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DISCRIMINATION COEFFICIENT OF VARIATION: AN IMPROVED MEASURE OF PRECISION OF ESTIMATES

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ABSTRACT

In this paper we propose a generalization of the usual coefficient of variation (CV) to address some of the known problems when used in criteria developed to determine precision of estimates. Some of the problems associated with CV include interpretation when the estimate is near zero, and the inconsistency in the interpretation about precision when computed for different one-to-one monotonic transformations. The proposed measure, termed discrimination coefficient of variation (DCV), generalizes CV using the hypothesis testing ideas involving length of the confidence interval (LCI) and length of the discrimination interval (LDI). This discrimination interval is used to check whether the sample is large enough or the confidence interval is short enough to discriminate with certain power between the current value and postulated change in the current value.

KEY WORDS: Effective sample size; Length of confidence interval; Length of discrimination interval; Suppression of Estimates

1. INTRODUCTION

Often as a result of processing a large data set, estimates in bulk are disseminated in the form of tables. To users, interpretation of these estimates could be misleading if no warning is given about their precision. Typically, standard errors are provided in a separate table if the user wants to look them up. However, even with information about estimated standard errors (se) some guidance is needed for the user to decide whether the estimate satisfies a suitable precision threshold. To this end, precision rules are used by the data producer to decide whether certain estimates should be suppressed or not, or whether they should be published with some qualifiers. For this purpose, a commonly used measure is coefficient of variation (also known as relative se), $CV(\hat{\theta})$ which is defined as the ratio of $se(\hat{\theta})$ over $\hat{\theta}$ assuming that $\hat{\theta} > 0$. Observe that it is natural to say that an estimate $\hat{\theta}$ is imprecise if its $se(\hat{\theta})$ is too large. However, the main problem is to decide how large is too large. To address this problem, a suitable standardization $se(\hat{\theta})$ of would be desirable so that it does not depend on the unit of measurement. Then relate the standardized $se(\hat{\theta})$ to the confidence interval (CI; generally based on $se(\hat{\theta})$) which has a direct practical interpretation in terms of whether $\hat{\theta}$ is precise enough to detect a certain change in θ. One way to do it is to consider the length of the confidence interval (LCI which is generally proportional to $se(\hat{\theta})$) relative to the point estimate $\hat{\theta}$. Observe that the usual CV is a form of standardized LCI because considering the large sample symmetric normal $(1-\alpha)$ confidence interval and denoting the upper α point under the standard normal distribution by z_α, we have

$$\hat{\theta} \pm z_{\alpha/2} se(\hat{\theta}) = \hat{\theta}(1 \pm z_{\alpha/2} CV(\hat{\theta})) \quad . \tag{1.1}$$

As a measure of precision of estimates, CV is commonly used as it has many merits.

- It is simple to understand.
- It is proportional to the large sample relative CI length.

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- It provides a standardized (or scale free) measure of precision around the mean.
- It allows for comparing two estimates with different means.
- It is useful for design or redesign of experiments and deciding about sample allocation.
- It provides caution to users regarding precision of published estimates.

However, there are several limitations of CV as a measure of precision. Some of these limitations arose in the context of publishing estimates of drug use prevalence from the National Survey on Drug Use and Health (NSDUH, previously with the acronym NHSDA) conducted by RTI International. In the following, special consideration is given to the case of estimating proportions as it is useful in highlighting the main problems in using CV. Also to distinguish the general parameter θ from the particular case of proportions, we replace $\hat{\theta}$ by $\hat{p}$ as appropriate.

(i) $CV(\hat{\theta})$ is not meaningful when $\hat{\theta} < 0$. This concern is only minor since we can easily redefine it as $CV(\hat{\theta}) = se(\hat{\theta}) / |\hat{\theta}|$.

(ii) $CV(\hat{\theta})$ is not useful when $\hat{\theta}$ is zero or near zero. For example, if $\hat{\theta}$ denotes a difference between two estimates, it may be close to zero but its $se(\hat{\theta})$ may not be small; see e.g. Kish (1965). In this case, CV could be high even for large samples. In the case of proportions, CV becomes extreme as $\hat{p}$ gets close to zero even for large samples. In both cases, the resulting CI may be precise enough to discriminate postulated changes in the current value of θ *or* p. However, this is not what would be implied by CV.

(iii) In the case of proportions, CV is not defined if $\hat{p}$ is 0 as $se(\hat{p})$ is also zero. Thus, it is unable to capture any relevant information in the estimator. However, CI (such as the one-sided CI based on exact binomial without using se) may provide meaningful information; see e.g., Jovanovic an Levy (1997).

(iv) $CV(\hat{p})$ is not symmetric in $\hat{p}$ about .50; i.e., $CV(\hat{p}) \neq CV(1-\hat{p})$. This concern is also somewhat minor as it can be easily addressed by restricting the definition of $CV(\hat{p})$ to the interval (0,.50] and then define $CV(\hat{p})$ as $CV(1-\hat{p})$ for $\hat{p}$ in the interval [.50, 1).

