal lengths of the constituent gauges:

$$L_p = \sum_{k=1}^{m} L_k, \tag{82}$$

and also

$$L_p^2 = \sum_{k=1}^{m} L_k^2 + 2\sum_{k=1}^{m-1}\sum_{q=k+1}^{m} L_k L_q. \tag{83}$$

The sum of correlated components then becomes:

$$\begin{aligned} u_c^2(l_\phi)\frac{(m-1)^2}{\beta^2} &= L_p^2 + \sum_{k=1}^{m} L_k^2 + 2\sum_{q=1}^{m}(-1)L_p L_q + 2\sum_{k=1}^{m-1}\sum_{q=k+1}^{m}(+1)L_k L_q \\ &= L_p^2 - 2L_p^2 + \sum_{k=1}^{m} L_k^2 + 2\sum_{k=1}^{m-1}\sum_{q=k+1}^{m} L_k L_q \\ &= L_p^2 - 2L_p^2 + L_p^2 \\ &= 0. \end{aligned}$$

The sum of the perfectly correlated components of the combined standard uncertainty associated with the pack experiment is exactly zero.