(v) In practice, an ad hoc cut-off value c_0 (such as .25 or .50) for CV is used to decide if the estimate is precise enough or not. For example, if CV exceeds c_0, then the estimate may be deemed unaccepatable. In the case of proportions, such a rule leads to extreme behavior in acceptable CIs when $\hat{p}$ is near .50 or the endpoints (0 or 1); see e.g., Folsom (1991). This can be explained as follows. For each estimate $\hat{p}$ acceptable by the CV-rule, there is a corresponding minimum sample size $n_{\min}(\hat{p})$ required to satisfy the CV-cut-off. (With complex surveys, $n_{\min}(\hat{p})$ is divided by the design effect to get the minimum effective sample size.) Suppose the cut-off value c_0 is chosen by setting $n_{\min}(\hat{p}) = 55$ when $\hat{p} = 0.1$; the choice of the pair (.10, 55) is used in NSDUH and is based on considerations for the corresponding LCI being reasonable. With this rule, one can compute $n_{\min}(\hat{p})$ as $\hat{p}$ varies. It turns out that the resulting CI can be very wide at or near .50 and very narrow near 0.

(vi) $CV(\hat{\theta})$ is not invariant to location and scale transformations of estimated proportions $\hat{\theta}$ or to any one-to-one monotonic transformations, in general. With log or logit transformations in the case of proportions, it leads to conflicting interpretation of precision compared to the untransformed case; see e.g., Chromy (2001). It can be explained as follows. While with CV($\hat{p}$), $n_{\min}(\hat{p})$ as a function of $\hat{p}$ decreases monotonically as $\hat{p}$ increases from 0 to 0.5; with CV(log $\hat{p}$), however, $n_{\min}(\log \hat{p})$ as a function of $\hat{p}$ is non-monotonic. In fact, $n_{\min}(\log \hat{p})$ first decreases as $\hat{p}$ approaches to about .20 and then increases as $\hat{p}$ approaches 0.5. With CV(logit $\hat{p}$), the problem of conflicting interpretation gets even worse because as $\hat{p}$ goes to 0.5, CV(logit $\hat{p}$) goes to infinity for a given n. Thus, using CV as a measure of precision, different monotonic transformations of $\hat{\theta}$ may give rise to quite different interpretations of precision for the same sample size.

The above limitations suggest that there might be a problem with the basic definition of CV itself and what is needed is a new measure of precision for the objectives in mind. One of the main objectives is to have a principled approach to the choice of the cut-off value c_0 so that both level and length of the CI are controlled. It may be true that the property of invariance to any monotonic smooth transformation may be too demanding for any measure of precision. However, it would be desirable to have a measure which allows for variation in its values for different monotonic transformations but not too drastically. To this end, we propose a new measure termed discrimination coefficient of variation (DCV) which generalizes the usual CV using hypothesis testing ideas such that both level and length of the CI are controlled. Section 2 contains a motivation of the proposed measure while Section 3 presents a description of DCV along with theoretical properties and a numerical illustration. Finally, Section 4 contains summary and discussion.

2. MOTIVATION OF THE PROPOSED MEASURE

Based on the limitations listed above, it is clear that the basic requirements of a measure of precision need to be clearly defined. That is, unlike CV, which is based on heuristic grounds, an alternative measure based on a principled approach is needed. It is assumed in the following that θ is scalar and takes values in an open subset of the real line. Possible generalizations to the case of multidimensional θ are discussed in the last Section. For unidimensional θ, we work with a bounded interval $[\underline{\theta},\overline{\theta}]$ in which $\hat{\theta}$-values of interest may lie. For example, in the case of proportions, the interval of interest can be taken as [.01, .50]. For the complementary interval [.50, .99], all the properties of the proposed measure carry over by symmetry arguments. We will also assume without loss of generality that $\underline{\theta}$ is nonnegative again by reason of symmetry.

First we observe that for an estimate $\hat{\theta}$ of a mean parameter θ to be precise, we need $LCI(\hat{\theta})$ to be small corresponding to a given confidence level $1-\alpha$; i.e., the sample size n should be large enough for the confidence interval to be tight. For large sample normal CI, $LCI(\hat{\theta})$ equals $2z_{\alpha/2}se(\hat{\theta})$ which implies that $se(\hat{\theta})$ should be small for $\hat{\theta}$ to be precise where $se(\hat{\theta}) = s(\hat{\theta})/\sqrt{n}$ assuming simple random sampling (if not, use the effective sample size obtained by dividing n by the design effect for complex samples) and $s(\hat{\theta})$ is the sample standard deviation which may depend on $\hat{\theta}$ in general as in the case of proportions. Now to define objectively a cut-off value signifying when $LCI(\hat{\theta})$ is too large, it is useful to link CI with hypothesis testing via the test inversion method. Note that at the design stage (i.e., before the issue of precision of $\hat{\theta}$ comes up at the estimation stage), one often uses the power analysis to determine n so that with power $1-\beta/2$, an alternative $\theta_1 (>\theta_0)$ can be detected while the two-sided level of the test is maintained at α. In other words, we want the sample size to be large enough for the test with Type I error α and Type II error $\beta/2$ to be able to discriminate the amount of change $\theta_1-\theta_0$ in θ with respect to the past level θ_0. Using the normal test for large samples, i.e., with the pivot $(\hat{\theta}-\theta_0)/se(\hat{\theta})$ having the standard normal distribution, it is well known that the minimum sample size (see e.g., Kupper and Hafner, 1989) for the desired amount of discrimination in change of θ is given by

$$n \geq \frac{s(\hat{\theta})^2}{(\theta_1-\theta_0)^2 c^2(\alpha,\beta)}, \quad c(\alpha,\beta) = (z_{\alpha/2}+z_{\beta/2})^{-1}. \tag{2.1}$$

Equivalently, at the estimation stage, this means that for $\hat{\theta}$ to be precise, the acceptable maximum $se(\hat{\theta})$ is given by

$$\frac{se(\hat{\theta})}{\theta_1-\theta_0} \leq c(\alpha,\beta) \tag{2.2}$$

Now, in terms of symmetric CI which is the case with large sample normal CI, above implies that

$$LCI(\hat{\theta}) \leq 2(\theta_1-\theta_0)c_*(\alpha,\beta), \quad c_*(\alpha,\beta) = z_{\alpha/2}c(\alpha,\beta). \tag{2.3}$$

For example, with $\alpha = .05, \beta = .50, z_{\alpha/2} = 1.96, z_{\beta/2} = .68$, we have $c(\alpha,\beta) \cong .38, c_*(\alpha,\beta) \cong .75$. If $\alpha = \beta = .05$, then $c(\alpha,\beta) \cong .25, c_*(\alpha,\beta) = .50$. The function $c(\alpha,\beta)$ serves as a cut-off for deciding when $se(\hat{\theta})$ or $LCI(\hat{\theta})$ is too high. Equivalently, $c^{-1}(\alpha,\beta)$ serves as a threshold for deciding when the estimate's precision (inversely proportional to $LCI(\hat{\theta})$) is too low.

In practice, both θ_1 (the current value) and $\theta_1 - \theta_0$ (the postulated change in the current value from past) are unknown although θ_1 can be estimated by $\hat{\theta}$. Moreover, the postulated amount of discrimination due to change in θ should vary with the current value θ_1 (or its estimate $\hat{\theta}$). This is for the simple reason that it would not be of much practical use to discriminate against small absolute changes in θ if $\hat{\theta}$ were large as compared to the situation when $\hat{\theta}$ were small or near zero. Thus, denoting by $\delta(\hat{\theta})$ the approximate postulated change in θ (i.e., $\theta_1 - \theta_0$), it follows that for large samples, we can say $\hat{\theta}$ is precise (in the sense that at one-sided levels $\alpha/2$, $\beta/2$ for the two types of error respectively, it can discriminate against an approximate postulated one-sided change of $\delta(\hat{\theta})$ as reflected in the current value θ_1) if

$$\frac{se(\hat{\theta})}{\delta(\hat{\theta})} \le c(\alpha,\beta) . \qquad (2.4)$$

It is interesting to note that the above criterion generalizes the usual $CV(\hat{\theta})$ in that if $\delta(\hat{\theta}) = \hat{\theta}$ (i.e., $\theta_1 = \hat{\theta}, \theta_0 = 0$), and $c(\alpha,\beta) = c_0$, then we get the left hand side of (2.4) identical to the usual $CV(\hat{\theta})$.

Based on above considerations, it follows that using testing ideas we can have an objective way to define the cut-off c_0 as well as a new interpretation of the denominator $\delta(\hat{\theta})$ ($= \hat{\theta}$ for the usual CV) as an approximate postulated change reflected in the current value that one may want to discriminate against. However, it is not clear how the function $\delta(\hat{\theta})$ should be specified in practice. Recall that with $\delta(\hat{\theta}) = \hat{\theta}$, the above measure might misbehave, i.e., besides other concerns, it may lead to the required sample size for satisfying a certain threshold $c(\alpha,\beta)$ to be too liberal for large $\hat{\theta}$ and too conservative for small $\hat{\theta}$. In other words, the corresponding CI may be too wide for large $\hat{\theta}$ and too narrow for small $\hat{\theta}$.

The basic idea proposed to overcome the above problem is as follows. Suppose we set values of $\delta(\hat{\theta})$ for selected values of $\hat{\theta}$ in the range $[\underline{\theta},\overline{\theta}]$ of interest that give rise to reasonable (i.e.,not too wide and not too narrow) CIs based on subject matter considerations. Then we construct a smooth function $\delta(\cdot)$ that goes through the selected points $(\hat{\theta},\delta(\hat{\theta}))$ and has certain desirable properties. Alternatively, one can define minimum n (denote by $n_{\min}(\hat{\theta})$) that is desirable for each selected $\hat{\theta}$ in order to get reasonable CIs, and then construct a smooth function $n_{\min}(\cdot)$ that passes through the selected points or the anchors $(\hat{\theta}, n_{\min}(\hat{\theta}))$. Note that the specification of the function $\delta(\hat{\theta})$ is equivalent to the specification of $n_{\min}(\hat{\theta})$ for given $c(\alpha,\beta)$ and $s(\hat{\theta})$ because $se(\hat{\theta}) = s(\hat{\theta})/\sqrt{n}$. We will specify the anchors directly in terms of $n_{\min}(\cdot)$ rather than $\delta(\cdot)$ because of ease in interpretability. Moreover, one could determine whether the selected values are affordable and practical using cost considerations. The main property that anchors must satisfy is based on the heuristic requirement that the larger $\hat{\theta}$ is, the easier (in the sense of sample size requirements for a given power) it should to detect or discriminate against meaningful changes in θ because the minimum size of discrimination deemed important in practice should increase with $\hat{\theta}$. This implies in view of (2.4) that the function $\delta(\cdot)$ (after suitable standardization to make it scale-free such as dividing it by $s(\hat{\theta})$) must be a nondecreasing function of $\hat{\theta}$ which, in turn, implies that the function $n_{\min}(\cdot)$ must be a nonincreasing function of $\hat{\theta}$.

Having more anchors $(\hat{\theta}, n_{\min}(\hat{\theta}))$ is certainly desirable as it gives more control on the behavior of the function $\delta(\cdot)$. In fact, the main reason for the limitations of the usual CV listed earlier can be ascribed to having only one anchor. For CV, the single anchor $(\hat{\theta}_*, n_{\min}(\hat{\theta}_*))$ for a generic value of $\hat{\theta}$ is used to define the cut-off value c_0 given by $c_0 = (s/\sqrt{n_{\min}(\hat{\theta}_*)})/\theta_*$. For example, in the case of estimating proportions for NSDUH, the effective $n_{\min}$ is taken as 55 for a generic choice of $\hat{\theta}$ as 10% and then c_0 is about 0.40 in the untransformed scale and .175 in the log scale. However, the above CV-rule provides no control on $n_{\min}(\cdot)$ for other values of $\hat{\theta}$ such as very small (1%, say) or very large (50%, say) for the estimate's precision to be acceptable. In practice, it may not be easy to define several anchors objectively. However, it may be possible to choose three typical values (low, moderate and high) in the range of θ under consideration and the corresponding values of $n_{\min}$ which may be considered adequate in practice to provide a reasonable control on the behavior of the function $\delta(\cdot)$. For example, for proportions, we can choose the three anchors as (.01, 125), (.10, 55), and (.50, 25) since the corresponding 95% CIs may be deemed as reasonable.

With three anchors, we propose to define $\delta(\cdot)$ as a polynomial of second degree given by

$$\delta(\hat{\theta}) = \gamma_0 + \gamma_1\hat{\theta} + \gamma_2\hat{\theta}^2 \tag{2.5}$$

where the $\gamma-$coefficients are specified by using the three conditions on the value of $\delta(\hat{\theta})$ corresponding to the three anchors. A diagnostic in the form of checking that the resulting function $n_{\min}(\cdot)$ is necessarily positive and nonincreasing in the range $[\underline{\theta}, \overline{\theta}]$ can be performed to ensure that the choice of the function $\delta(\cdot)$ is reasonable and its values are nonnegative. In this regard, some fine-tuning of the selection of anchors may be necessary in practice.

Sometimes, instead of working on the original scale $\hat{\theta}$, one may wish to work on a transformed scale $h(\hat{\theta})$ (where $h(\cdot)$ is a monotone smooth function) for improved normal approximation as well as for meaningful CIs; e.g., often the logit transformation is used in the case of proportions. It is observed that regardless of the choice of the transformation function $h(\cdot)$, the $\delta(\cdot)$-values at the anchors $(\hat{\theta}, n_{\min}(\hat{\theta}))$ are specified such that the ratio $se(h(\hat{\theta}))/\delta(h(\hat{\theta}))$ at $n_{\min}(\hat{\theta})$ equals $c(\alpha,\beta)$. Thus the ratio $se(h(\hat{\theta}))/\delta(h(\hat{\theta}))$ for any other realized n will be $\sqrt{n_{\min}/n}$ at anchor values of $\hat{\theta}$ because $se(h(\hat{\theta}))$ is assumed to have the form $s(h(\hat{\theta}))/\sqrt{n}$. It follows that the ratio $se(h(\hat{\theta}))/\delta(h(\hat{\theta}))$ has the desirable property of being invariant to the choice of the transformation scale $h(\cdot)$ at least at the $\hat{\theta}$-values chosen for the anchors. However, it would not be invariant for other values of $\hat{\theta}$. Clearly if several $\hat{\theta}$-values for anchors are chosen in specifying the function $\delta(\cdot)$ (say, by fitting a spline), then the proposed criterion $se(h(\hat{\theta}))/\delta(h(\hat{\theta}))$ would be (nearly) invariant for all of those $\hat{\theta}$-values. In the next section it is shown that the proposed measure remains invariant for location and scale transformations and approximately so in general if $\hat{\theta}$ represents a small change with respect to a reference point θ_0. Finally, it may be noted that in defining the anchors as well as the function $n_{\min}(\cdot)$, it may be better in practice to use the original scale of $\hat{\theta}$ for ease in interpretability.

3. DCV: THE PROPOSED MEASURE

Given an estimate $\hat{\theta}$ of a mean parameter θ and the corresponding estimated standard error $se(h(\hat{\theta}))$ $(=s(h(\hat{\theta}))/\sqrt{n})$, consider the transformation $h(\hat{\theta})$ performed in the interest of improving the validity of the normal approximation for moderate samples. Thus we assume $h(\hat{\theta}) \sim N(h(\theta), s^2(h(\hat{\theta})/n)$.

3.1 Definition of $DCV[h(\hat{\theta})]$

Now to define DCV of $h(\hat{\theta})$, we need three quantities: length of the confidence interval $LCI(h(\hat{\theta}))$, length of the discrimination interval $LDI(h(\hat{\theta}))$, and the threshold level for acceptability of an estimate's precision $c_*^{-1}(\alpha,\beta)$. These quantities can be obtained from the following steps.

Step I (Construct the Confidence Interval for a given confidence level). Choose confidence level $(1-\alpha)$, and then construct a suitable CI for $h(\theta)$. Under large sample normality, we have the symmetric CI as $h(\hat{\theta})\pm z_{\alpha/2}se(h(\hat{\theta}))$. Thus $LCI(h(\hat{\theta}))$ is $2z_{\alpha/2}se(h(\hat{\theta}))$ which depends on the confidence level and the sample size n for a given $s(h(\hat{\theta}))$. This step provides control on the level of the CI.

Step II (Specify the threshold for acceptable precision for a given power level for discrimination). Choose the power level $(1-\beta)$**,** and then define the threshold $c_*^{-1}(\alpha,\beta)$ as $z_{\alpha/2}/(z_{\alpha/2}+z_{\beta/2})$.

Step III (Construct the Discrimination Interval for the given confidence and power levels). This step provides control on the length of the CI obtained from Step I by ensuring that the sample size is large enough to discriminate against the anticipated change using the precision threshold obtained from Step II. To construct a symmetric discrimination interval $h(\theta_0)\pm\delta(h(\hat{\theta}))$ for a hypothetical past value $h(\theta_0)$, which gives $LDI(h(\hat{\theta}))$ as $2\delta(h(\hat{\theta}))$, choose three anchors $\{h(\hat{\theta}_i), n_{\min}(h(\hat{\theta}_i))\}, i=1,2,3$ such that the corresponding CIs in the original scale of θ are deemed to be reasonable, i.e., not too wide and not too narrow. The $\hat{\theta}$-values in the anchors must be in the range $[\underline{\theta},\overline{\theta}]$ and their choice is based on subject matter considerations. Now compute the corresponding anticipated change $\delta(h(\hat{\theta}_i))$ specified indirectly by the anchors as

$$\delta(h(\hat{\theta}_i))=\frac{s(h(\hat{\theta}_i))}{\sqrt{n_{\min}(h(\hat{\theta}_i))}}c^{-1}(\alpha,\beta)\,. \tag{3.1}$$

Next compute the $\gamma-$coefficients of the discrimination function $\delta(\cdot)$ (2.5) with $\hat{\theta}$ replaced by $h(\hat{\theta})$ such that it satisfies (3.1) at the chosen anchor values of $\hat{\theta}$. Check that the resulting function $\delta(\cdot)$ is positive for all values of $\hat{\theta}$ in the region $[\underline{\theta},\overline{\theta}]$, and plot $\delta(h(\hat{\theta}))/s(h(\hat{\theta}))$ as a function of $\hat{\theta}$ to check that it is nondecreasing. Alternatively, for ease in interpretability, plot $n_{\min}(h(\hat{\theta}))$ as a function of $\hat{\theta}$ to check that it is nonincreasing where $n_{\min}(h(\hat{\theta}))$ can be computed using a relation similar to (3.1) as

$$n_{\min}[h(\hat{\theta})]=\left(\frac{s[h(\hat{\theta})]}{\delta[h(\hat{\theta})]}c^{-1}(\alpha,\beta)\right)^2\,. \tag{3.2}$$

If the behavior of the function $n_{\min}(h(\hat{\theta}))$ is deemed unreasonable in the sense that it is not monotone decreasing or that the required values of $n_{\min}$ for certain regions of $\hat{\theta}$ are too extreme (high or low), then the choice of anchors should be revised.

Step IV (Compute $DCV[h(\hat{\theta})]$). Given $LCI(h(\hat{\theta}))$ from Step I and $LDI(h(\hat{\theta}))$ from Step III, compute $DCV[h(\hat{\theta})]$ as

$$DCV[h(\hat{\theta})]=\frac{LCI[h(\hat{\theta})]}{LDI[h(\hat{\theta})]} \tag{3.3}$$

and declare $\hat{\theta}$ as precise if $DCV[h(\hat{\theta})]\le c_*(\alpha,\beta)$. With symmetric CI and DI, the precision rule reduces to $se[h(\hat{\theta})]/\delta[h(\hat{\theta})]\le c(\alpha,\beta)$, which is akin to the usual rule $CV(\hat{\theta})\le c_0$.

3.2 Properties of $DCV[h(\hat{\theta})]$

These are listed below.

(a) The usual $CV(\hat{\theta})$ is a special case of $DCV[h(\hat{\theta})]$. It is easily seen by setting $\delta[h(\hat{\theta})] = h(\hat{\theta})$ and $h(\hat{\theta}) = \hat{\theta}$.

(b) $DCV[h(\hat{\theta})]$ is invariant to location and scale transformations unlike $CV(\hat{\theta})$ which is not invariant to location transformation. This can be shown as follows. Suppose $h(\hat{\theta}) = a_h + b_h\hat{\theta}$ where $b_h \neq 0$. Then $se[h(\hat{\theta})] = |b_h| se(\hat{\theta})$ and

$$\begin{aligned} \delta[h(\hat{\theta})] &= \gamma_{h0} + \gamma_{h1}(a_h + b_h\hat{\theta}) + \gamma_{h2}(a_h + b_h\hat{\theta})^2 \\ &= \gamma'_{h0} + \gamma'_{h1}\hat{\theta} + \gamma'_{h2}\hat{\theta}^2 \end{aligned} \quad (3.4)$$

where $\gamma'_{h0} = \gamma_{h0} + a_h\gamma_{h1} + a_h^2\gamma_{h2}$, $\gamma'_{h1} = b_h(\gamma_{h1} + 2a_h\gamma_{h2})$, and $\gamma'_{h2} = b_h^2\gamma_{h2}$. Now the fact that $\gamma_{h0}, \gamma_{h1}, \gamma_{h2}$ by construction render $se[h(\hat{\theta}_i)] / \delta[h(\hat{\theta}_i)]$ as invariant over $h(\cdot)$ for $\hat{\theta}_i$, $i = 1,2,3$ in the anchors, implies that for any linear transformation $h(\cdot)$, $\gamma'_{hj} / |b_h|$, $j = 0,1,2$ are invariant. This completes the proof.

(c) $DCV[h(\hat{\theta})]$ is approximately invariant for any monotone twice continuously differentiable transformation $h(\cdot)$ whenever the DI (discrimination interval) is small around θ_0. To see this note that by second order Taylor expansion of $\delta[h(\hat{\theta})]$ around $\hat{\theta} = \theta_0$, we can express it approximately as a $\delta(\cdot)$ function for a linear transformation, and so by property (b), the result follows.

(d) The specification of the region $[\underline{\theta}, \overline{\theta}]$ in $DCV[h(\hat{\theta})]$ provides the flexibility for ensuring symmetry about any point in the region. For proportions, $DCV[h(\hat{\theta})]$ can be easily made symmetric about $\hat{\theta} = .50$; i.e., $DCV[h(\hat{\theta})] = DCV[h(1-\hat{\theta})]$ choosing the region $[\underline{\theta}, \overline{\theta}]$ of interest as $[\underline{\theta}, .50]$ where $\underline{\theta}$ is assigned a suitable value such as .01. Now, for $\hat{\theta}$ lying in the complementary interval $[1-\overline{\theta}, 1-\underline{\theta}]$, $DCV[h(\hat{\theta})]$ is defined as $DCV[h(1-\hat{\theta})]$.

(e) With suitable anchors in defining the function $\delta(\cdot)$, extreme behavior in the acceptable precision rule for proportions near end points arising from using $CV(\hat{\theta})$ as mentioned in the introduction can be avoided with $DCV(\hat{\theta})$.

(f) The problem of conflicting precision interpretations for proportions with $CV(\log\hat{\theta})$ and $CV(\log it\,\hat{\theta})$ in relation to $CV(\hat{\theta})$ can also be avoided by controlling the shape of the corresponding $\delta(\cdot)$ functions using suitable anchors.

(g) We note that $DCV[h(\hat{\theta})]$ does not suffer from the problem of zero denominator because $\delta(\cdot)$ is defined to be strictly positive.

3.3 A Numerical Illustration of $DCV[h(\hat{\theta})]$

For the case of proportions (i.e., when $\theta = p$), we consider linear ($h(p) = p$), log ($h(p) = \log p$), and logit ($h(p) = \log it(p)$) transformations. The interval of interest for the estimated parameter $\hat{p}$ is taken as [.01,.50] and for the complementary interval [.50, .99], measures of precision can be obtained by symmetry. For illustration of the proposed measure, we consider $CV(\hat{p})$, $CV(\log\hat{p})$, and $DCV(\log it\,\hat{p})$. We do not consider $CV(\log it\,\hat{p})$ because of the problem of zero divisor at $\hat{p} = .50$. The problems of extreme CIs under the acceptable precision rule based on $CV(\hat{p})$ and that of conflicting interpretation with $CV(\log\hat{p})$-based precision rule can be overcome with the corresponding DCV -measures. However, here we show only the behavior of the precision rule based on $DCV(\log it\,\hat{p})$ as it is sufficient to illustrate the DCV -advantage. Besides, it is the logit transformation that is commonly used in practice for building CIs for proportions because of built-in range restrictions on the parameter. In fact, for illustrating implications on CIs under precision rules based on $CV(\hat{p})$, $CV(\log \hat{p})$, and of course $DCV(\log it\,\hat{p})$, we first construct normal CIs for $\log it\,p$, and then invert to the original scale to obtain CIs for p.

Fig. 1(a) shows the extreme behavior of the function $n_{\min}(\hat{p})$ as $\hat{p}$ varies from .01 to .50 under the usual precision rule based on $CV(\hat{p})$. The $CV(\hat{p})$-precision rule requires only one anchor which was chosen as $(\hat{p} = .10, n_{\min}(\hat{p}) = 55)$. The cut-off value c_0 for this rule is .4045. It is seen that this rule is very conservative at low values of $\hat{p}$ requiring a minimum sample size of 605 at $\hat{p} = .01$ and is very liberal at high values of $\hat{p}$ with the minimum sample size requirement being only 6 at $\hat{p} = .50$. The corresponding megaphone effect on CI-width is shown in Fig 1(b). Fig. 2(a) shows the aberrant behavior of $n_{\min}(\hat{p})$ under the $CV(\log \hat{p})$-precision rule where $(\hat{p} = .10, n_{\min}(\hat{p}) = 55)$ is again used as the anchor. The cut-off value c_0 for this rule is .175. The aberrant behavior is due to the fact that $n_{\min}(\hat{p})$ decreases below 55 as $\hat{p}$ increases to about .20 and then starts increasing to a maximum of 68 as $\hat{p}$ approaches .50. This contradicts the minimum sample size requirement implied by the $CV(\hat{p})$-precision rule. It is also counter-intuitive to the notion that for a given precision level, the required minimum sample size should decrease as $\hat{p}$ increases to .50 because the corresponding practically meaningful discrimination amount for change should also increase. In order to make $n_{\min}(\hat{p})$ non-increasing, Chromy (2001) suggested the 'rule of 68' so that the adjusted $n_{\min}(\hat{p})$ never goes below 68. Fig 2(b) shows the overly narrow CI-width for the adjusted $CV(\log \hat{p})$-precision rule with a constant minimum sample size of 68 for all $\hat{p}$ starting around .05. For lower values of $\hat{p}$, this rule (with a minimum sample size of 152 at $\hat{p} = .01$), however, seems more reasonable than the $CV(\hat{p})$-precision rule. Finally, Fig 3(a) shows the behavior of $n_{\min}(\hat{p})$ under the $DCV(\log it\,\hat{p})$-precision rule when three anchors corresponding to (.01, 125), (.10, 55), and .50, 25) are used. The corresponding CI-width shown in Fig. 3(b) no longer has the extreme megaphone shape but is more tempered due to a gradual decrease in the minimum sample size requirement as governed by the anchors. Here, the values for the confidence and the power levels are chosen as .95 and .50 respectively which imply that $c_*(\alpha, \beta)$ equals .75.

4. SUMMARY AND DISCUSSION

DCV (Discrimination Coefficient of Variation) was proposed as a new measure of precision to provide a generalization of and a principled alternative to the commonly used CV derived from intuitive considerations. Using testing ideas, for a given estimator, DCV depends on LCI (length of the confidence interval) and LDI (length of the discrimination interval) constructed from the estimator such that prescribed confidence and power levels are achieved. DI (discrimination interval) is somewhat analogous to CI, and is defined as the postulated change in the parameter that can be discriminated against with certain power. DCV is defined as the ratio of LCI over LDI and overcomes several limitations of CV by explicitly controlling both level and length of the CI by using an objective criterion for defining the precision threshold for LCI relative to LDI.

Although DCV is not completely free from subjective considerations such as the choice of anchors to better control the precision rule over the range of values of the estimator, it is done to a much lesser extent than in the case of CV. Some practical guidance is provided to make these choices as objective as possible. The usual CV is, in fact, a special case of DCV, and also uses anchors but only one. It was shown that limitations of CV result from not having sufficient anchors. Through an illustrative example, it was shown that DCV can be easily computed in practical applications, and in view of its several desirable theoretical properties, it is expected to work well in practice. An important feature of DCV is the property of invariance to smooth monotonic transformations to the estimator at the anchor values and approximate invariance for others. With many anchors, however, this limited invariance property can be enhanced.

The large sample normal CI was used in defining the DCV-precision rule although the underlying idea of defining a suitable cut-off for the ratio of LCI over LDI is quite general. In other words, the construction of CI need not be based on standard error of the estimator, and sampling distributions other than normal can be used to ensure that confidence and power levels are satisfied. In the case of proportions with small samples or when the estimator is zero, for example, one can use exact binomial to define CI; see Jovanovic and Levy (1997) for the rule of three for finding a 95% one-sided CI when the estimated proportion is zero. Such extensions of DCV to small samples require further investigation. It may be of interest to note that the notion of DCV easily lends itself to its generalization to the multi-dimensional case, although component-wise DCVs will always be of interest to practitioners in view of their simplicity and interpretability. In particular, for a vector estimate $\hat{\theta}$ with the estimated covariance matrix $V(\hat{\theta})$ and the corresponding discrimination vector $\delta(\hat{\theta})$, the multivariate DCV can be naturally defined as the inverse of the quadratic form $\delta'(\hat{\theta})\, V(\hat{\theta})^{-1}\delta(\hat{\theta})$. The cut-off for the precision rule can be obtained from the inverse of the noncentrality parameter under the large sample chi-square distribution of the quadratic test statistic $(\hat{\theta}-\theta_0)'V(\hat{\theta})^{-1}(\hat{\theta}-\theta_0)$ so that both confidence and power levels are satisfied.

Finally we remark that in defining DCV it was assumed that $se(\hat{\theta})$ is of the form $s(\hat{\theta})\big/\sqrt{n}$. In practice, often the $se(\hat{\theta})$ may not have this simple form. For example, with clustered data or with complex sample surveys such situations may arise. However, one can still use approximately this simple form after adjusting by a multiplicative factor defined as the square root of the dispersion effect or design effect which can be estimated from past data and treated as known. In the NSDUH application which led to the present study, estimated proportions for drug use prevalence are of interest among other estimates. Although the binomial variance is not directly applicable due to a complex sampling design, one can still use the binomial working assumption with a suitable adjustment via design effect. More generally, for any ratio estimate lying between 0 and 1, DCV for proportions can be used under the binomial working assumption with a suitable adjustment for dispersion effect.

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Fig. 1(a) $n_{\min}(\hat{p})$ under $CV(\hat{p})$ -rule

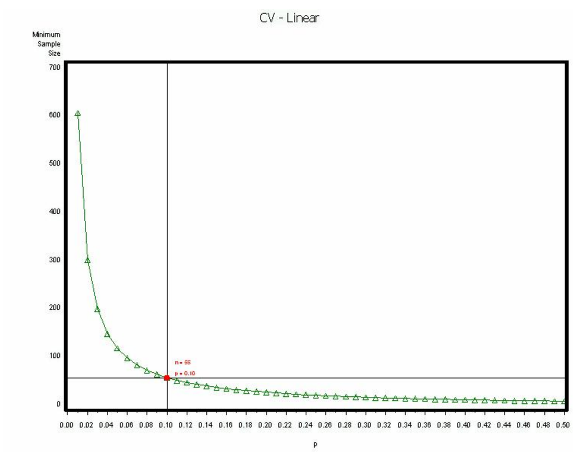

Fig. 2(a) $n_{\min}(\hat{p})$ under $CV(\log\,\hat{p})$ -rule

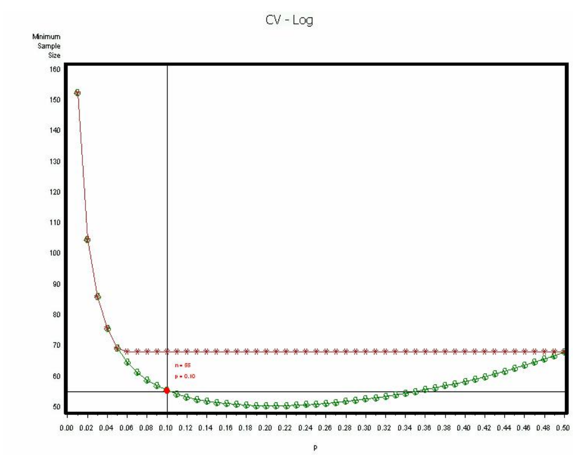

Fig. 3(a) $n_{\min}(\hat{p})$ under $DCV(\log it\ \hat{p})$ -rule

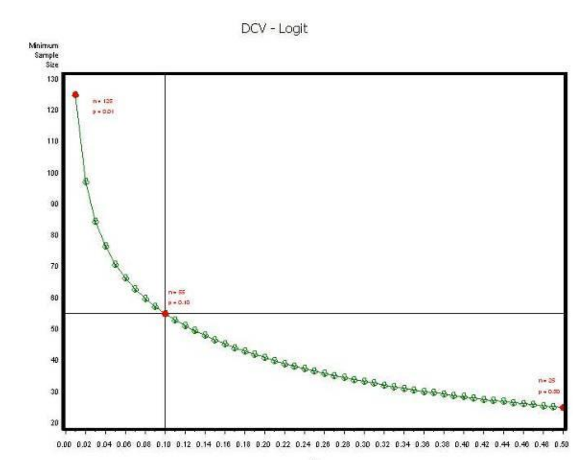

Fig. 1(b) 95% CI with $n_{\min}(\hat{p})$ under $CV(\hat{p})$ -rule

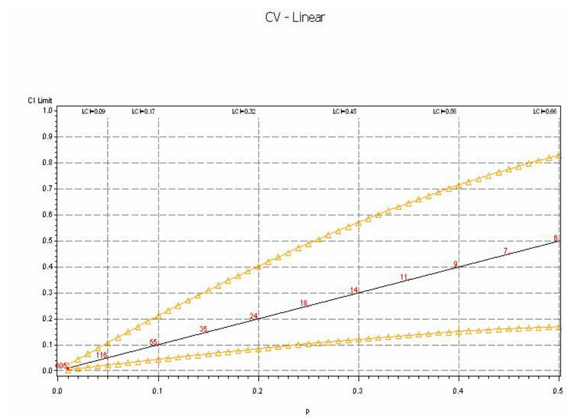

Fig. 2(b) 95% CI with $n_{\min}(\hat{p})$ under $CV(\log\,\hat{p})$ -adj

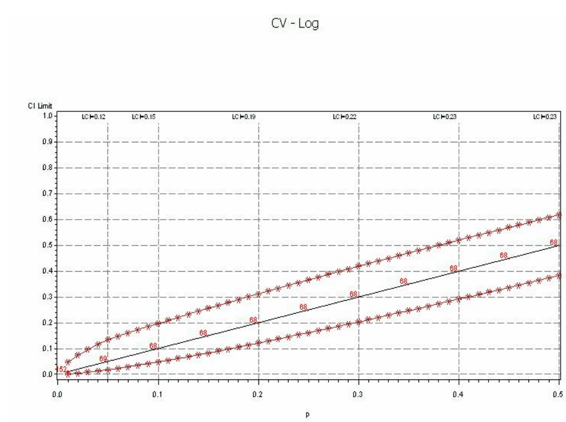

Fig. 3(b) 95% CI with $n_{\min}(\hat{p})$ under $DCV(\log it\ \hat{p})$ -rule